PHYSICAL REVIEW D VOLUME 49, NUMBER 12 15 JUNE 1994

Noise and fluctuations in semiclassical gravity

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(Received 24 November 1993)

We continue our earlier investigation of the back reaction problem in semiclassical gravity with the Schwinger-Keldysh or closed-time-path (CTP) functional formalism using the language of the decoherent history formulation of quantum mechanics. Making use of its intimate relation with the Feynman-Vernon influence functional method, we examine the statistical mechanical meaning and show the interrelation of the many quantum processes involved in the back reaction problem, such as particle creation, decoherence, and dissipation. We show how noise and fluctuation arise naturally from the CTP formalism. We derive an expression for the CTP effective action in terms of the Bogoliubov coefficients and show how noise is related to the fluctuations in the number of particles created. In so doing we have extended the old framework of semiclassical gravity, based on the mean field theory of Einstein equation with a source given by the expectation value of the energy-momentum tensor, to that based on a Langevin-type equation, where the dynamics of the fluctuations of spacetime is driven by the quantum fluctuations of the matter field. This generalized framework is useful for the investigation of quantum processes in the early Universe involving fluctuations, vacuum instability, and phase transition phenomena as well as the nonequilibrium thermodynamics of black holes. It is also essential to an understanding of the transition from any quantum theory of gravity to classical general relativity.

PACS number(s): 04.62.+v, 03.65.Sq, 05.40.+j, 98.80.Cq

I. INTRODUCTION AND SUMMARY

The *central theme* of this paper is to show how the back reaction problem in semiclassical gravity [1,2] can be viewed in the light of a quantum open system [3] and how the concepts and techniques of nonequilibrium statistical mechanics can be fruitfully applied to this theory for the description of quantum statistical processes in the early Universe [4]. This idea has been used recently to expound the dissipative nature of effective quantum field theories [5,6], some basic issues of quantum cosmology [7–11] and quantum mechanics [12,13].

The *primary aim* of this paper is to elevate the theory of semiclassical gravity from the old level based on the semiclassical equation with a source given by the vacuum expection value of the energy-momentum tensor of quantum matter fields associated with particle creation [14] whose back reaction leads to dissipation [15] in the dynamics of spacetime, to a new level based on an Einstein-Langevin equation with a stochastic source given by the fluctuations in the matter field, where the effects of noise and fluctuations are also incorporated in the processes of decoherence and dissipation.

The *main topics* of investigation reported in this paper are noise and fluctuations in quantum fields associated with particle creation in cosmological spacetimes; and dissipation in the dynamics of spacetime due to the back reaction of these quantum processes.

The *specific findings* of this paper are: (i) explicitly showing the relation of particle creation with decoherence through the noise kernel in the influence functional and the Bogoliubov coefficients in the theory of quantum fields in curved spacetime. (ii) delineating the character of noise from the coupling of the quantum field to the background spacetime, and (iii) deriving the noise terms (in addition to the average of the energy-momentum tensor) in the semiclassical Einstein equation as a stochastic source and relating the fluctuations of energy density to the fluctuations in the number of particles created.

The *principal method* used here is that of the Schwinger-Keldysh or closed-time-path (CTP) functional formalism [16]. This is the method we used before (with a Bianchi type-I universe as model) [17] in deriving a real and causal equation of motion for the cosmological back reaction problem. There we identified a nonlocal kernel in the dissipative term and showed that the integrated dissipative power in the dynamics of spacetime is equal to the energy density of the total number of particles created. This clearly established the dissipative nature of quantum processes such as particle creation [18]. We now describe the progression of ideas and the evolution of the background leading to the present work, which addresses the other part of this problem (which actually existed in our original results, but was not the focus of attention in our earlier investigation), i.e., noise and fluctuations.

Two earlier papers written by one of us outlined the usefulness of adopting the quantum open system point of view for understanding the dissipative nature of quantum fields and semiclassical gravity [5] and some basic issues of quantum cosmology [7]. Paper [5] noticed the missing role played by noise in the equation of motion for the effective system, and advocated that a Langevin-type equation should be used in place of the conventional semiclassical Einstein equation. It was also predicted there that for quantum fields under general conditions a colored noise source should appear in the driving term. The other two conjectures put forth in that paper, i.e., the existence of a fluctuation-dissipation relation for nonequilibrium quantum systems which can be used to understand back reaction problems in semiclassical gravity, and the existence of dissipative behavior in effective field theories, will be taken up in later investigations [19,20,6].

Paper [7] pointed out the interrelation of quantum and statistical processes such as decoherence [21–27], correlation [28], particle creation (as amplification of vacuum fluctuations [14]), noise, fluctuation [29–38], dissipation [8,15,17,18,39–41], and their role in the evolution of the effective system (which can be the classical limit of quantum mechanics [24] or the semiclassical spacetime dynamics from quantum cosmology [11,9]). The pairwise relation of these processes have been explored since then by many authors in various contexts. For example, that noise governs decoherence was seen in all the analysis of environment-induced decoherence [21]. This, together with the fluctuation-dissipation theorem which relates noise to dissipation, implies that there is a limit to the degree of decoherence and the accuracy of defining the classical trajectory [24]. There is also a balance between decoherence and the build-up of correlation between the canonical variables of a quantum system in reaching the classical limit [9]. The relation of particle creation and decoherence is explored in [42] as is also implicit in [9]. The relation of noise and fluctuations in particle number is studied in [43] for the quantum statistics of cosmological particle creation.

The main features of the quantum open system paradigm are well illustrated by the quantum Brownian model. Using the Feynman-Vernon [44] formalism, and extending the work of Caldeira and Leggett [45] and Grabert *et al.* [46] to a general environment, Hu, Paz, and Zhang [30,31] looked into the nature of colored noises from the environment and the nonlocal dissipation they engender on the dynamics of the system. In this formalism the effects of noise and dissipation can be extracted from the noise and dissipation kernels as the real and imaginary parts of the influence functional, their interrelation manifesting simply as the fluctuation-dissipation theorem [47] obtained as a categorical functional relation. If one views the quantum field as the environment and spacetime as the system in the quantum open system paradigm, then the statistical mechanical meaning of the back reaction problem in semiclassical cosmology can be understood more clearly [5]. In particular, one can identify noise with the coarse-grained quantum fields, derive the semiclassical Einstein equation as a Langevin equation, and understand the back reaction process as the manifestation of a fluctuation-dissipation relation [19].

One gratifying by-product in this earlier process of search and discovery is that the influence functional method [44] used in the context of nonequilibrium statistical mechanics is largely equivalent to the Schwinger-Keldysh, or the closed-time-path (CTP) method [16] developed in quantum field theory. This for us is particularly useful, because not only does one recover from the influence functional (IF) the dissipation kernel in the equation of motion of the CTP, but one can now clearly identify the meaning of the noise kernel already existent in the CTP effective action and find the corresponding stochastic source in the semiclassical equation of motion. We will indeed borrow the physical insight provided by the IF formalism to analyze the results obtained by the CTP method.

The character and function of noise in some common quantum field processes have been studied before in a different context. For example, [30] treated colored noise from a non-ohmic environment [31] dealt with colored noise from nonlinear coupling, [34] discussed particle creation as the result of parametrically amplified quantum noise. The quantum origin of noise and fluctuations basic to the gravitational-instability theory of structure formation is discussed in [32]. The relation of stochastic and thermal field is explained in [48] while that between quantum noise and thermal radiance in accelerated observers and spacetimes with horizons for the Unruh [51] and Hawking [52,53] effects is discussed in [49,50,54,33,34]. We will discuss the back reaction problem in semiclassical gravity in terms of the fluctuation-dissipation relation in later publications [19,20]. Dissipation in quantum cosmology arising from the neglected inhomogeneous modes in a minisuperspace approximation leading to an effective Wheeler-DeWitt equation was discussed in [40,55]. One could extract the noise corresponding to the coarsed-grained modes of spacetime excitations and define a gravitational entropy, as discussed in [4]. One could also deduce a fluctuation-dissipation relation in quantum cosmology, exemplified by the back reaction problem in a Bianchi type-I universe [19]. Sharing the same goal as this paper but taking a different approach is the work of [56], in which the colored noise associated with quantum fields is identified by means of a cumulant expansion on the influence functional and an Einstein-Langevin equation for the back reaction problem was derived in semiclassical cosmology. A recent paper of Kuo and Ford [57] also addresses fluctuations in semiclassical gravity. They work with the energy-momentum tensor in the canonical formalism. Their approach and results should have points of contact with ours (see Sec. V below).

The following is a brief description of the contents of this paper. In order to demonstrate the stochasticity of semiclassical evolution induced by quantum fluctuations, we shall analyze a cosmological model in which a free, real scalar field is coupled to the scale factor of a Friedmann-Robertson-Walker (FRW) universe. One can think of this as the semiclassical limit of the corresponding model in quantum cosmology, this transition having been studied by many authors (the latest complete work is that of [9], in which are listed some of the earlier references). Using the conceptual framework of the consistent or decoherent histories approach to quantum cosmology [22,23], we consider histories where the matter field is fully coarse grained. From this we obtain a closed, exact expression for the decoherence functional between two such histories, that is, between two different specifications of the

FRW conformal factor, as a function of time, this being the only remaining degree of freedom. (This result was obtained also in [9,56].) This expression will allow us to show that decoherence is directly related to the differential in particle creation between one and the other history.

From this we shall then discuss the dynamics of fluctuations in the scale factor around its expectation value, as seen by an observer who does not have access to the full information of the scalar field except for its overall effect on the dynamics of the system. We shall show that this dynamics is aptly described by a Langevin-type equation, where the usual semiclassical corrections to the matter energy-momentum tensor are supplemented by stochastic terms. Moreover, we shall deduce from the formalism itself the noise autocorrelation function. We stress that this is the correct way of treating fluctuations of quantum fields as noise [58]. Quantum fluctuations in the inflaton field viewed as seeds for galaxy formation is a very attractive program [59,60]. But the existing practice is flawed in at least two respects [32]: (1) the correct deduction of the origin and nature of noise from quantum fluctuations; and (2) the correct treatment of quantum to classical transition in the long-wavelength perturbation modes via decoherence considerations. We show how the quantum bath variables after averaging effectively contribute a stochastic source with a correlation function determined by the nature of the bath and the coupling. In the most general cases one expects colored and non-Gaussian noises to appear. The habitual way of simply reinterpreting the quantum scalar field as a fluctuating classical Gaussian source with its mean square value set equal to the corresponding quantum average value is incorrect except at the coincidence limit.

As can be seen from the summary above, the moral of our story is that the nature of semiclassical dynamics can only be appreciated in full by combining concepts and techniques from quantum and statistical field theory. For this reason, we shall begin in Sec. II with a brief summary of the closed time path effective action [16] and the influence functional [44] formalisms. We shall show how the decoherence functional formulation provides a natural framework for the application of these concepts to our problem. In Sec. III we apply these formalisms to the cosmological model described above, arriving at an exact expression for the decoherence functional in terms of the Bogoliubov coefficients. This expression makes obvious the connection between decoherence and particle creation.

In Sec. IV, we analyze the semiclassical dynamics experienced by an observer confined to one of the decohering histories. For definiteness, we shall focus on cosmic evolutions which are close to a solution of the usual (deterministic) semiclassical Einstein equations. We shall show how the structure of the decoherence functional implies the presence of noise in this dynamics, and determine its statistical properties. In this light we issue a warning that the usual procedure of treating the spontaneous fluctuations in the field as a classical Gaussian stochastic variable [60] has only limited validity. In Sec. IV C we analyze the nonlocal nature of noise and dissipation by examining a simple set of histories which depart only slightly from the Minkowski space and discuss how the colored nature of noise depends on the coupling of the field to spacetime. In Sec. V, we explain the physical origin of noise as fluctuations in particle creation number. We show that the fluctuations in the energy-momentum tensor calculated in the CTP formalism can also be obtained from the fluctuations in the number of particles via simple quantum field theory arguments. In Sec. VI we summarize our findings and discuss the implications of our results. A few technical details for the derivation of the main results in the text are put in the Appendix.

II. METHODS IN QUANTUM AND STATISTICAL FIELD THEORIES

As described in the Introduction, two methods have been used effectively for the description of the back reaction problem: the closed time path effective action (CTP, or Schwinger-Keldysh) formalism [16] for obtaining a causal and real equation of motion; and the influence functional (IF, or Feynman-Vernon [44]) method for treating a quantum open system, in identifying the noise in the environment and the dissipation in the effective equation of motion for the system. We give here a brief description of these formalisms and their interconnection. We also sketch the decoherent history formulation of quantum mechanics as we will use this conceptual framework to apply the IF and CTP formalisms to the analysis of semiclassical gravity theory.

A. The influence functional approach to nonequilibrium field theory

The IF approach [44] is designed to deal with a situation in which the system S described, say, by the x fields is interacting with an environment E, described by the q fields (in another common statistical mechanical nomenclature these are also called the relevant and irrelevant parts, respectively). The full quantum system is described by a density matrix $\rho(x,q;x',q',t)$. If we are only interested in the state of the system as influenced by the overall effect, but not the precise state, of the environment, then the reduced density matrix $\rho_r(x,x',t) = \int dq\ \rho(x,q;x',q,t)$ would provide the relevant information. (The subscript r stands for reduced.) It is propagated in time from t_i by the propagator $\mathcal{J}_r$:

$$\rho_r(x,x',t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i\ \mathcal{J}_r(x,x',t \mid x_i,x'_i,t_i)\ \rho_r(x_i,x'_i,t_i\ ). \tag{2.1}$$

Assuming that the action of the coupled system decomposes as $S = S_s[x] + S_e[q] + S_{\rm int}[x,q]$, and that the initial density matrix factorizes (i.e., takes the tensor product form), $\rho(x,q;x',q',t_i) = \rho_s(x,x',t_i)\rho_e(q,q',t_i)$, the propagator for the reduced density matrix is given by

$$\mathcal{J}_r(x,x',t|\; x_i,x_i',t_i) = \int_{x_i}^{x_f} Dx \int_{x_i'}^{x_f'} Dx'\; e^{i(S_s[x]-S_s[x']+S_{\rm IF}[x,x',t])},$$

where $S_{\rm IF}$ (called $\delta\mathcal{A}$ in [30]) is the influence action related to the influence functional $\mathcal{F}$ defined by

$$\begin{aligned}\mathcal{F}[x,x'] &\equiv e^{iS_{\rm IF}[x,x',t]}\\ &\equiv \int dq_f\; dq_i\; dq_i' \int_{q_i}^{q_f} Dq \int_{q_i'}^{q_f} Dq'\; e^{i(S_e[q]+S_{\rm int}[x,q]-S_e[q']-S_{\rm int}[x',q'])}\rho_e(q_i,q_i',t_i). \end{aligned} \tag{2.2}$$

$S_{\rm IF}$ is typically complex; its real part $\mathcal{R}$, containing the dissipation kernel D, contributes to the renormalization of S_s, and yields the dissipative terms in the effective equations of motion. The imaginary part $\mathcal{I}$, containing the noise kernel N, provides the information about the fluctuations induced on the system through its coupling to the environment. Since the connection between these kernels and their effect on the physical processes of dissipation and fluctuation has been discussed at length elsewhere (cf. Ref. [30]), we shall limit ourselves here only to a schematic summary.[1]

The main features of the influence action follow from the elementary properties $S_{\rm IF}(x,x') = -S_{\rm IF}(x',x)^*$ and $S_{\rm IF}(x,x) = 0$, which can be deduced from its definition Eq. (2.2), and derived in the final analysis from the unitarity of the underlying quantum theory of the closed system. If we decompose $S_{\rm IF}$ in its real and imaginary parts, $S_{\rm IF} = \mathcal{R} + i\mathcal{I}$, then $\mathcal{R}(x,x') = -\mathcal{R}(x',x)$, $\mathcal{I}(x,x') = \mathcal{I}(x',x)$, and $\mathcal{R}(x,x) = \mathcal{I}(x,x) = 0$. Keeping only quadratic terms, we may write

$$S_{\rm IF}(x,x') = \int dt\; dt' \left(\frac{1}{2}(x-x')(t)D(t,t')(x+x')(t') + \frac{i}{2}(x-x')(t)N(t,t')(x-x')(t')\right), \tag{2.3}$$

where D and N stand for the real dissipation and noise kernels, respectively ($D \equiv 2\eta$, $N \equiv 2\nu$, in the notations of [30]). It is convenient to express D as $D(t,t') = -\partial_{t'}\gamma(t,t')$, and rewrite

$$S_{\rm IF}(x,x') = \int dt\; dt' \left(\frac{1}{2}(x-x')(t)\gamma(t,t')(\dot{x}+\dot{x}')(t') + \frac{i}{2}(x-x')(t)N(t,t')(x-x')(t')\right). \tag{2.4}$$

The physical meaning of the γ kernel may be elucidated by deriving the mean field equation of motion for the mean value of the system variable $\bar{x}$. It is

$$\frac{\partial S_s}{\partial \bar{x}(t)} + \int dt'\; \gamma(t,t')\frac{d\bar{x}(t')}{dt'} = 0. \tag{2.5}$$

The term containing γ represents the back reaction of the environment on the system. It causes the dissipation of energy from the system by an amount (integrated over the whole history of the system)

$$\Delta E = \int dt\; dt'\; \gamma(t,t')\dot{\bar{x}}(t)\dot{\bar{x}}(t'). \tag{2.6}$$

Thus we see that the even part of the kernel γ is associated with dissipation, while the odd part can be assimilated to a nondissipative environment-induced change in the system dynamics. In quantum field-theoretic applications, the odd part of γ will contain formally infinite terms which can be absorbed in the classical action for the system via standard renormalization procedures [16]. For simplicity, we shall assume that only the even part of γ is left after renormalization has been carried out.

In general, the γ and N kernels are nonlocal; however, their main features are manifest already under the local approximation $\gamma \sim \gamma_0\delta(t-t')$, $N \sim N_0\delta(t-t')$. The influence action then takes the form

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[1]This simplified schematic discussion is really just for the illustration of main ideas, not for precision and completeness. The reader is referred to [30–34] for details on the discussion of the process of decoherence in quantum to classical transition, the origin and nature of quantum noise, the fluctuation-dissipation relation and the explicit derivations of the master, Fokker-Planck, and Langevin equations for the models of a Brownian particle in a general environment and interacting quantum fields in cosmological spacetimes.

$$S_{\rm IF}(x,x') = \int dt \left(\frac{1}{2}(x-x')(t)\gamma_0(\dot{x}+\dot{x}')(t) + \frac{i}{2}(x-x')(t)N_0(x-x')(t) \right). \tag{2.7}$$

Assuming an action functional of the simple form $S_s[x] \sim \int\{\frac{1}{2}\dot{x}^2 - V(x)\}$, it is straightforward to derive the master equation for the reduced density matrix [44,45]:

$$i\frac{\partial \rho_r}{\partial t} \sim \left[\left(-\frac{1}{2}\partial_x^2 + V(x)\right) - \left(-\frac{1}{2}\partial_{x'}^2 + V(x')\right) - i\frac{\gamma_0}{2}(x-x')\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) - i\frac{N_0}{2}(x-x')^2\right]\rho_r. \tag{2.8}$$

The object "closest" (see [28]) to a classical distribution function is the Wigner function [61]

$$f_W(X,p) = \int dy\; e^{ipy}\rho_r\left(X+\frac{y}{2}, X-\frac{y}{2}\right), \tag{2.9}$$

where $X \equiv (1/2)(x+x'), y \equiv x - x'$. The master equation (2.8) implies (to lowest order in a Kramers-Moyal expansion) the Fokker-Planck equation [62]

$$\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial X} - V'\frac{\partial}{\partial p}\right) f_W = \left(\gamma_0\frac{\partial}{\partial p}p + \frac{N_0}{2}\frac{\partial^2}{\partial p^2}\right) f_W \tag{2.10}$$

(where $V' = dV/dx$). From this equation one can see clearly the stochasticity in the semiclassical dynamics. However, it is better to defer further discussion to Sec. II C below, until we have introduced the notion of the decoherence functional. Suffice it to observe here that the Fokker-Planck equation admits the equilibrium solution

$$f_W^{\rm eq} \sim e^{-(2\gamma_0/N_0)[(p^2/2)+V(x)]} \tag{2.11}$$

from which a fluctuation-dissipation theorem $N_0 = 2\gamma_0\langle p^2\rangle_{\rm eq}$ can be derived. If the environment acts as a heat bath, then $\langle p^2\rangle_{\rm eq} \sim k_B T$, and this reduces to the Einstein-Kubo formula for the dispersion coefficient.

B. The closed-time-path functional formalism in quantum field theory

In the CTP approach, our goal is not to follow the dynamics of the full density matrix, or even the system part, but only the expectation values of the fields as they unfold in time. This evolution is governed by a real and causal equation of motion, which is obtained from the CTP effective action by a variational principle.

Let ψ be the fields in the theory, and $\bar{\psi}$ their expectation values for any given initial states. Consider pairs of histories (ψ,ψ') defined on all spacetime, with the property that $\psi(T^0) = \psi'(T^0)$ for a given very large time T^0 (in practice, we shall implicitly take the limit $T^0 \to \infty$). Assume for simplicity (more general choices are also possible [63]) that the fields were originally in their vacuum state $|0{\rm in}\rangle$. Then we can introduce external sources J, J', and construct the CTP generating functional

$$Z[J,J'] = e^{iW[J,J']} = \langle 0{\rm in}|\tilde{T}(e^{-i\int J'\psi})T(e^{i\int J\psi})|0{\rm in}\rangle, \tag{2.12}$$

where T ($\tilde{T}$) stands for time (antitime) ordering. Observe that the generating functional W is totally defined once the in state $|0{\rm in}\rangle$ is chosen and that $W \equiv 0$ whenever $J = J'$. Now introduce the path integral representation

$$Z[J,J'] = e^{iW[J,J']} = \int D\psi\; D\psi'\; e^{i(S[\psi]-S^*[\psi']+J\psi-J'\psi')}. \tag{2.13}$$

The expectation values can be obtained as

$$\bar{\psi} = \frac{\delta W}{\delta J}, \quad \bar{\psi}' = -\frac{\delta W}{\delta J'}. \tag{2.14}$$

The physically relevant situation under consideration corresponds to setting $J = J' = 0$.

The CTP effective action is just the Legendre transform of W

$$\Gamma_{\rm CTP}[\bar{\psi},\bar{\psi}'] = W[J,J'] - J\bar{\psi} + J'\bar{\psi}', \tag{2.15}$$

where now the sources are thought of as functionals of the background fields $\bar{\psi},\bar{\psi}'$. In particular, the equations of motion are the inverses of Eqs. (2.14):

$$\frac{\delta\Gamma_{\rm CTP}}{\delta\bar{\psi}} = -J, \quad \frac{\delta\Gamma_{\rm CTP}}{\delta\bar{\psi}'} = J'. \tag{2.16}$$

The physical situations correspond to solutions of the homogeneous equations at $\bar{\psi} = \bar{\psi}'$. These equations are real and causal. Moreover, $\Gamma_{\rm CTP}[\bar{\psi},\bar{\psi}'] = -\Gamma^*_{\rm CTP}[\bar{\psi}',\bar{\psi}]$, and $\Gamma_{\rm CTP}[\bar{\psi},\bar{\psi}] \equiv 0$. As the generating functional itself, the CTP effective action is totally defined once the initial quantum state is given.

To apply this formalism to the situation above, we should substitute the ψ field by the pair (x,q). When the physical situation requires treating the x and q fields asymmetrically, as is the case when, say, only the system field x is relevant, we do not couple the q field to an external source. (In a perturbative evaluation of the CTP generating functional, this means discarding all graphs with q fields on some external leg.) Comparing the path integral expression for the generating functional with the IF approach described earlier, we find

$$e^{iW[J,J']} = \int Dx\; Dx'\; e^{i(S_s[x]-S_s[x']+Jx-J'x'+S_{\rm IF}[x,x',+\infty])}. \tag{2.17}$$

Conversely, we may describe the influence action as the CTP effective action for the quantum q fields interacting with external c-number x fields specialized to the expectation values of its arguments.

In the semiclassical approximation, one can neglect Feynman graphs containing closed x field loops, corresponding to quantum effects of the x fields. Then the path integral and the Legendre transformation may be computed explicitly, yielding

$$\Gamma_{\rm CTP}[x,x'] \approx S_s[x] - S_s[x'] + S_{\rm IF}[x,x',+\infty]. \quad (2.18)$$

This equation shows the connection between the CTP effective action and the influence functional. From this we may derive the semiclassical equations of motion for the expectation values of the x field. We see that the noise kernel does not contribute to these equations, because, it being even under the exchange of x and x', its variation vanishes at the coincidence point. However, we shall show below, as is also clear from the master equation point of view [30], that the noise kernel determines the dynamics governing the deviations from the expectation value.

As a simple example of the foregoing, let us consider a model where the system variable x is coupled to an array of environment coordinates $\{q_n\}$, the action being $S[x,q_n] = S_s[x] + \sum_n \{S_e[q_n] + \int dt\ \Xi[q_n]x\}$ (models of this kind were considered by Schwinger [16] in his analysis of quantum Brownian motion, and by many authors afterward).

The CTP effective action takes the form $\Gamma_{\rm CTP}[x,x'] = S_s[x] - S_s[x'] + \Gamma[x,x']$, Γ being related to $S_{\rm IF}$ through Eq. (2.18). Keeping only quadratic terms in the CTP effective action, we write

$$e^{i\Gamma[x,x']} = e^{i\int dt\, dt'\ \{G_{++}(t,t')x(t)x(t')+G_{+-}(t,t')x(t)x'(t')+G_{-+}(t,t')x'(t)x(t')+G_{--}(t,t')x'(t)x'(t')\}}. \quad (2.19)$$

On the other hand, under the semiclassical approximation for the system variable, we find

$$e^{i\Gamma[x,x']} = \left\langle 0\text{in}\left|\prod_n \tilde{T}(e^{-i\int dt\ \Xi[q_n]x'})T(e^{i\int dt\ \Xi[q_n]x})\right|0\text{in}\right\rangle. \quad (2.20)$$

Taking the variational derivatives of these equations with respect to x and x' at $x = x' = 0$, we find

$$G_{++}(t,t') = \sum_n i\langle 0\text{in}|T(\Xi[q_n(t)]\Xi[q_n(t')])|0\text{in}\rangle, \quad (2.21)$$

$$G_{+-}(t,t') = \sum_n (-i)\langle 0\text{in}|(\Xi[q_n(t')]\Xi[q_n(t)])|0\text{in}\rangle, \quad (2.22)$$

$$G_{-+}(t,t') = \sum_n (-i)\langle 0\text{in}|(\Xi[q_n(t)]\Xi[q_n(t')])|0\text{in}\rangle, \quad (2.23)$$

$$G_{--}(t,t') = \sum_n i\langle 0\text{in}|\tilde{T}(\Xi[q_n(t)]\Xi[q_n(t')])|0\text{in}\rangle. \quad (2.24)$$

Introducing the kernels

$$G(t,t') = \sum_n i\langle 0\text{in}|[\Xi[q_n(t)],\Xi[q_n(t')]]|0\text{in}\rangle, \quad (2.25)$$

$$G_1(t,t') = \sum_n \langle 0\text{in}|\{\Xi[q_n(t)],\Xi[q_n(t')]\}|0\text{in}\rangle, \quad (2.26)$$

where, as usual, square (curly) brackets denote (anti) commutators, we find

$$\Gamma[x,x'] = \int dt\, dt'\ \{[x-x'](t)G(t,t')\theta(t-t')[x+x'](t') + (i/2)[x-x'](t)G_1(t,t')[x-x'](t')\} \quad (2.27)$$

(θ being the step function) which assumes the same pattern discussed above in the framework of the influence action approach [cf. Eq. (2.3), after identifying $D = 2G$ and $N = G_1$].

C. The consistent histories formulation of quantum mechanics

Let us now relate these concepts and techniques in statistical field theory to the more recent studies of the quantum to classical transition problem via the consistent histories formulation of quantum mechanics [22–24].

In the consistent or decoherent histories approach, the complete description of a coupled x, q system is given in terms of fine-grained histories $x(t), q(t)$. These histories are quantum in nature; i.e., it is possible in principle to observe interference effects between different generic histories. A classical description is acceptable only at the level of coarse-grained histories, and to the extent that interference effects between these histories become unobservable. Let us adopt the simple coarse-graining procedure of leaving the q field unspecified. Then each coarse-grained history is labeled by a possible evolution of the x field, and the interference effects between histories are measured by the decoherence functional (DF)

$$\mathcal{D}[x,x'] = e^{i(S_s[x]-S_s[x'])}\int dq_i\, dq_i'\, dq_f \int Dq\ \ Dq'\ e^{i(S_e[q]+S_{\rm int}[x,q]-S_e[q']-S_{\rm int}[x',q'])}\rho_e(q_i,q_i',t_i) \qquad (2.28)$$

which is the fundamental object of the theory. (For a more formal definition see [22,23].) The coarse-grained histories $x(t)$ can be described classically if and only if the decoherence functional is approximately diagonal, that is, $\mathcal{D}[x,x'] \simeq 0$ whenever $x \neq x'$. The conditions leading to this in quantum mechanics is the focus of many current studies, to which we refer the readers for the details. For quantum cosmology the issue is complicated by the problem of time, and there even the definition of the decoherence functional can be ambiguous [11]. In the problem of transition from quantum cosmology to semiclassical gravity, a WKB time is usually assumed. In a work thematically related to this Paz and Sinha [9] showed that an influence functional appears naturally from a reduced density matrix by tracing out the matter fields. They discussed the decoherence between WKB branches of the wave function and tried to relate it to the notion of decoherence between spacetime histories. We assume in this work that this essential step can be taken in some satisfactory way and start our discussion at the semiclassical gravity level with the form of the decoherence functional[2]

$$\mathcal{D}[x,x'] = e^{i(S_s[x]-S_s[x']+S_{\rm IF}[x,x',\infty])} = e^{i\Gamma_{\rm CTP}[x,x']}. \qquad (2.29)$$

Notice that aleady at this formal level decoherence can occur only when the noise kernel is nonzero, which signals the presence of spontaneous fluctuations in the system.

We now arrive at the crucial point of our analysis, namely, the proper description of the dynamics of a single decohered history (that is, one particular decohered history chosen at random from the heap of all possible consistent ones). For an observer confined (by necessity or by choice) to the level of coarse-grained descriptions, dynamical evolution must be described in terms of mutually exclusive histories, all interference effects having been suppressed below the accuracy of his observation devices. For example, if he chooses to describe the evolution of the system in terms of its Wigner function f_W (introduced in Sec. II A), he will now interpret it as an actual ensemble average, describing the joint evolution of the bundle of coarse grained histories. Correspondingly, he will regard Eq. (2.10) as a classical Fokker-Planck equation. Now the classical random process described by Eq. (2.10) is not deterministic; rather, it describes the evolution of an ensemble of particles whose individual orbits obey the Langevin-type equations

$$\dot{x} = p, \quad \dot{p} = -V' - \gamma p + \xi, \qquad (2.30)$$

where ξ represents a noise term with autocorrelation $\langle \xi(t)\xi(t')\rangle = N(t,t')$. (The ordinarily assummed Gaussian and white nature of the noise follows only from a quadratic and local noise kernel, which describes rather special cases in cosmological situations, see [32,33]). Thus, the observer confined to a coarse grained history will conclude that semiclassical evolution is stochastic. Note that the statistical properties of this random evolution are totally determined by the decoherence functional (or equivalently, the closed time path effective action, or the influence functional); no *ad hoc* assumptions on the behavior of quantum fluctuations are necessary.

As noticed by Feynman and collaborators ([44]), there is a shortcut to Eqs. (2.30): One can rewrite the part in the influence action containing the noise kernel as

$$e^{-\frac{1}{2}\int\ dt\ (x-x')N(x-x')} \equiv \int D\xi\ e^{i\int\ dt\ \xi(x-x')}e^{-\frac{1}{2}\int\ dt\ \xi N^{-1}\xi}. \qquad (2.31)$$

Therefore the action of the environment on the system may be described by adding the external source term $-\int\ x\xi$ to the system action S_s, and averaging over external sources with the proper weight [29,31,34]. Variation of this effective action directly yields the Langevin equations (2.30). This is how noise can be understood as a stochastic force from the environment acting on the system.

We are now ready to explore the consequences and implications of these methods and ideas in the context of semiclassical gravity. As a first observation, and in order to connect with the more familiar language of quantum field theory in curved spacetime, we show that decoherence and noise are closely linked to particle creation, this being the main dissipative mechanism in our problem.

[2]In coarse graining away the environment variable q as in the simple Calderia-Leggett-type models [21], there is no decoherence in the decoherent history sense [22,23] unless one makes a further coarse graining of $x(t)$, such as specifying the ranges of values of x at different times. This is necessary to ensure the consistency or decoherence condition which requires the validity of the probability sum rules for a set of histories. For the condition for *a set of* histories to decohere is that the nondiagonal elements of the decoherence functional vanish *for all* pairs of histories in the set. This extra coarse graining on x was explained in [24,26]. In so doing the simple form of the decoherence functional (2.29) may become more complicated than necessary for the analysis of the semiclassical gravity domain. However, Gell-Mann and Hartle [24] had offered a partial solution to this problem, which we will assume for the purpose of using the decoherence functional in the semiclassical gravity form. We thank Juan Pablo Paz for stressing this point.

III. DECOHERENCE FUNCTIONAL IN TERMS OF THE BOGOLIUBOV COEFFICIENTS: PARTICLE CREATION AND DECOHERENCE

We shall now carry out an analysis of noise, fluctuations, and dissipation with the well-studied model of a Friedmann-Robertson-Walker (FRW) universe filled with a quantum scalar field.

The metric for our model is

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \sum_{i=1}^{3}(dx^i)^2\right), \qquad (3.1)$$

where $\eta = \int dt/a$ is the conformal time. (We assume spatial flatness only for definiteness, this plays no role in the analysis below.) We shall use the conventions of [64] throughout.

The scalar curvature for this model is

$$R = \frac{2(n-1)}{a^2}\left(\frac{\ddot{a}}{a} + \frac{(n-4)}{2}\left(\frac{\dot{a}}{a}\right)^2\right), \qquad (3.2)$$

where an overdot means a derivative with respect to η, the conformal time, and n denotes the spacetime dimension. We are interested in the four-dimensional case, of course, but for the time being we may leave n unspecified.

The Einstein-Hilbert action for general relativity is

$$S_e = \text{const} \times \left(-\int d\eta\; a^{n-4}\dot{a}^2\right), \qquad (3.3)$$

where the dimensional constant is $m_P^2(n-1)(n-2)L^{n-1}$, this last factor being the "volume" of a surface of homogeneity, and m_P the Planck's mass (in full consideration of renormalization, a factor μ^{n-4}, where μ is the renormalization scale, should also be included).

Consider a real scalar free field with arbitrary mass m and coupling to curvature ξ_n. In terms of the canonical field variable Φ, the action is

$$S_f = -\tfrac{1}{2}\int d^n x\; \sqrt{-g}\{g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + (m^2 + \xi_n R)\Phi^2\}. \qquad (3.4)$$

Specializing to our model, introducing the conformally related field variable $\Phi = a^{-1}\phi$, and discarding some total derivatives, we find

$$S_f = \tfrac{1}{2}\int d^n x\; \{\dot{\phi}^2 - (\nabla\phi)^2 - M^2\phi^2\}, \qquad (3.5)$$

where

$$M^2 = \left[m^2 + \left(\xi_n - \frac{(n-2)}{4(n-1)}\right)R\right]a^2. \qquad (3.6)$$

From the discussions in Sec. II C one can adopt the necessary procedures linking semiclassical gravity with quantum cosmology or follow the spirit of quantum field theory in curved spacetime and begin the discussion of semiclassical gravity with (2.29). Thus we assume that the decoherence functional between different histories $a_+(\eta), a_-(\eta)$ of the conformal factor, after ϕ is totally coarse-grained away, takes the form

$$\mathcal{D}[a_+, a_-] = \int D\phi_+ D\phi_-\; e^{i(S_g[a_+] - S_g[a_-] + S_f[a_+,\phi_+] - S_f[a_-,\phi_-])}. \qquad (3.7)$$

Here in the gravitational action $S_g = S_e + S_{\rm tr}$ we have included the trace anomaly-generating terms $S_{\rm tr}$ arising from the Jacobian of the $\Phi \to \phi$ transformation. As usual [65]

$$S_{\rm tr} = \frac{L^{n-1}}{2880\pi^2}\int d\eta\; \left[-3\left(\frac{\ddot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^4\right]. \qquad (3.8)$$

The histories are assumed to match at some point $\eta = \eta^0$ in the far future, and the integration is over field histories such that $\phi_+(\eta^0) = \phi_-(\eta^0)$. Further, we must choose the boundary conditions (and/or the measure) in the distant past to ensure convergence of the path integral. For the purpose of this paper we shall adopt the simplest procedure of assuming that for either evolution a_+ and a_-, M^2 vanishes in the distant past. Thus the boundary conditions can be fixed by the same procedure as in a flat space time path integral, where again we shall use the simplest criterion of tilting the path of integration in the complex η plane, in such a way that the $+$ branch acquires a negative slope, and the $-$ branch a positive slope. If we think of the integration path as a closed time loop, going from past to future on the $+$ branch, and returning on the $-$ one, this means that the imaginary part of η is nonincreasing throughout [66].

To continue, let us decompose the field in plane waves (or other spatial modes compatible with the symmetry of space):

$$\phi(\mathbf{x}, t) = \int \frac{d^{n-1}\mathbf{k}}{(2\pi)^{n-1}}\; e^{i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}(t), \qquad (3.9)$$

where $k = |\mathbf{k}|$. The amplitude of the $\mathbf{k}$th mode obeys a wave equation of the type

$$\ddot{\phi}_{\mathbf{k}} + \omega_k^2\phi_{\mathbf{k}} = 0, \qquad (3.10)$$

where $\omega_k^2 = k^2 + M^2$. We shall omit the subindices $\mathbf{k}$ henceforth.

As is well known [1,14], the quantization of the scalar field proceeds by further decomposing each Fourier amplitude in its positive and negative frequency parts, defined by a suitable choice of time parameter. This is accomplished by developing the corresponding mode on a basis of solutions of the Klein-Gordon equation Eq. (3.10), so normalized that the positive frequency function has unit Klein-Gordon norm, the negative frequency function has norm -1, and they are mutually orthogonal in the Klein-Gordon inner product. Such a basis of solutions constitute a particle model. Properly normalized particle models are related to each other through Bogoliubov transformations. Let us observe that, each function of the particle model being a solution of Eq. (3.10), the particle model may be defined by simply giving the corresponding Cauchy data on an arbitrary Cauchy surface. Further identification of the coefficient of the positive frequency function in the development of the field, as a destruction operator, allows for the second quantization of the theory. The particle model is also associated to a vacuum state, which is the single common null eigenvector of the destruction operators, and to a Fock basis, built from the vacuum through the action of the creation operators.

It is also well known that in a generic dynamic space time, there is no single particle model which can be identified outright with the physical concept of "particle"; however, oftentimes it is possible to employ a variety of criteria (such as minimization of the particle number as detected by a free falling particle detector, Hamiltonian diagonalization, conformal invariance, analytical properties in the Euclidean section of the space time, if any, etc.) to single out a preferred particle model in the distant past (or "in" particle model), and another in the far future, or "out" particle model. In general, these models are not equivalent, the vacuum of one model being a multiparticle state in the other.

In our problem, the choice of boundary conditions for the path integral above amounts to a definite choice of the in particle model, and the in quantum state, in each branch of the closed time path. Indeed, because $M^2 \to 0$ in the distant past on either branch, the field becomes conformally invariant there, so that quantization can be carried out as in Minkowski spacetime. Now our procedure of deforming the time path into the complex plane would pick up the Minkowski vacuum; so in a generic spacetime, we are defining the initial state to be the conformal vacuum, and the in particle model to be the conformal one. As shown in the previous section, the choice of initial state defines the CTP effective action. Making the provisos discussed there we may write

$$\mathcal{D}[a_+,a_-] = e^{i\{S_g[a_+]-S_g[a_-]+\Gamma[a_+,a_-]\}}, \tag{3.11}$$

where Γ is the influence or effective action for the scalar field, evaluated at vanishing field background, and the conformal factor being treated as an external field.

Since the CTP effective action is independent of the out quantum states, we have more freedom in choosing an out particle model. It is convenient to choose a common out particle model for both evolutions (that is, the Cauchy data on the matching surface $\eta = \eta^0$ are the same although the actual basis functions will be different). The positive-frequency time dependent amplitude functions $f_\pm$ for the conformal model in each branch are related to those F of the out model by $f_\pm = \alpha_\pm F + \beta_\pm F^*$ at $\eta = \eta^0$, where $\alpha_\pm, \beta_\pm$ are the Bogoliubov coefficients in each branch, obeying the normalization condition $|\alpha_\pm|^2 - |\beta_\pm|^2 = 1$. The CTP effective action in Eq. (3.11) is found to be

$$\Gamma = \left(\frac{i}{2}\right) \ln[\alpha_-\alpha_+^* - \beta_-\beta_+^*]. \tag{3.12}$$

We give two independent proofs of this formula in the Appendix (Secs. 1 and 2). We also show that it leads to real and causal corrections to Einstein's equations in Appendix Sec. 3. This expression is exact. (A similar expression can be obtained from the influence functional for cosmological models [56].)

The lesson for us is that there can be decoherence ($\mathrm{Im}\Gamma > 0$) if and only if there is particle creation in different amounts in each evolution. (This is also implicit in [9].) Indeed, we can always choose the out model so that $\alpha_+ = 1$, $\beta_+ = 0$, yielding $\Gamma = (i/2)\ln\alpha_-$. The condition for decoherence in this case is then $|\alpha_-| > 1$. But since $|\alpha_-|^2 = 1 + |\beta_-|^2$, this can only happen if there is particle creation between these two particle models.

For this simple model this result suggests that the physical mechanism underlying decoherence in the decoherent history scheme of Gell-Mann and Hartle [22,23] is the same as in the environment-induced scheme [21] based on a reduced density matrix obtained by projecting [67] from the full density matrix and tracing over the environmental degrees of freedom. (For the connection between these two schemes see [26].) If the system and environment are correlated (i.e., that the full density matrix cannot be decomposed into a tensor product of system and enviromment states), this tracing procedure will leave the system in a mixed state.

The correlations between the system and the environment may be present in the initial conditions, or they may arise dynamically. Since in our initial condition the system (a) and the environment (ϕ) are uncorrelated, decoherence occurs only when correlations are generated in the dynamics. For free fields, as in this model, correlations between the scale factor and the fields are generated through particle creation. (For example, consider a combined tensor product quantum state where the field is in its vacuum state for some value of the scale factor. Although the field state would react to adiabatic changes in a, the combined state will remain a tensor product unless particle creation occurs.) The problems of correlations engendered by particle creation and interaction and their role in entropy generation have been considered in [68,69,43].

One may observe that since Γ becomes identically zero when its arguments coincide, one seems to get the same probabilities for all coarse-grained histories. In actuality this only means that further coarse graining may be necessary to obtain a set of histories compatible with the actual description of our Universe.

IV. EQUATION OF MOTION, NOISE, AND FLUCTUATION

A. Equation of motion

Recall from Sec. II that the expectation value of the conformal factor obeys the equation

$$\left.\frac{\delta S_g}{\delta a}+\frac{\delta\Gamma[a_+,a_-]}{\delta a_+}\right|_{a_+=a_-=a}=0. \tag{4.1}$$

Being causal and nonlocal, this equation cannot be derived from an action functional. Let us now consider the dynamics of small fluctuations $\delta a_\pm$ around a solution a of the semiclassical equations above. To do this, we shall start by computing the CTP effective action for the field.

As in the previous section, we shall choose as particle model that which reduces to the conformal model in the distant past. Since the unperturbed evolution is the same on either branch of the closed time path, this condition defines a single unperturbed *in* particle model. Projecting this model to the far future, we obtain also an *out* particle model. We shall adopt this choice, which reduces the unperturbed Bogoliubov coefficients to 1 and 0. Since we just want the effective action up to quadratic order in the perturbations, with this choice we only need the perturbed $\beta_\pm$ coefficients to linear order, and the perturbed $\alpha_\pm$ ones to second order.

Let f be the positive frequency function of the unperturbed particle model defined above, and let $f_\pm$ be the positive frequency functions of the perturbed conformal particle models. Then $f_\pm$ has an expansion $f_\pm = f + f_\pm^{\rm I} + f_\pm^{\rm II} + \cdots$ in powers of the perturbation (denoted here by the superscript Roman numerals). By construction, the Cauchy data for $f_\pm$ are the same as for f, therefore the correction terms must vanish in the distant past.

Introduce the notation

$$\Delta\omega_\pm^2=\int d\eta'\,\frac{\delta\omega^2}{\delta a(\eta')}\delta a_\pm(\eta') \tag{4.2}$$

for the correction to ω^2 due to the perturbation. The identity

$$\frac{\delta\Delta\omega^2}{\delta\delta a(\eta)}=\frac{\delta\omega^2}{\delta a(\eta)} \tag{4.3}$$

follows from this definition. In terms of $\Delta\omega^2$, we find

$$f_\pm^I(\eta)=-\int_{-\infty}^{\eta}d\eta'\;G(\eta,\eta')\Delta\omega_\pm^2(\eta')f(\eta'), \tag{4.4}$$

where G is the retarded propagator

$$G(\eta,\eta')=i[f^*(\eta')f(\eta)-f(\eta')f^*(\eta)]\theta(\eta-\eta') \tag{4.5}$$

which is of course independent of the actual choice of particle model. Using the explicit expression for G, we obtain

$$f_\pm^I=\left(1-i\int_{-\infty}^{\eta}d\eta'\;|f(\eta')|^2\Delta\omega_\pm^2(\eta')\right)f(\eta)+\left(i\int_{-\infty}^{\eta}d\eta'\;f(\eta')^2\Delta\omega_\pm^2(\eta')\right)f^*(\eta). \tag{4.6}$$

From this we can read off the $\beta_\pm$ coefficients to the desired accuracy

$$\beta_\pm=i\int_{-\infty}^{+\infty}d\eta\;f(\eta)^2\Delta\omega_\pm^2(\eta)+O[(\Delta\omega_\pm^2)^2]. \tag{4.7}$$

Iterating this procedure, we get the α coefficients [70]

$$\alpha_\pm=\left(1+\int_{-\infty}^{+\infty}d\eta\;f^*(\eta)^2\Delta\omega_\pm^2(\eta)\int_{-\infty}^{\eta}d\eta'\;f(\eta')^2\Delta\omega_\pm^2(\eta')+O[(\Delta\omega_\pm^2)^3]\right)\exp\left(i\int_{-\infty}^{+\infty}d\eta\;|f(\eta)|^2\Delta\omega_\pm^2(\eta)\right). \tag{4.8}$$

Observe that indeed the normalization condition is satisfied. Inserting these expressions back in the formula for the effective action, we find a term of first order in the perturbation, which is canceled by the variation of the classical action (since we assume the background evolution is a solution of the semiclassical equations of motion for the expectation values), and a quadratic term

$$\begin{aligned}\Gamma[\delta a_+,\delta a_-]=\frac{i}{2}\Bigg(&\int_{-\infty}^{+\infty}d\eta\;f(\eta)^2\Delta\omega_+^2(\eta)\int_{-\infty}^{\eta}d\eta'\;f^*(\eta')^2\Delta\omega_+^2(\eta')\\&+\int_{-\infty}^{+\infty}d\eta\;f^*(\eta)^2\Delta\omega_-^2(\eta)\int_{-\infty}^{\eta}d\eta'\;f(\eta')^2\Delta\omega_-^2(\eta')\\&-\int_{-\infty}^{+\infty}d\eta\;f(\eta)^2\Delta\omega_-^2(\eta)\int_{-\infty}^{\eta}d\eta'\;f^*(\eta')^2\Delta\omega_+^2(\eta')\Bigg)\end{aligned} \tag{4.9}$$

leading to the equations of motion for the expectation value of the perturbation

$$\int d\eta'\;\frac{\delta^2S_g}{\delta a(\eta)\delta a(\eta')}\delta a(\eta')+\int d\eta'\;\frac{\delta\omega^2(\eta')}{\delta a(\eta)}\int d\eta''\;D(\eta',\eta'')\Delta\omega^2(\eta'')=0, \tag{4.10}$$

where

$$D(\eta',\eta'')=\frac{i}{2}[f(\eta')^2f^*(\eta'')^2-f^*(\eta')^2f(\eta'')^2]\theta(\eta'-\eta''). \tag{4.11}$$

As expected, this equation is real, causal, and nonlocal [17]. The boundary conditions are that δa must vanish in the distant past. However, this is the equation only for the expectation value of the perturbation, and since we are perturbing around a solution of the semiclassical equations of motion, the only solution with those boundary conditions is the trivial one. What we want to describe is the effective dynamics of the conformal factors alone, which, as we discussed in detail in Sec. II, is stochastic in nature and described by a Langevin equation. This equation, in turn, is best derived following Feynman's procedure [44]. To implement Feynman's method, it is convenient to introduce the symbols $\{X\} \equiv (X_+ + X_-)$ and $[X] \equiv (X_+ - X_-)$. In this notation, the effective action reads

$$\Gamma = \frac{1}{2}\int d\eta\, d\eta'\, [\Delta\omega^2(\eta)]D(\eta,\eta')\{\Delta\omega^2(\eta')\} + \frac{i}{2}\int d\eta\, d\eta'\, [\Delta\omega^2(\eta)]N(\eta,\eta')[\Delta\omega^2(\eta')], \quad (4.12)$$

where

$$N(\eta,\eta') = \tfrac{1}{4}\{f(\eta)^2 f^*(\eta')^2 + f^*(\eta)^2 f(\eta')^2\}. \quad (4.13)$$

We can see a sort of "division of labor" here: the first term, the dissipation kernel, determines the equation of motion, but does not contribute to decoherence, while the second term, the noise kernel, does not affect the equations of motion, but is responsible for decoherence.

It may be argued that, if one summed over all modes, one could get exact decoherence, since the decoherent terms could diverge [9]. This effect is, however, generally believed to be unphysical. Indeed, more physically relevant coarse-graining strategies seem to avoid this pitfall [25].

The failure of our observer to reduce the first term to a difference between functionals of each history separately was expected, since we knew that no such functional could lead to the proper equations of motion. Physically, it is the dissipative nature of semiclassical evolution which precludes its formulation in terms of an action principle. We shall see an example of this below. However, an observer in the coarse-grained history could still think of this first term as arising from both a classical action and a dissipative function (see Ref. [71], entry 121).

B. Noise

To understand the meaning of the second term better, recall that the decoherence functional has the form [44]

$$\mathcal{D}[\delta a_+, \delta a_-] = e^{\frac{1}{2}\int d\eta d\eta' [\Delta\omega^2](\eta)D(\eta,\eta')\{\Delta\omega^2\}(\eta')} e^{-\frac{1}{2}\int d\eta d\eta' [\Delta\omega^2](\eta)N(\eta,\eta')[\Delta\omega^2](\eta')}. \quad (4.14)$$

Performing the functional Fourier transform

$$e^{-\frac{1}{2}\int d\eta d\eta' [\Delta\omega^2](\eta)N(\eta,\eta')[\Delta\omega^2](\eta')} = \int D\xi\; \mathcal{P}[\xi]\; e^{-i\int d\eta [\Delta\omega^2](\eta)\xi(\eta)} \quad (4.15)$$

the decoherence functional may be understood as the result of averaging the functional

$$e^{\frac{1}{2}\int d\eta d\eta' [\Delta\omega^2](\eta)D(\eta,\eta')\{\Delta\omega^2\}(\eta')} e^{-i\int d\eta [\Delta\omega^2](\eta)\xi(\eta)} \quad (4.16)$$

over all possible values of a stochastic external source ξ, with probability distribution $\mathcal{P}[\xi]$. To this order of expansion ξ is a Gaussian variable which produces a stochastic source on the right-hand side of the equation for δa: namely,

$$\int d\eta'\, \frac{\delta^2 S_g}{\delta a(\eta)\delta a(\eta')}\delta a(\eta') + \int d\eta'\, \frac{\delta\omega^2(\eta')}{\delta a(\eta)} \int d\eta''\, D(\eta',\eta'')\Delta\omega^2(\eta'') = \int d\eta'\, \frac{\delta\omega^2(\eta')}{\delta a(\eta)}\xi(\eta'). \quad (4.17)$$

Because of the nonlocality of the noise kernel, the noise is generally nonwhite; it is also generally non-Gaussian [32,56,58]. Indeed, its Gaussian nature in our example is merely a result of our having stopped at quadratic order in the expansion of the effective action. The important thing to notice here is that the formalism itself saves one the trouble (or embarrassment) of making *ad hoc* and oftentimes inconsistent guesses about the nature of the noise. For linear perturbations, the noise is Gaussian, with autocorrelation

$$C(\eta,\eta') = \int d\eta'' \int d\eta''' \frac{\delta\omega^2(\eta'')}{\delta a(\eta)} N(\eta'',\eta''') \frac{\delta\omega^2(\eta''')}{\delta a(\eta')}. \quad (4.18)$$

C is, of course, the expectation value of the product of the noise at times η and η'. Equation (4.18) follows from taking two derivatives of Eq. (4.15) with respect to $\Delta\omega^2$, then setting this to zero.

It is a remarkable fact that, for histories, the more classical they become, the noisier they are. This point was emphasized in [24,9]. Mathematically this follows from decoherence and noise being determined by the same kernel N. Physically, in our context, it follows from the fact that noise and decoherence are both related to particle creation and back reaction. Indeed, noise is just the difference between the stochastic process of particles as they are actually created, and the smoothed-out average effect represented by the expectation value.

C. Fluctuations

The assumption that quantum fluctuations in the fundamental fields can somehow transmute into classical stochastic fluctuations is central to the stochastic inflation program [59] and underlies most theories of galaxy formation via the perturbation of quantum fields [60]. Although widely accepted and applied, the crucial point in this program, this transmutation or transgression, has never been satisfactorily proven. (For a critique of this view, see [32].) The usual prescription is to consider ϕ as a classical stochastic Gaussian variable, with an autocorrelation chosen to match the quantum two point functions. Let us see what can go wrong with this *ad hoc* assumption.

It is helpful to again look at this problem from a slightly different angle. As is well known, the single equation for the conformal factor we have derived here is equivalent to the trace of the full Einstein equations [72]. More concretely, from the definition

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_f}{\delta g^{\mu\nu}} \tag{4.19}$$

one gets

$$\frac{\delta S_f}{\delta a} = -a^3 T^\mu_\mu. \tag{4.20}$$

Therefore, as a Heisenberg operator, the trace of the energy-momentum tensor (not counting the trace anomaly terms already included in S_g) is given by

$$T^\mu_\mu(\eta) = \frac{1}{2a^3} \int d\eta' \, \frac{\delta\omega^2(\eta')}{\delta a(\eta)} \phi^2(\eta'). \tag{4.21}$$

Comparing these formulas to Eq. (4.17) we see that the stochastic source ξ corresponds to the random fluctuations $\frac{1}{2}(\phi^2 - \langle\phi^2\rangle_q)$, where $\langle\ \rangle_q$ denotes the expectation value of an observable with respect to the in vacuum, computed from the usual rules in quantum field theory. In our approach, the expectation value $\langle\phi^2\rangle_q$ is automatically included in the nonstochastic part of the effective action.

Let $\langle\ \rangle_c$ denote the classical ensemble average over the different values of the source. Since $\langle\xi\rangle_c = 0$, we find $\langle\phi^2\rangle_c = \langle\phi^2\rangle_q$. Moreover, from $\langle\xi(\eta)\xi(\eta')\rangle_c = N(\eta,\eta')$, we find

$$\langle\phi^2(\eta)\phi^2(\eta')\rangle_c - \langle\phi^2(\eta)\rangle_c\langle\phi^2(\eta')\rangle_c = 4N(\eta,\eta') \tag{4.22}$$

$$= \{f(\eta)^2 f^*(\eta')^2 + f^*(\eta)^2 f(\eta')^2\}. \tag{4.23}$$

If we compare this result to the corresponding quantum average, we find

$$\langle\phi^2(\eta)\phi^2(\eta')\rangle_c = \tfrac{1}{2}\langle\{\phi^2(\eta)\phi^2(\eta') + \phi^2(\eta')\phi^2(\eta)\}\rangle_q. \tag{4.24}$$

The need to symmetrize the quantum average could be expected, since there is no analogue of noncommuting variables in classical stochastic dynamics.

Before discussing further the implications of this equation, let us try to recover this result by simply viewing the field ϕ as a classical stochastic field, as is done in almost all discussions on this subject (for a review, e.g. [60]). As the autocorrelation $\langle\phi(\eta)\phi(\eta')\rangle$ should be real and even in its arguments, the only choice is to identify it with the Hadamard function, yielding

$$\langle\phi(\eta)\phi(\eta')\rangle_{c'} = \frac{1}{2}G_1(\eta,\eta')$$

$$= \tfrac{1}{2}\{f(\eta)f^*(\eta') + f^*(\eta)f(\eta')\} \tag{4.25}$$

(we use the subindex c' to distinguish these averages from those discussed above). But then the Gaussian character of this variable implies

$$\langle\phi^2(\eta)\phi^2(\eta')\rangle_{c'} - \langle\phi^2(\eta)\rangle_{c'}\langle\phi^2(\eta')\rangle_{c'} = \tfrac{1}{2}G_1(\eta,\eta')^2 \tag{4.26}$$

which fails to reproduce the quantum average, even after symmetrization [cf. Eq. (4.24)]. On this count it can be seen that the conventional view on quantum fluctuations is flawed. (It also misses out the full complexity of the issue of quantum to classical transtion, see [32].) Fortunately Eq. (4.26) can yield the correct result in the coincidence limit $\eta' = \eta$. For this reason, the usual scenarios for the generation of primordial fluctuations in inflationary cosmology can remain valid if the proper form of the noise correlator is used.

The equality between the specific kinds of classical and quantum averages defined in Eq. (4.24) warrants that several familiar results from quantum field theory in curved spacetime will also be valid in the semiclassical approximation. For example, the mean square value of the spontaneous fluctuations of a massless, minimally coupled scalar field in de Sitter spacetime will grow linearly with cosmological time [73]. This result is consistent with the view that these fluctuations can be represented as white noise [33,34], associated with the thermal radiation at the Hawking temperature of the de Sitter universe [52,53]. We shall demonstrate this equivalence between quantum and semiclassical results for a more complex example in Sec. V.

On consideration of self-interacting quantum field theories, which would not be one-loop exact (as is the case for the free field theory we are discussing here), the quantum to classical correspondence would not necessarily hold beyond one loop. In these conditions, however, it is possible to improve systematically the accuracy of the semiclassical approximation by a suitable choice of coarse-graining procedure, as in the "correlation histories" approach. We have discussed these issues elsewhere [25].

D. Nonlocal kernels and colored noise

In the above we have derived an expression for the dissipation D and noise kernel N in terms of the positive frequency components of the amplitude functions of the conformal *in* particle model of a given consistent cosmology. To examine the structure of these kernels, we shall now specialize to a particularly simple but illustrative example, by choosing the background-consistent evolution to be just the Minkowski spacetime. That is, we are interested in the physics of small departures from the special case of Robertson-Walker conformal factor $a^2 = 1$ under the influence of, say, a real, massive, free scalar field.

In this simple case, the unperturbed positive frequency modes are just

$$f_k(\eta) = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \eta}, \tag{4.27}$$

where the natural frequency is $\omega_k^2 = k^2 + m^2$ (because of the spherical symmetry the modes can be labeled by $k \equiv |\mathbf{k}|$). The sum over all modes, with the dimensionless measure $V d^3\mathbf{k}/(2\pi)^3$ can be conveniently expressed in terms of the positive frequency Wightman function

$$G_+((\eta, \mathbf{x}), (\eta', \mathbf{x}')) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} f_{\mathbf{k}}(\eta) f_{\mathbf{k}}^*(\eta') e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{4.28}$$

We find

$$\int \frac{V d^3\mathbf{k}}{(2\pi)^3} f_{\mathbf{k}}^2(\eta) f_{\mathbf{k}}^{2*}(\eta') = \int d^3\mathbf{x} d^3\mathbf{x}' \; G_+^2((\eta, \mathbf{x}), (\eta', \mathbf{x}')). \tag{4.29}$$

which, by virtue of translation invariance, becomes

$$V \int d^3\mathbf{x} \; G_+^2((\eta, 0), (\eta', \mathbf{x})). \tag{4.30}$$

The positive frequency Wightman function also admits the representation

$$G_+(x^\mu, x'^\nu) = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu(x^\mu - x'^\mu)} 2\pi\delta(p^2 + m^2)\theta(p_0) \tag{4.31}$$

from which we get the well-known relation

$$\int \frac{V d^3\mathbf{k}}{(2\pi)^3} f_{\mathbf{k}}^2(\eta) f_{\mathbf{k}}^{2*}(\eta') = \frac{V}{(4\pi)^2} \int d\omega \; e^{-i\omega(\eta-\eta')} \sqrt{1 - \frac{4m^2}{\omega^2}} \theta(\omega - 2m). \tag{4.32}$$

In the massless case, the last integration is trivial, yielding

$$\frac{V}{(4\pi)^2} \left[-iPV \left(\frac{1}{\eta - \eta'} \right) + \pi\delta(\eta - \eta') \right]. \tag{4.33}$$

Since the noise kernel is just one half of the real part of this expression, we get immediately

$$N(\eta, \eta') = \frac{V}{32\pi} \delta(\eta - \eta'). \tag{4.34}$$

Thus we see that a massless free field is associated with a purely white noise (which makes physical sense, since there is no dimensionful scale to define a memory time). This result is relevant more generally to the study of near-conformal fields on arbitrary background Robertson-Walker spacetimes. As long as the departure from conformal invariance is small, we can use the conformal modes in the formal expressions from the previous subsections which are equivalent to those of a Minkowski massless field.

For $m \neq 0$, the integral is not easily done, but we can reason as follows (see Ref. [74]): When the lapse $\eta - \eta'$ is small, the integral will be dominated by high frequencies. But in this regime, the mass is unimportant, so we still get the δ functionlike singularity. The mass begins to play a role for finite time separation. In particular, for very large time separations $\eta - \eta' \gg m^{-1}$, the integral is dominated by the low frequencies (close to the branch point). In this regime, the noise kernel becomes

$$N(\eta, \eta') \sim \frac{V}{4} \sqrt{\frac{\pi}{m}} \left(\frac{\cos[2m(\eta - \eta') - \pi/4]}{(\eta - \eta')^{3/2}} \right). \tag{4.35}$$

It is tempting to conjecture that the δ-functionlike singularity in the noise kernel is due to the contribution of those $\mathbf{x}$ such that $(\eta', \mathbf{x})$ lies close to the past light cone of the point $(\eta, \mathbf{x})$. Indeed, for $\eta \geq \eta'$ we have the exact expression

$$G_+ = \frac{-m}{8\pi} (-\sigma^2)^{-1/2} H_1^{(2)}[m(-\sigma^2)^{1/2}], \tag{4.36}$$

where σ^2 is the four-dimensional geodesic distance between the arguments of the propagator. As $\sigma^2 \to 0$, this yields a mass independent leading singularity, which is indeed equivalent to the whole massless propagator. Thus the integration over $\mathbf{x}$ of this term alone reproduces the δ functionlike behavior of the noise kernel.

This result in turn suggests that the δ-functionlike behavior will be common to all Hadamard vacua in curved spacetimes, as these share the singularity structure of the Minkowski propagators [1]. The details of the particular evolution are coded in the nonsingular tail of the noise kernel.

The dissipation kernel for a scalar field in flat space time is analyzed in Ref. [32], where the fluctuation-dissipation theorem is also stated.

V. STOCHASTIC SOURCES IN THE ENERGY DENSITY

The established theory of semiclassical gravity is based on the Einstein equation for classical metric with a source given by the vacuum expectation values of the energy-momentum tensor of a quantum field. A major proposal we advance in this paper is that noise terms should be added as a stochastic source to the semiclassical Einstein equation beyond the usual order of approximation, turning it into an Einstein-Langevin equation. These noise terms arise from the difference between the average amount of vacuum polarization and particle creation (measured by the expectation value of the energy momentum tensor) and the actual value of the same quantities in a specific history. By comparison to our generalized semiclassical theory the usual semiclassical Einstein theory can be viewed as a mean-field theory. It is well known that mean field theory is inadequate in addressing the full effect of fluctuations and instability, as in studies of critical phenomena [75]. To the extent the transition from classical general relativity to quantum gravity may involve instability and phase transitions, the old theory is ill prepared for such an analysis. We will discuss the ramifications of this generalized theory in future works.

In this section we shall explain the nature of noise by relating it to fluctuations in particle number and vacuum polarization, using simple models in introductory quantum field theory in curved spacetimes. We shall calculate the amount of particle creation in a cosmological evolution with asymptotically static regions. Simpler still, we assume that the evolution never deviates much from Minkowski spacetime. We shall do this in two ways, first by deploying the machinery from the previous sections, and then by a straightforward analysis based on elementary quantum field theory. We shall show that both analysis give the same result in their common range of applicability. This will explain the meaning of the noise term in the Einstein-Langevin equation.

A. CTP effective action and the energy-momentum tensor

In the last section we have seen how the variation of the CTP effective action with respect to the conformal factor yields the trace of the stress-energy tensor. We also noted that the presence of a stochastic source in the semiclassical equations for the conformal factor is equivalent to a random component in the trace of the energy momentum tensor of the field. In a FRW universe we also know that the trace determines the full energy momentum tensor [72]. In particular, the energy density can be related to the trace by

$$T_0^0(\eta) = \frac{1}{a^4}\left(\text{const} + \int^{\eta} d\eta'\, a^3 \dot{a} T_\mu^\mu\right). \tag{5.1}$$

Let us now consider an asymptotically static, near-flat evolution, where the mode frequencies ω^2 are composed of a constant part ω_0^2, and a small time varying component $\Delta\omega^2$ (we retain this notation for the fluctuating part for simplicity, although properly speaking we do not treat it as a perturbation, but as part of the background).

In the *out* region, where $a = 1$ again, the energy density has a "deterministic" part

$$\rho_d = \text{const} - \int d\eta d\eta' \frac{d\Delta\omega^2}{d\eta}(\eta) D(\eta,\eta')\Delta\omega^2(\eta') \tag{5.2}$$

which, after integration by parts, becomes

$$\rho_d = \text{const} + \int d\eta d\eta' \Delta\omega^2(\eta)\frac{\partial D(\eta,\eta')}{\partial\eta}\Delta\omega^2(\eta'), \tag{5.3}$$

and a "stochastic" part

$$\rho_s = -\int d\eta\ \Delta\omega^2(\eta)\frac{\partial\xi(\eta)}{\partial\eta}. \tag{5.4}$$

Consequently, the mean deviation from the average value is

$$\langle\rho_s^2\rangle = \int d\eta d\eta'\ \Delta\omega^2(\eta)\frac{\partial^2 N(\eta,\eta')}{\partial\eta\partial\eta'}\Delta\omega^2(\eta'). \tag{5.5}$$

The dissipation kernel D is given by

$$D(\eta,\eta') = -\text{Im}\left(V\int d^3\mathbf{x}\ G_+^2((\eta,0),(\eta',\mathbf{x}))\theta(\eta-\eta')\right). \tag{5.6}$$

Using the representation

$$\theta(\eta-\eta') = i\int \frac{d\omega}{2\pi}\ \frac{e^{-i\omega(\eta-\eta')}}{\omega+i\epsilon} \tag{5.7}$$

and the formulas from Sec. IV D D reduces to

$$D(\eta,\eta') = \frac{V}{64\pi^3}\int d\omega\ e^{-i\omega(\eta-\eta')}\int_{4m^2}^{\infty}\frac{dt}{t-(\omega+i\epsilon)^2}\sqrt{1-\frac{4m^2}{t}}, \tag{5.8}$$

which yields the average energy density

$$\rho_d = \frac{V}{32\pi^2}\int_{2m}^{\infty} d\nu\ \nu\sqrt{1-\frac{4m^2}{\nu^2}}|\Delta\omega^2(\nu)|^2, \tag{5.9}$$

where

$$\Delta\omega^2(\nu) = \int d\eta\ e^{-i\nu\eta}\Delta\omega^2(\eta). \tag{5.10}$$

In turn, the noise kernel $N(\eta,\eta')$ is

$$N(\eta,\eta') = \tfrac{1}{4}\mathrm{Re}\left(V\int d^3\mathbf{x}\; G_+^2((\eta,0),(\eta',\mathbf{x}))\right) \quad (5.11)$$

and reduces to

$$N(\eta,\eta') = \frac{V}{(8\pi)^2}\int d\omega\; e^{i\omega(\eta-\eta')}\sqrt{1-\frac{4m^2}{\omega^2}}\theta(|\omega|-2m) \quad (5.12)$$

which yields the fluctuations in the energy density

$$\langle\rho_s^2\rangle = \frac{V}{32\pi^2}\int_{2m}^{\infty} d\nu\;\nu^2\sqrt{1-\frac{4m^2}{\nu^2}}|\Delta\omega^2(\nu)|^2. \quad (5.13)$$

Since in the asymptotic region there is no vacuum polarization, these fluctuations can be ascribed solely to fluctuations in the number of created particles. We shall show now that it is indeed so, by computing these fluctuations independently. In this way, we shall demonstrate the consistency of our approach with familiar results from quantum field theory in curved spaces, in this simple example.

B. Fluctuations in particle number

Let us study the same problem, that is, fluctuations in the energy density in the *out* region of a near-flat, asymptotically-static evolution, by using simple arguments from quantum field theory in curved spacetimes.

Physically there is energy in the *out* region because particles have been created. Indeed they are created with a definite spectrum. The probability of finding $(2n+1)$ particles in mode k vanishes, and the probability of finding $2n$ particles is [14]

$$p_{2n} = \frac{(2n)!}{2^{2n}(n!)^2}\frac{|\beta|^{2n}}{|\alpha|^{2n+1}}. \quad (5.14)$$

With the help of the elementary series

$$\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)x^n = \frac{1}{\sqrt{1-x}} \quad (5.15)$$

it is immediate to get the average number of created particles

$$\langle n\rangle = |\beta|^2 \quad (5.16)$$

and the fluctuation in number

$$\langle n^2\rangle - \langle n\rangle^2 = 2|\alpha|^2|\beta|^2. \quad (5.17)$$

(We could also obtain these results by computing the *in* vacuum expectation values of the *out* particle number operator.)

Observe that all modes have (positive) frequency greater than m, and that the number of modes with frequencies between ν and $\nu+d\nu$ is

$$\frac{4\pi Vk^2dk}{(2\pi)^3} = \frac{V}{2\pi^2}\sqrt{1-\frac{m^2}{\nu^2}}\nu^2 d\nu \quad (5.18)$$

so that the average energy density becomes

$$\rho_d = \frac{V}{2\pi^2}\int_m^{\infty} d\nu\nu^3\sqrt{1-\frac{m^2}{\nu^2}}|\beta|^2 \quad (5.19)$$

and the fluctuations in energy density are given by

$$\langle\rho_s^2\rangle = \frac{V}{\pi^2}\int_m^{\infty} d\nu\nu^4\sqrt{1-\frac{m^2}{\nu^2}}|\alpha|^2|\beta|^2. \quad (5.20)$$

Although the particle number for a particular mode is certainly not a Gaussian variable, the energy will be, by virtue of the central limit theorem.

In the simple case in question, the Bogoliubov coefficients are (see Sec. IV) approximated by $\alpha \approx 1$, and

$$\beta(\nu) = \left(\frac{i}{2\nu}\right)\Delta\omega^2(2\nu). \quad (5.21)$$

Introducing this expression in the formulas above, we immediately recover the results from Sec. V A. Of course, we already noted in Refs. [17,18] that the energy dissipated from the conformal factor is equal to the energy of the created particles. The new ingredient found here is that fluctuations in energy density, which constitutes noise in the semiclassical Einstein equation, also have a simple physical interpretation.

The above analysis is only a trivial application of a very powerful tool, designed to illustrate the meaning of some new aspects in these methods. Of course, one does not need such heavy formalisms to treat these simple models. In a more complex problem the new method we are proposing here not only allows us to compute the fluctuations in the asymptotically static regions, it also tells us how to feed back those fluctuations into the evolution of the conformal factor, even at intermediate times, where there may not be a well-defined particle number operator. It is also very important that once the physical situation is defined (e.g., what is the initial state, and what will be coarse grained over), the formalism will generate the correct results with self-consistency without the need of making *ad hoc* assumptions or adjustments along the way. In some examples given in Sec. IV we have seen the danger of taking a problem at face value in seeking convenient solutions to deeper issues.

VI. DISCUSSION

Our earlier papers [17,18] showed how the Schwinger-Keldysh (CTP) method can be successfully applied to treat particle creation and back reaction in semiclassical cosmology. We obtained a real and causal equation of motion and showed how particle creation can be viewed as a dissipative process. In this paper we have developed further this method to incorporate the treatment of noise and fluctuations, relating them to decoherence and particle creation. With the help of the Feynman-Vernon influence functional method we can understand better the statistical mechanical meaning of the quantum processes involved and expound the origin and nature of noise and fluctuations associated with quantum fields (and eventually that of spacetime) in semiclassical gravity. Our findings lead us to propose a generalized theory of semiclassical gravity where stochastic source terms corresponding to the fluctuations in the number of particle creation appear in addition to the usual term corresponding to the vacuum expectation value of the energy-momentum tensor.

As is shown here explicitly through some simple examples, these new methods make it possible to display the full interconnections between particle creation, decoherence, noise, fluctuation, and dissipation [7]. Our analysis also confirms previous hints on the connection between decoherence and particle creation [9,42], and the balance between decoherence and the stability and predictability of classical evolution [24]. Perhaps the most important finding of this work is that semiclassical evolution is inherently stochastic, and that its statistical properties may be rigorously derived from the closed-time-path effective action or the influence functional.

From a theoretical point of view these new methods are essential in gaining a fuller understanding of the intricacies of the relation between quantum and semiclassical physics, and in particular, semiclassical gravity and cosmology. Two aspects stand out. First, their *formal structure* bestows upon us a complete description of the system and its environment, with almost no room for any *ad hoc* assumptions or piecemeal adjustments. Its logical extension leads us to new discoveries and insights, such as the noise terms in the equations of motion or the entropy of the open system from the reduced density matrix. Second, their *intrinsic power* makes it possible to treat complex problems such as the quantum to classical transition and cosmological back reaction which requires a self-consistent description of particle creation, decoherence, noise, fluctuations, and dissipation on the same footing.

Extending the examples given here to more realistic situations, this formalism points the way to a more complete and accurate treatment of problems involving quantum processes in the very early Universe, such as structure formation and phase transition problems. The concept and methodology can also be applied to the study of black hole thermodynamics in a dynamical setting. The CTP effective action and the nonequilibrium IF are particularly suitable to address the back reaction problem of semiclassical black hole collapse. There, at the verge of the domain of validity of the semiclassical gravity theory, we expect to see similar stochastic behavior associated with the quantum field vacuum becoming more prominant as the gravitational field increases in strength. As is known from critical phenomena studies, the prominance of the fluctuation terms signals the onset of instability (of the ground state) of the old phase, here described by Einstein's gravity, and the transition to the new phase—possibly described by a theory of quantum gravity [75]. We will discuss these issues in future works [56,19,20].

In a broader light, recognition of this unavoidable statistical feature of semiclasical evolution may affect drastically our understanding of these "medium energy" phenomena, which can be viewed in a more general sense as "mesoscopic" physics. A sketch of these ideas can be found in [33,6]. The actual application of these methods to concrete model building in gravitation and cosmology or other subjects usually hinges upon the identification of meaningful coarse-grained descriptions adequate to each particular setting [24,41]. The results of this paper can help to explicate these issues as well.

On general grounds, the consistent histories approach to quantum physics assumes that one has at his/her disposal a fine-grained description of the system of interest, which is subsequently coarse grained to leave only the physically relevant variables. However, what constitutes a meaningful choice of fine-grained histories depends on the scale of the problem: a nucleon may be regarded as an "elementary" particle at energies of MeV's, while it will reveal its composite character at energies of GeV's. Strictly speaking, there is no ground to believe that an ultimate absolute fine grained description of the Universe exists, which would seem to void the application of this approach in cosmological problems.

Our results suggest a practical test to choose the correct level of description for a given problem, namely, that a description of a physical system may be considered fine grained insofar as the ever-present dissipative and stochastic elements in the dynamics are small compared to the characteristic scale of energy and dimension of observation. This happens for finer scales of measurement and higher degree of accuracy. At a coarser scale of measurement or with a lesser degree of accuracy one effectively averages over certain set of (irrelevant) variables and would necessarily recognize the appearance of dissipation and fluctuations phenomena in the coarse-grained description (see [41] for further discussions on this point). Thus, for a given accuracy, it is possible to show rigorously that a given set of histories can indeed be treated as fine-grained, even if the underlying levels of description are unknown.

The application of this criterion along with a generalized concept of what constitutes a "history" (e.g., allowing for collective [41] or hydrodynamic [24] variables, and/or correlations [25] as parts of the specification of a history) may help us focus on the key issues for understanding the nature and origin of the semiclassical regime.

ACKNOWLEDGMENTS

We thank Andrew Matacz for discussions and Juan Pablo Paz and Sukanya Sinha for useful comments on a preliminary draft of this paper. E.C. was partially supported by the Directorate General for Science Research and Development of the Commission of the European Communities under Contract No. C11-0540-M(TT), and by CONICET, UBA, and Fundación Antorchas (Argentina). B.L.H. was supported in part by the National

Science Foundation (NSF) under Grant No. PHY 91-19726. This collaboration was partially supported by CONICET and NSF under the Scientific and Technological Exchange Program between Argentina and USA. E.C. wishes to thank D. Hobill and the Kananaskis Center of the University of Calgary, where parts of this work were completed, for their kind hospitality. B.L.H. wishes to thank O. Bertolami, M. Bento, and J. Mourao for their warm hospitality during his visit to the Institute Superior Technico of Lisbon, where part of the results of this work was first reported.

APPENDIX

In this appendix we shall analyze the exact expression (3.12) for the closed time path effective action given in Sec. III. We first give two independent proofs of it (Secs. A 1 and A 2), and then show that the resulting contribution to the semiclassical equations of motion is real and causal (Sec. A 3).

1. Proof based on elementary field theory

This proof is based on the observation that the definition of the CTP effective action, as given in Sec. II, may be reduced to

$$e^{i\Gamma} = \sum_n \langle 0\text{in}|n\text{out}\rangle_- \langle n\text{out}|0\text{in}\rangle_+ . \tag{A1}$$

Manipulating the Bogoliubov coefficients it is easy to show that

$$\langle (2n+1)\text{out}|0\text{in}\rangle = 0 \tag{A2}$$

and

$$\langle 2n\text{out}|0\text{in}\rangle = \left(\frac{\sqrt{(2n)!}}{2^n(n!)}\right)\left(\frac{\beta^n}{\alpha^{(2n+1)/2}}\right)^* \tag{A3}$$

(we have chosen the relative phases of the in and out vacua to match the results from perturbation theory; this choice fixes all the rest).

Using this in the formula for Γ and applying the summation formula Eq. (5.15) we get the desired result (3.12).

2. Proof based on functional analysis

This proof is based on the functional integral expression

$$e^{i\Gamma} = \int D\phi_+ D\phi_-\ e^{i(S_f[a_+,\phi_+]-S_f[a_-,\phi_-])}. \tag{A4}$$

Remember that the positive branch has a negative slope in the complex η plane, and the negative branch has a positive slope. Also that the CTP histories are continuous across the turning point.

Let ϕ be the common value of ϕ_+ and ϕ_- at the Cauchy surface in the future, and let $\bar{\phi}_\pm$ be the classical solution, in each branch, that vanishes in the past and matches $\bar{\phi}$ in that Cauchy surface (we are assuming we have already decomposed the integral by Fourier modes, so we have a zero-dimensional field theory). Then each history can be written as $\phi_\pm = \bar{\phi}_\pm + \varphi_\pm$, where $\varphi_\pm$ vanishes both in the past and the future, and we get

$$e^{i\Gamma} = \int d\phi\ e^{i(S_f[a_+,\bar{\phi}_+]-S_f[a_-,\bar{\phi}_-])} \int D\varphi_+\ e^{i(S_f[a_+,\varphi_+])} \int D\varphi_-\ e^{-i(S_f[a_-,\varphi_-])}. \tag{A5}$$

Because it is quadratic, the action on a classical solution reduces to

$$S_f[a_\pm,\bar{\phi}_\pm] = \tfrac{1}{2}[\bar{\phi}_\pm\dot{\bar{\phi}}_\pm(+\infty) - \bar{\phi}_\pm\dot{\bar{\phi}}_\pm(-\infty)]. \tag{A6}$$

But because of the boundary conditions, this yields

$$S_f[a_+,\bar{\phi}_+] - S_f[a_-,\bar{\phi}_-] = \tfrac{1}{2}\phi[\dot{\bar{\phi}}_+ - \dot{\bar{\phi}}_-]. \tag{A7}$$

Let $h_\pm$ be solutions of the Klein-Gordon equation on each branch, vanishing in the past (i.e., h_+ is negative frequency and h_- is positive frequency) and satisfying an arbitrary normalization (e.g., having unit Klein-Gordon norm). If the turning point is located at $\eta = \eta^0$,

$$\bar{\phi}_\pm = \phi\ \left(\frac{h_\pm(\eta)}{h_\pm(\eta^0)}\right). \tag{A8}$$

The integral over ϕ in the expression for Γ is then an ordinary Gaussian integral, which yields

$$\left[\frac{1}{2\pi i}\left(\frac{\dot{h}_+(\eta^0)}{h_+(\eta^0)} - \frac{\dot{h}_-(\eta^0)}{h_-(\eta^0)}\right)\right]^{-1/2}. \tag{A9}$$

The remaining functional integrals are of the usual in-out type. Following Ref. [76], we know that, e.g.,

$$\int D\varphi_\pm\ e^{i(S_f[a_\pm,\varphi_\pm])} = \left(\text{Det}\frac{\delta^2 S_f}{\delta\phi_\pm^2}\right)^{-1/2}. \tag{A10}$$

And that, in turn

$$\left(\text{Det}\frac{\delta^2 S_f}{\delta\phi_\pm^2}\right) = \text{const} \times\ h_\pm(\eta^0). \tag{A11}$$

Using these expressions for the remaining functional integrals we are led to

$$\Gamma = \left(\frac{i}{2}\right)\ln[h_-(\eta^0)\dot{h}_+(\eta^0) - h_+(\eta^0)\dot{h}_-(\eta^0)] + \text{const}. \tag{A12}$$

Finally, rewriting the *in* functions $h_\pm$ in terms of the *out* particle model and the Bogoliubov coefficients, we get the desired formula.

3. Proof of the causal and real property

We now present the argument why the equations of motion deriving from the effective action are necessarily real and causal. The contribution from Γ to the equations of motion takes the form

$$\frac{i}{2}\left(\alpha\frac{\partial\alpha^*}{\partial a(\eta)}-\beta\frac{\partial\beta^*}{\partial a(\eta)}\right). \tag{A13}$$

Reality follows from the identity $|\alpha|^2-|\beta|^2=1$, which implies

$$\mathrm{Re}\left(\alpha\frac{\partial\alpha^*}{\partial a(\eta)}-\beta\frac{\partial\beta^*}{\partial a(\eta)}\right)=0. \tag{A14}$$

Thus, the equations of motion are i times something purely imaginary, and therefore themselves real.

To see that the equations are causal, we need to show that they are not changed if we perturb the evolution at the future of time η. To see this, assume we choose a particle model at some time $\eta'\geq\eta$, but still earlier than the time when perturbation sets in. Then the *in-out* Bogoliubov coefficients can be expressed in terms of the Bogoliubov coefficients between the *in* particle model and the η' model, and those linking the η' model to the *out* one:

$$\alpha=\alpha_{\mathrm{in},\eta'}\alpha_{\eta',\mathrm{out}}+\beta_{\mathrm{in},\eta'}\beta^*_{\eta',\mathrm{out}}, \tag{A15}$$

$$\beta=\alpha_{\mathrm{in},\eta'}\beta_{\eta',\mathrm{out}}+\beta_{\mathrm{in},\eta'}\alpha^*_{\eta',\mathrm{out}}. \tag{A16}$$

Using this in the equations of motion, observing the η'-*out* coefficients do not depend on the metric before η', we get

$$\frac{i}{2}\left(\alpha_{\mathrm{in},\eta'}\frac{\partial\alpha^*_{\mathrm{in},\eta'}}{\partial a(\eta)}-\beta_{\mathrm{in},\eta'}\frac{\partial\beta^*_{\mathrm{in},\eta'}}{\partial a(\eta)}\right) \tag{A17}$$

which in turn does not depend on the metric after η'. Since η' may be chosen arbitrarily close to η, this proves causality.

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