On black hole entropy

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Two techniques for computing black hole entropy in generally covariant gravity theories including arbitrary higher derivative interactions are studied. The techniques are Wald's Noether charge approach introduced recently, and a field redefinition method developed in this paper. Wald's results are extended by establishing that his local geometric expression for the black hole entropy gives the same result when evaluated on an arbitrary cross section of a Killing horizon (rather than just the bifurcation surface). Further, we show that his expression for the entropy is not affected by ambiguities which arise in the Noether construction. Using the Noether charge expression, the entropy is evaluated explicitly for black holes in a wide class of generally covariant theories. For a Lagrangian of the functional form $\tilde{L} = \tilde{L}(\psi_m, \nabla_a \psi_m, g_{ab}, R_{abcd}, \nabla_e R_{abcd})$, it is found that the entropy is given by $S = -2\pi \oint (Y^{abcd} - \nabla_e Z^{e:abcd}) \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \bar{\epsilon}$, where the integral is over an arbitrary cross section of the Killing horizon, $\hat{\epsilon}_{ab}$ is the binormal to the cross section, $Y^{abcd} = \partial \widetilde{L} / \partial R_{abcd}$, and $Z^{e:abcd} = \partial \tilde{L} / \partial \nabla_e R_{abcd}$. Further, it is shown that the Killing horizon and surface gravity of a stationary black hole metric are invariant under field redefinitions of the metric of the form $\bar{q}_{ab} \equiv q_{ab} + \Delta_{ab}$, where Δ_{ab} is a stationary tensor field that vanishes at infinity and is regular on the horizon (including the bifurcation surface). Using this result, a technique is developed for evaluating the black hole entropy in a given theory in terms of that of another theory related by field redefinitions. Remarkably, it is established that certain perturbative, first order, results obtained with this method are in fact exact. A particular result established in this fashion is that a scalar matter term of the form $\nabla^{2p}\phi \nabla^{2q}\phi$ in the Lagrangian makes no contribution to the black hole entropy. The possible significance of these results for the problem of finding the statistical origin of black hole entropy is discussed.

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I. INTRODUCTION

Black hole thermodynamics seems to hint at some profound insight into the character of gravity in general, and quantum gravity in particular. The hope is that further study will reveal something about the nature of quantum gravity. One direction to pursue is to investigate the stability of black hole thermodynamics under perturbations of the classical (Einstein) theory that are expected on general grounds as a result of quantum effects.

Whatever the ultimate nature of quantum gravity, there should be an effective Lagrangian that describes the dynamics of a classical "background field" for sufficiently weak fields at sufficiently long distances. Such a low energy effective action will presumably be generally covariant, and will have higher curvature terms, and also higher derivative terms in the metric and all other matter fields. For example, such interactions naturally arise

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from renormalization in the context of quantum field theory [1], and in the construction of an effective action for string theory [2]. While such actions are pathological if taken as fundamental, they can define benign perturbative corrections to Einstein gravity with ordinary matter actions. Let us also add at this point that many of the recent candidates for a theory of quantum gravity, especially those which attempt to unify gravity with other interactions, are theories in higher dimensional spacetimes. Thus in the following investigation, we will allow space-time to have an arbitrary dimension D.

The question we would like to address is whether the laws of black hole thermodynamics are consistent with all such effective actions, or whether perhaps consistency with these laws picks out a preferred class of potential "corrections" to classical gravity. Most recent efforts have been devoted to calculating the entropy Sfor explicit black hole solutions in various theories. It is well known that, in general, the standard relation S = A/(4G) of Einstein gravity no longer applies [3, 4]. Until recently though, it had not even been established (except for special cases, e.g., Lovelock gravity [5]) that the entropy can be expressed as a local functional evaluated at the horizon. Now several researchers have shown that S is indeed local in general, and have provided vari-

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ous techniques to compute it [6-8]. In this paper, we will compute S for a wide class of theories, using both Wald's Noether charge technique [6], and a method exploiting field redefinitions that we developed prior to the recent appearance of more powerful and general techniques [6, 7]. In addition, we will extend the results of [6] by showing why ambiguities in the Noether charge construction do not affect the entropy, and establishing an expression for the entropy, which is valid on *arbitrary* cross sections of the horizon (rather than just on the bifurcation surface).

A key concept in what follows is the notion of a Killing horizon. A Killing horizon is a null hypersurface whose null generators are orbits of a Killing field. In fourdimensional Einstein gravity, Hawking proved that the event horizon of a stationary black hole is a Killing horizon [9]. This proof cannot obviously be extended to higher curvature theories (however, no counterexamples are known, including some nonstatic solutions [10]). If the horizon generators of a Killing horizon are geodesically complete to the past (and if the surface gravity is nonvanishing), then the Killing horizon contains a (D-2)-dimensional spacelike cross section B on which the Killing field χ^a vanishes [11]. B is called the *bifurca*tion surface. Such a bifurcation surface is fixed under the Killing flow, and lies at the intersection of the two null hypersurfaces that comprise the full Killing horizon. (For example, in the maximally extended Schwarzschild black hole space-time, the bifurcation surface is the 2sphere of area $16\pi G^2 M^2$ at the origin of Kruskal U-V coordinates.) The techniques employed for computing black hole, entropy in this paper all apply only to black holes with bifurcate Killing horizons. For a space-time containing an asymptotically stationary black hole that forms by a collapse process there is certainly no bifurcation surface. However, the operative assumption is then that the stationary black hole, which is asymptotically approached can be extended to the past to a space-time with a bifurcate Killing horizon. Investigation of the validity of this fundamental assumption is important but will be left to other work.

The zeroth law of black hole thermodynamics states that for a stationary event horizon the surface gravity κ , which is proportional to the black hole's temperature (in any theory of gravity [12]), is constant over the entire horizon. If χ^a is the null generator of the horizon, κ is defined by $\chi^b \nabla_b \chi^a = \kappa \chi^a$. This constancy of κ on Killing horizons has been proven for Einstein gravity [13] with matter satisfying the dominant energy condition, but this proof does not readily extend to the present context of higher curvature theories. If one *assumes* (as we do) a bifurcate Killing horizon, however, then constancy of the surface gravity is easily seen to hold independently of any field equations. Conversely, if the surface gravity is constant and nonvanishing on a Killing horizon, then the horizon can be extended to a bifurcate horizon [11].

The first law of black hole thermodynamics takes the form

$$\frac{\kappa}{2\pi}\delta S = \delta M - \Omega^{(\alpha)}\,\delta J_{(\alpha)} - \cdots \quad . \tag{1}$$

 $M, J_{(\alpha)}, \text{and } \Omega^{(\alpha)}$ are the black hole mass, canonical angular momentum, and the angular velocity of the horizon [14]. The ellipsis indicates possible contributions from variations of other extensive parameters which characterize the black hole (e.g., electric or magnetic charge). For Einstein gravity, the entropy S is one-quarter the surface area of the horizon: S = A/(4G). Equation (1) then has the rather remarkable feature that it relates variations in properties of the black hole as measured at asymptotic infinity to a variation of a geometric property of the horizon. Given the recent results of Refs. [6, 7], one now knows that although the precise expression for the entropy is altered, it remains a quantity localized at the horizon for arbitrary theories of gravity, and so this aspect of the first law is preserved.

The remainder of the paper is organized as follows. In Sec. II, we describe Wald's result that the entropy is a Noether charge [6], generalize it to arbitrary horizon cross sections, and discuss the extension to nonstationary black holes. We also apply this technique to compute S for a certain wide class of Lagrangians. Section III introduces the field redefinition technique, and illustrates it with examples. Section IV presents a discussion of our results and their possible implications for the problem of the statistical origin of black hole entropy. Throughout the paper, we consider only asymptotically flat spaces, and we employ the conventions of [15].

II. ENTROPY AS A NOETHER CHARGE

Wald [6] has recently derived a general formula for the entropy of a stationary black hole based on a Lagrangian derivation of the first law of black hole mechanics. The results of [6] apply to black holes with bifurcate Killing horizons in any diffeomorphism invariant theory in any space-time dimension. (The entire discussion of this section refers to stationary black holes of this type, except in Sec. II C.)

Given the symmetry of diffeomorphism invariance, there is a conserved Noether current associated to any vector field (see below). Wald begins with the Noether charge associated with the horizon generating Killing field χ^a , defined by integrating the Noether current over a spacelike hypersurface extending from asymptotic infinity to the bifurcation surface in a stationary black hole solution to the field equations. (According to our results below, the inner boundary could be chosen as any spacelike slice of the horizon.) Upon considering a variation of the fields away from the background solution, he finds an identity relating a surface term at infinity to another on the horizon, in the form expected for the first law. The terms at infinity yield precisely the mass and angular momentum variations appearing in Eq. (1) [16]. From the other boundary term, Wald thus finds that the entropy is given by

$$S = 2\pi \oint Q , \qquad (2)$$

where the integral is over the bifurcation surface of the

horizon. The (D-2)-form Q is the "Noether potential" (defined below) associated with the Killing field $\tilde{\chi}^a$ that is null on the horizon and normalized to have unit surface gravity. As we shall show below, one obtains the same result for the entropy if Q is integrated over *any* cross section of the horizon.

The entropy (2) is not expressed in terms of *only* the dynamical fields and their derivatives, since by construction Q involves the Killing field $\tilde{\chi}^a$ and its derivatives. (The Killing field is of course determined by the metric, but by an integral operation rather than differential ones.) However, as noted in [6], the explicit dependence on $\tilde{\chi}^a$ can be eliminated as follows. First, one uses the identity $\nabla_c \nabla_a \tilde{\chi}_b = -R_{abcd} \tilde{\chi}^d$ (which holds for any Killing vector) to eliminate any second or higher derivatives of $\tilde{\chi}^a$, leaving only $\tilde{\chi}^a$ and $\nabla_a \tilde{\chi}_b$. The term linear in $\tilde{\chi}^a$ contributes nothing at the bifurcation surface, since $ilde{\chi}^a$ vanishes there. Moreover, at the bifurcation surface one has $\nabla_a \tilde{\chi}_b = \hat{\epsilon}_{ab}$, where $\hat{\epsilon}_{ab}$ denotes the binormal to B. This allows one to substitute $\hat{\epsilon}_{ab}$ for $\nabla_a \tilde{\chi}_b$ in the expression for Q. Thus, at least at the bifurcation surface, all explicit reference to the Killing field can be eliminated from Q. Let us denote by Q the form that is obtained from Q in this fashion. Then the expression $2\pi \oint Q$ evaluated at the bifurcation surface correctly displays the entropy as a "local" geometric functional of the metric, the matter fields and their derivatives. It will be shown below that in fact $S = 2\pi \oint \hat{Q}$ on an arbitrary cross section of the Killing horizon.

Wald's construction for the entropy has tremendous advantages compared with methods previously available. One works with an arbitrary Lagrangian, and there is no need to find the corresponding Hamiltonian, as in Ref. [8]. Further, there is no need to identify a preferred surface term in the action. Adding a total derivative to the Lagrangian does not affect the entropy (2), as will be shown below. Another feature to be noted is that no "Euclideanization" is required.

In the remainder of this section, we first sketch the Noether charge construction [17,18], and explain why the attendant ambiguities do not affect the entropy. Then, we show how the entropy can be expressed as an integral over an arbitrary cross section of the horizon, rather than over the bifurcation surface. Next, we discuss the possible definitions of entropy for nonstationary black holes. Finally, we explicitly compute the black hole entropy for a wide class of Lagrangians.

A. The Noether potential Q

The (D-2)-form Q may be defined as the potential for the corresponding "Noether current" (D-1)-form J, in the case that J = dQ. The symmetry relevant for the black hole entropy is diffeomorphism invariance. The Noether current associated with the diffeomorphism generated by a vector field ξ^a is defined as follows [17]. Let Lbe a Lagrangian D-form built out of some set of dynamical fields, including the metric, collectively denoted here by ψ . Under a general field variation $\delta\psi$, the Lagrangian varies as

$$\delta L = E \cdot \delta \psi + d\theta(\delta \psi), \tag{3}$$

where the centered dot denotes a summation over the dynamical fields including contractions of tensor indices, and E = 0 are the equations of motion. [The ambiguity $\theta \rightarrow \theta + d\gamma$ allowed by (3) is inconsequential—see below.]

The diffeomorphism invariance of the theory is ensured if, under field variations induced by diffeomorphisms $\delta \psi = \mathcal{L}_{\xi} \psi$, one has $\delta L = \mathcal{L}_{\xi} L = d i_{\xi} L$ [19]. The Noether current J associated with a vector field ξ^a is defined by

$$J = \theta(\mathcal{L}_{\xi}\psi) - i_{\xi}L, \tag{4}$$

where θ is defined by (3). One easily sees that dJ = 0, modulo the equations of motion, as a consequence of the diffeomorphism covariance of the Lagrangian. Thus, modulo the equations of motion, we have J = dQ, for some Q, at least locally. Much more can be said, however, as a consequence of the fact that J is closed for all vector fields ξ^a . Namely, there exists a unique, globally defined Q satisfying J = dQ, that is a local function of the dynamical fields and a linear function of ξ^a and its derivatives [18]. Moreover, Ref. [18] presents an inductive algorithm for constructing Q in such a situation. We call Q the Noether potential associated with ξ^a . The Noether charge for a spacelike (D-1)-dimensional hypersurface Mis given by $\int_M J$, and hence, in this case, reduces to the boundary integral $\int_{\partial M} Q$. Thus the black hole entropy (2) is 2π times the contribution to the Noether charge coming from the boundary at the horizon.

There are three stages at which ambiguity can enter the above construction of the Noether charge. First, an exact form $d\alpha$ can be added to the Lagrangian without changing the equations of motion. This induces an extra term $\mathcal{L}_{\xi}\alpha$ in $\theta(\mathcal{L}_{\xi}\psi)$, and therefore extra terms $di_{\xi}\alpha$ and $i_{\xi}\alpha$ appear in J and Q, respectively. Now in the entropy (2), ξ is chosen to be the Killing field $\tilde{\chi}^a$, and Q is evaluated at the bifurcation surface where $\tilde{\chi}^a=0$. Thus the extra term $i_{\tilde{\chi}\alpha}$ makes no contribution to the entropy. In fact the extra term vanishes everywhere on the horizon, as will be shown below, leaving $2\pi \oint Q$ unmodified for any cross section of the Killing horizon.

The second ambiguity arises because θ is defined by (3) only up to the addition of a closed form β . Assuming β is closed for all variations $\delta \psi$ (and that it vanishes when $\delta \psi = 0$), then the result of [18] quoted above implies that β has the form $d\gamma$. Thus J and Q are modified by the addition of $d\gamma(\mathcal{L}_{\xi}\psi)$ and $\gamma(\mathcal{L}_{\xi}\psi)$, respectively. With ξ equal to the Killing field $\tilde{\chi}^a$, the extra term $\gamma(\mathcal{L}_{\tilde{\chi}}\psi)$ vanishes because $\mathcal{L}_{\tilde{\chi}}\psi = 0$ for the background fields ψ in a stationary solution. In this case, it is immediately clear that this ambiguity will not affect $2\pi \oint Q$ for any slice of the horizon.

The third ambiguity arises because Q is defined by J = dQ only up to the addition of a closed form σ . With the same assumptions as for β in the previous paragraph, we similarly conclude that σ is exact. Since the integral of an exact form over a closed surface vanishes, Q and $Q + \sigma$ yield the same entropy.

B. Arbitrary horizon cross sections

There are several reasons why it is important to be able to evaluate the black hole entropy as an integral over an arbitrary slice of the Killing horizon rather than only at the bifurcation surface. For one thing, if the (approximately stationary) black hole formed from collapse, then the bifurcation surface is not even a part of the space-time. As a practical matter, it may be inconvenient to have to determine the geometry and matter fields in the vicinity of the bifurcation surface, for instance, if the solution is known in a coordinate system that does not extend all the way to there. The primary reason though arises because one wants to have a definition of the entropy that applies to nonstationary black holes. A clear prerequisite for such a definition is that it yield the same result for any cross section of a stationary black hole horizon.

In the case of general relativity, Eq. (2) yields onequarter the area of the bifurcation surface for the entropy. It is a simple consequence of stationarity that all cross sections of the Killing horizon are isometric, so, in particular, they have the same area. Thus one can deduce that the entropy is one-quarter the area of *any* cross section of the horizon. Similarly, in Lovelock gravity, it was found [5] that the entropy depends only upon the intrinsic geometry of the bifurcation surface. Since all cross sections of the horizon are isometric, the entropy of Lovelock black holes is given by an intrinsic expression, which can be evaluated over any cross section with equal results.

For a general theory, the entropy (2) does not depend only upon the intrinsic geometry of the horizon, so it is not immediately clear what form it will take on arbitrary cross sections of the horizon. Nevertheless, it is easy to see that $\oint Q$ is in fact the same for all cross sections of a stationary horizon. The difference between the integrals over two cross sections is given by the integral of dQ = J over the segment of the horizon that is bounded by them. For a stationary space-time, Eq. (4) yields $J = -i_{\tilde{\chi}}L$, whose pullback to the horizon vanishes, since $\tilde{\chi}^a$ is tangent to the horizon. Thus the entropy is indeed given by (2) with the integral taken over *any* cross section of the horizon.

Can the explicit dependence of the entropy on the Killing field be eliminated on any slice of the horizon, as was possible at the bifurcation surface? Recall that, as explained above, when eliminating the Killing field from Q to obtain \tilde{Q} , Wald used the fact that $\tilde{\chi}^a$ vanishes at the bifurcation surface and the fact that $\nabla_a \tilde{\chi}_b$ is the binormal, neither of which are true on an arbitrary cross section. Nevertheless, we will now show that although $\tilde{\chi}^a$ does not vanish the term proportional to $\tilde{\chi}^a$ vanishes, and although $\nabla_a \tilde{\chi}_b$ is not the binormal the difference between it and the binormal makes no contribution. That is, the entropy is given by $2\pi \oint \tilde{Q}$ on any cross section of the horizon.

When any higher derivatives of the Killing field are eliminated by use of the Killing identity, the Noether charge takes the general form

$$\hat{Q} = B_a \tilde{\chi}^a + C_{ab} \nabla^a \tilde{\chi}^b \,, \tag{5}$$

where B_a and C_{ab} are tensor-valued (D-2)-forms that are invariant under the Killing flow.

The pullback to the horizon of the form $B_a \tilde{\chi}^a$ necessarily vanishes everywhere on a Killing horizon if B_a is regular at the bifurcation surface. To see why, note that the Killing field is tangent to the horizon, and therefore it defines a flow of the horizon into itself. The form $B_a \tilde{\chi}^a$ is invariant under the Killing flow, so its pullback is an invariant form on the horizon submanifold. If it vanishes at one point on a given horizon generator, it vanishes everywhere along that generator. Now it vanishes on the bifurcation surface, where $\tilde{\chi}^a$ vanishes, provided none of the tensor fields out of which B_a is built are singular there (regularity at the bifurcation surface is an implicit assumption already in Wald's derivation [6]). Furthermore, all horizon generators terminate at the bifurcation surface. However, the Killing "flow" does not flow anywhere at the bifurcation surface. Nevertheless, one can argue by continuity that the components of the form $B_a \tilde{\chi}^a$ are arbitrarily small in good coordinates on the horizon sufficiently close to the bifurcation surface, and transforming to the Killing coordinate along the flow will only make them smaller. Therefore we can indeed conclude that the pullback of $B_a \tilde{\chi}^a$ to a bifurcate Killing horizon vanishes, provided B_a is regular at the bifurcation surface. Thus this term will make no contribution to the black hole entropy on any slice.

Now consider the second term in Eq. (5). Since the Killing field is hypersurface orthogonal at the Killing horizon, we have $\nabla_a \tilde{\chi}_b = w_{[a} \tilde{\chi}_{b]}$ for some w_a defined on the horizon. On an arbitrary cross section we therefore have $\nabla_a \tilde{\chi}_b = \gamma \hat{\epsilon}_{ab} + s_{[a} \tilde{\chi}_{b]}$, where $\hat{\epsilon}_{ab}$ is the binormal to the cross section, γ is some function, and s^a is some spacelike vector tangent to the cross section. Contracting both sides of this equation with $\tilde{\chi}^a$ we find that $\gamma = 1$, so we have

$$\nabla_a \tilde{\chi}_b = \hat{\epsilon}_{ab} + s_{[a} \tilde{\chi}_{b]}. \tag{6}$$

The same reasoning as in the previous paragraph leads to the conclusion that the pullback of the covariant tensorvalued form $C_{ab}\tilde{\chi}^b$ vanishes on the horizon [20]. Thus the second term in (6) does not make any contribution to the pullback of \hat{Q} , and therefore makes no contribution to the entropy. The conclusion is that when pulled back to any point of the Killing horizon, $\hat{Q} = C_{ab}\hat{\epsilon}^{ab} = \tilde{Q}$. Therefore in fact the entropy (2) is also given by $2\pi \oint \tilde{Q}$ over any cross section of the horizon.

In the above arguments as in Ref. [6], one makes the assumption that the horizon is a regular bifurcate Killing horizon, and that all the fields (not just the metric) are regular at the bifurcation surface. This is a rather strong assumption, since there certainly exist stationary spacetimes with Killing horizons that do not possess a regular bifurcation surface. How do we know that the stationary solution to which a black hole settles down might not be of this type? Recent work by Racz and Wald [11] has something very important to say on this point. Namely, if the surface gravity is constant and nonvanishing over a patch of a Killing horizon (and the patch includes a spacelike cross section of the horizon), then there exists a stationary extension of the space-time around that patch, which extends all the way back to a regular bifurcation surface. This result makes it seem likely that one can dispense with the need to go back to the bifurcation surface, and instead give a *local* argument for the vanishing of the pullback of the tensors $B_a \tilde{\chi}^a$ and $C_{ab} \tilde{\chi}^b$ to the horizon, provided one assumes the surface gravity is constant and nonvanishing. In fact, such an argument exists, and will be presented elsewhere [21].

The arguments of Ref. [11] referred only to the metric, so they do not guarantee that all of the (matter) fields can also be extended in a nonsingular way back to the bifurcation surface, although that may actually be true. Certainly one knows that all scalars formed from the background matter fields and their derivatives (as well as the metric and the curvature) will be nonsingular at the bifurcation surface, since they are constant under the Killing flow. This is not sufficient, however, since a tensor can diverge even if all scalars formed by contracting it with tensors from some particular class are regular. For scalar matter fields, we have shown that stationarity is sufficient to enforce the desired regularity at the bifurcation surface [21]. In other cases, regularity may require that additional conditions be placed on the matter fields (rather like the assumption of constant surface gravity for the metric). In the context of the local argument above, these results are required so that the vanishing of the pullbacks still holds in the presence of matter fields.

C. Nonstationary black holes

The above arguments establish that, provided a stationary black hole has a regular Killing horizon, there is no ambiguity in the entropy, and it can be evaluated with an integrand of the same form on any cross section of the horizon. In the nonstationary case, there are three obvious candidate forms for the entropy:

$$S_{1} = 2\pi \oint Q(\xi^{a}, \nabla^{a}\xi^{b}, \ldots),$$

$$S_{2} = 2\pi \oint \hat{Q}(\xi^{a}, \nabla^{[a}\xi^{b]}),$$

$$S_{3} = 2\pi \oint \tilde{Q}(\hat{\epsilon}_{ab}) .$$
(7)

The integrand in the first expression is the full potential Q produced by the Noether charge construction. As indicated, Q may depend on arbitrarily high order derivatives of the vector field ξ^a . In S_2 , all of the higher derivatives have been eliminated from Q via identities that would hold if ξ^a were a Killing vector, yielding \hat{Q} . Hence the only remaining dependence on the vector field is on ξ^a and $\nabla^{[a}\xi^{b]}$. In the last expression S_3 , the term proportional to ξ^a is dropped, and $\nabla_{[a}\xi_{b]}$ is replaced by the binormal $\hat{\epsilon}_{ab}$ to the particular slice over which the integral is to be evaluated, yielding \tilde{Q} in the integrand. As discussed in the preceding subsection, all three of these expressions yield identical results when pulled back to a bifurcate Killing horizon, with ξ^a equal to the horizon generating Killing field.

Now if one wishes to define the entropy of a nonstationary black hole, it is not so clear what to do. There is apparently no preferred choice of vector field out of which to construct S_1 and S_2 . Further, the ambiguities explained in Sec. II A can no longer be dismissed, and all of three of the expressions may contain significant ambiguities. In the absence of a deeper understanding of black hole entropy, it would seem that there is no fundamental criterion that might be imposed in defining the entropy of a nonstationary black hole, other than it prove convenient for deriving results about the change of entropy in dynamical processes. Of course, if an entropy that satisfies the second law can be defined, that would be a preferred definition.

Actually, for nonstationary perturbations of stationary solutions, the entropy is well-defined. Using the Noether charge approach [6], the first law has been established for variations from a stationary to an arbitrary, nonstationary solution. In this law, the entropy of the nonstationary solution is defined as S_1 , evaluated at the bifurcation surface of the stationary background. However, it turns out that in fact one has $\delta S_1 = \delta S_2 = \delta S_3$ in this case [6]. Moreover, as required by consistency with the first law, the ambiguities in Q discussed in Sec. II A do not affect S_1 at the bifurcation surface for these nonstationary perturbations. [This is obvious for the first and third ambiguities but requires a short computation for the second one, where Q is modified by the addition of $\gamma(\mathcal{L}_{\tilde{\chi}}\psi)$ [6].]

Wald proposed S_3 as the natural candidate for the entropy of a general nonstationary black hole, since it is a local geometrical expression [6]. Our results of the preceding section show that this proposal at least has the merit that for a stationary black hole, it gives the correct entropy on any slice of a Killing horizon. It also seems to meet the criterion of convenience when compared to the alternatives S_1 and S_2 , since no external vector field is required. Of course, one must still resolve the inherent ambiguities in \tilde{Q} in some fashion.

To define S_1 or S_2 in a dynamical process joining two (approximately) stationary black hole states, one must choose some vector field, which agrees with the initial and final horizon-generating Killing fields. It would seem natural to demand that the vector field also be tangent to the event horizon generators through the intermediate nonstationary interval (although this could only be approximately the case, since the original Killing horizon must be somewhat inside the event horizon). With such a vector field, one can construct Q and \hat{Q} , and test S_1 and S_2 as candidates for the black hole entropy. Having chosen an arbitrary vector field to define Q, one still has J = dQ. Thus during the dynamical process, the changes in S_1 from slice to slice will be given by the flux of the Noether current through the intervening segment of the horizon [6], and the total change in the entropy between the two stationary stages will be given by the total flux of Noether current. It would be interesting if one could establish that $(S_2 \text{ or}) S_3$, with some particular choice of slicing of the horizon, coincides with S_1 for a particular choice of vector field. In this case, the change of $(S_2 \text{ or})$ S_3 could also be connected to a flux of Noether current.

D. Explicit entropy expressions

In this subsection we will find an explicit expression for the entropy in a wide class of theories. (We only retain the contributions denoted as \tilde{Q} , above.) As stated above, Ref. [18] presents an inductive algorithm, which can be used for constructing the Noether charge Q. Two useful facts, which arise from this construction, and which will simplify our calculations below, are if the maximum number of derivatives of ξ^a in J is k, then (i) the maximum number in Q is k-1, and (ii) the term in Q with k-1 derivatives is algebraically determined by the term in J with k derivatives.

Let us first compute the entropy for a general Lagrangian of the functional form $L = L(\psi_m, \nabla_a \psi_m)$ g_{ab}, R_{abcd} , that is involving no more than second derivatives of the space-time metric g_{ab} , and first derivatives of the matter fields, denoted by ψ_m . Since ultimately the potential form is pulled back to the horizon, where the (pulled-back) term linear in the Killing field χ^a vanishes, we need only determine the part of Q involving at least first derivatives of ξ^a . Hence we only require the part of J with at least second derivatives of ξ^a . From (4), this is the part of $\theta(\mathcal{L}_{\mathcal{E}}\psi)$ with at least second derivatives of ξ^{a} , and then it follows from (3) that the latter is given by the part of δL involving at least second derivatives of the field variations. For the class of Lagrangians, we are considering, such terms can only arise from variations of the Riemann tensor.

To implement the form notation in computations, it is convenient to introduce the notation $\epsilon_{a_1\cdots a_m}$, which denotes the volume form $\epsilon_{a_1\cdots a_D}$ regarded as a tensor valued (D-m)-form. Note that there are therefore many notations for the same tensor field, and we use whichever one is convenient at any given juncture. A result that we will make use of is

$$d(W^{a_1\cdots a_m}\epsilon_{a_1\cdots a_m}) = m(\nabla_b W^{a_1\cdots a_{m-1}b})\epsilon_{a_1\cdots a_{m-1}}.$$
 (8)

With the *D*-form ϵ , the Lagrangian form may be written as $L = \tilde{L}\epsilon$, where $\tilde{L} = \tilde{L}(\psi_m, \nabla_a \psi_m, g_{ab}, R_{abcd})$ is a scalar function. Now varying *L*, one has

$$\delta L = Y^{abcd} (-2\nabla_a \nabla_c \delta g_{bd}) \epsilon + \cdots$$

= $\nabla_a (-2Y^{abcd} \nabla_c \delta g_{bd} \epsilon) + \cdots$, (9)

where the ellipsis indicates terms that make no relevant contributions to Q, and the tensor Y^{abcd} is defined by

$$Y^{abcd} \equiv \partial \widetilde{L} / \partial R_{abcd}$$

Note that the tensor Y^{abcd} has all symmetries of Riemann tensor (i.e., $Y^{abcd} = Y^{[ab][cd]}$, $Y^{abcd} = Y^{cdab}$, and $Y^{a[bcd]} = 0$; the last will not be used in the following), and hence the indices on $\nabla_a \nabla_c \delta g_{bd}$ need not be explicitly antisymmetrized. Comparing Eqs. (3) and (9), we see

$$\theta = -2Y^{abcd} \nabla_c \delta g_{bd} \,\epsilon_a + \cdots \tag{10}$$

and thus the Noether current is given by

$$J = -2Y^{abcd} \nabla_c (\nabla_b \xi_d + \nabla_d \xi_b) \epsilon_a + \cdots$$
$$= -2Y^{abcd} \nabla_{(c} \nabla_{b)} \xi_d \epsilon_a + \cdots, \qquad (11)$$

where now the ellipsis refers to terms with less than second derivatives of ξ^{a} .

At this point one can apply the formulas of [18] to write down the leading term in Q, but it is just as easy to read it off directly from (11) above. To do so, we must massage (11) so that the leading term appears as an exterior derivative, together with terms that involve less than two derivatives of ξ^a . We have

$$J = -2Y^{abcd} \nabla_{(c} \nabla_{b)} \xi_{d} \epsilon_{a} + \cdots$$

= $-2Y^{abcd} \nabla_{b} \nabla_{c} \xi_{d} \epsilon_{a} + \cdots$
= $\nabla_{b} (-2Y^{abcd} \nabla_{c} \xi_{d} \epsilon_{a}) + \cdots,$

and so

$$Q(\xi) = -Y^{abcd} \nabla_c \xi_d \,\epsilon_{ab} + \cdots, \tag{12}$$

where we have dropped the terms, which vanish for $\xi^a = 0$. The entropy is given by $S = 2\pi \oint Q(\tilde{\chi})$, where the integral may be evaluated on any cross section of the horizon. One may also construct \tilde{Q} by making the replacement $\nabla_c \tilde{\chi}_d \to \hat{\epsilon}_{cd}$, which then yields

$$S = -2\pi \oint Y^{abcd} \hat{\epsilon}_{cd} \epsilon_{ab}$$

= $-2\pi \oint Y^{abcd} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \bar{\epsilon}.$ (13)

In the second line we have introduced $\bar{\epsilon}$, the induced volume form on the horizon cross section, and used the identity $\epsilon_{ab} = \hat{\epsilon}_{ab}\bar{\epsilon}$, which holds when ϵ_{ab} is pulled back as a (D-2)-form to the horizon cross section. (We have not distinguished the various expressions for S here, since they yield identical results on a Killing horizon.) This rather general result (13) was also derived in Refs. [7, 22].

It is a simple exercise to show that Eq. (13) reproduces S = A/(4G) for Einstein gravity, as well as the expression for the entropy of Lovelock black holes derived by Hamiltonian methods in Ref. [5]. While in these examples the black hole entropy depends only on the intrinsic geometry of the horizon, generically the entropy (13) depends on both the intrinsic and *extrinsic* geometry. It is a straightforward exercise to extend this result, for example, to Lagrangians including first derivatives of the Riemann tensor, i.e., $\tilde{L} = \tilde{L}(\psi_m, \nabla_a \psi_m, g_{ab}, R_{abcd}, \nabla_e R_{abcd})$. The final result reduces to

$$S = -2\pi \oint \left(Y^{abcd} - \nabla_e Z^{e:abcd} \right) \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \bar{\epsilon}, \tag{14}$$

where we have introduced the tensor $Z^{e:abcd} = \partial \tilde{L} / \partial \nabla_e R_{abcd}$. In principle, this result could have been more complicated because the Noether current now includes third derivatives of the vector ξ^a , but one finds that the (anti)symmetries of $Z^{e:abcd}$ reduce the order of the derivatives in all such terms. Recently, Iyer and Wald have produced a result, more general than Eq. (14), for Lagrangians containing arbitrarily high order derivatives of the curvature [6].

Finally, we would like to provide an explicit illustration of the ambiguities that arise in these expressions for black hole entropy. Consider the following interaction involving the metric and a scalar field:

$$L_i = [\nabla^a \nabla^b \phi \nabla_a \nabla_b \phi - (\nabla^2 \phi)^2 + R_{ab} \nabla^a \phi \nabla^b \phi] \epsilon$$

Despite the fact that second derivatives of the scalar appear in L_i , by the arguments above, it is not hard to see that those terms make only contributions to Q with no derivatives of ξ^a . Hence Eq. (13) may still be applied, and from $Y_i^{abcd} = -\nabla^{[a}\phi g^{b][c}\nabla^{d]}\phi$, one finds a contribution to the entropy:

$$S_i = 2\pi \oint g_{ab}^{\perp} \nabla^a \phi \nabla^b \phi \ \bar{\epsilon}, \tag{15}$$

where g_{ab}^{\perp} is the metric in the subspace normal to the cross section on which this expression is evaluated. Now, in fact, L_i is a total derivative:

$$L_i = d\alpha_i = d[(\nabla^a \nabla^b \phi \nabla_b \phi - \nabla^a \phi \nabla^2 \phi)\epsilon_a].$$

By the arguments of Sec. II A, this only produces a contribution $Q_i = i_{\xi}\alpha_i$ to the Noether potential with no derivatives of ξ^a , which yields a vanishing contribution to the entropy, $S'_i = 0$. At one level, this apparent contradiction is resolved by the observation that S_i vanishes as well. To see why, note that $g_{ab}^{\perp} = \tilde{\chi}_a \beta_b + \beta_a \tilde{\chi}_a$, where β^a is a vector orthogonal to the cross section that satisfies $\beta^a \beta_a = 0$ and $\beta^a \tilde{\chi}_a = 1$ at the horizon. The integrand in Eq. (15) evidently vanishes, since it is proportional to $\tilde{\chi}^a \nabla_a \phi = \mathcal{L}_{\tilde{\chi}} \phi = 0$, so one does have $S_i = 0$ when evaluated on a slice of a Killing horizon. On a nonstationary horizon, the Lie derivative would not vanish, and hence one would expect in general that $S_i \neq S'_i = 0$.

The discrepancy between the forms of Q_i inferred from (12) and from Sec. II A, arises because the ambiguity $\theta_i \rightarrow \theta_i + d\gamma_i$ in the definition of θ_i has implicitly entered our calculations. The discussion in Sec. II A, by which $L_i = d\alpha_i$ would yield $Q_i = i_{\xi}\alpha_i$, asserts that $\theta(\mathcal{L}_{\xi}\psi) = \mathcal{L}_{\xi}\alpha_i$. This form explicitly disagrees with the result in Eq. (10) if one inserts Y_i^{abcd} as given above, and $\delta g_{bd} = \nabla_b \xi_d + \nabla_d \xi_b$. However, it is not hard to show that this latter result can be reexpressed as $d\gamma_i(\mathcal{L}_{\xi}\psi)$ up to terms that make no contribution to \tilde{Q} . Alternatively, one may consider Q in Eq. (12) directly

. .

$$Q(\xi) = -Y_i^{abcd} \nabla_c \xi_d \epsilon_{ab} + \cdots$$

= $\nabla^{[a} \phi g^{b][c} \nabla^{d]} \phi \nabla_c \xi_d \epsilon_{ab} + \cdots$
= $\left(\nabla^a \phi \nabla^b [\mathcal{L}_{\xi} \phi] - \frac{1}{2} \nabla^a \phi \nabla_c \phi [\mathcal{L}_{\xi} g]^{cb} \right) \epsilon_{ab} + \cdots$
= $\gamma_i (\mathcal{L}_{\xi} \psi) + \cdots$

where as usual we drop terms with no derivatives on ξ^a . Thus the contribution to the entropy calculated in Eq. (15) could be eliminated via the ambiguity in the definition of θ . This explicitly illustrates that these ambiguities must be resolved in establishing a unique definition of black hole entropy for nonstationary horizons.

III. FIELD REDEFINITIONS

In this section we introduce a new technique, based on field redefinitions, for computing black hole entropy. If a field redefinition can be used to relate the actions, which govern two theories, then the entropies of black holes in these theories turn out to be related via the same field redefinition. Hence one can determine the entropy for a new theory by using a field redefinition to transform it to a theory for which the entropy is already known. This technique is useful because field redefinitions can introduce (or remove) certain higher curvature interactions and other higher derivative terms in a gravitational action. In general, the expression for black hole entropy [in particular, S = A/(4G) for Einstein gravity] is modified by such field redefinitions [23].

The field redefinition technique would not be very practical if it were not for the remarkable fact that the leading order perturbative result is in fact exact, as one can infer from the general form, which the entropy density takes in a Noether charge derivation. We will illustrate this perturbative procedure and its justification with an example below.

The validity of the field redefinition technique rests on the fact that both the asymptotic structure of the space-time and the horizon structure are left intact by the field redefinitions we consider. To understand this point further, suppose a metric \bar{g}_{ab} is defined by $\bar{g}_{ab} \equiv g_{ab} +$ Δ_{ab} , where g_{ab} is an asymptotically flat black hole metric, and Δ_{ab} is a tensor field constructed from g_{ab} and/or other tensor fields with the property that it vanishes at infinity. For example, Δ_{ab} might be a multiple of R_{ab} , the Ricci tensor of g_{ab} . Then, provided the tensor field Δ_{ab} falls off fast enough at infinity, the mass and angular momentum of the space-time given by \bar{g}_{ab} will be the same as that for g_{ab} . Moreover, if g_{ab} is a stationary black hole space-time with a bifurcate Killing horizon generated by a Killing vector χ^a , and if $\mathcal{L}_{\chi}\Delta_{ab} = 0$, then χ^a is a Killing field for \bar{g}_{ab} as well, and \bar{g}_{ab} has the same Killing horizon and surface gravity as g_{ab} . The condition that Δ_{ab} be invariant under the Killing field is satisfied in our application, since Δ_{ab} will be constructed from the metric and matter fields in a stationary solution of some theory. The fact that the Killing horizon and surface gravity are common to both g_{ab} and \bar{g}_{ab} requires further explanation.

First we reiterate that χ^a is clearly a Killing field for \bar{g}_{ab} , provided it generates a symmetry of both g_{ab} and any fields entering Δ_{ab} . The bifurcation surface B is defined by the metric-independent equation $\chi^a = 0$, so it must coincide for the two metrics. The (D-2)-dimensional surface B is spacelike with respect to g_{ab} by assumption, and in fact also with respect to \bar{g}_{ab} . To see why, note that the 2-form $\bar{\nabla}_a \bar{\chi}_b$ (where $\bar{\chi}_b \equiv \bar{g}_{bc} \chi^c$) is orthogonal to B, and is timelike: $(\bar{\nabla}_a \bar{\chi}_b)(\bar{\nabla}^a \bar{\chi}^b) = -(\bar{\nabla}_a \chi^b)(\bar{\nabla}_b \chi^a) = -(\nabla_a \chi^b)(\nabla_b \chi^a) = -2\kappa^2$, where κ is the surface gravity of the Killing horizon with respect to g_{ab} . To obtain the second equality we evaluated on B, where χ^a vanishes. This computation shows not only that B is spacelike with respect to \bar{g}_{ab} agrees with that of g_{ab} .

Now if a Killing field vanishes on a spacelike (D-2)surface B, then the null hypersurface generated by the null geodesic congruences that start out orthogonal to B is a Killing horizon [24]. In fact, the Killing horizon generated in this fashion for the metric \bar{g}_{ab} coincides with that for g_{ab} . This follows because, although the light cones defined by g_{ab} and \bar{g}_{ab} do not in general agree, it so happens that the Killing field is null with respect to \bar{g}_{ab} everywhere on the Killing horizon \mathcal{H} of g_{ab} . To see why, note that $\bar{g}_{ab}\chi^a\chi^b = \Delta_{ab}\chi^a\chi^b$, which defines a scalar that is constant along the orbits of the Killing field. As long as Δ_{ab} is regular at the bifurcation surface, where χ^a vanishes, this scalar must therefore vanish everywhere on \mathcal{H} .

Now consider the first law of black hole mechanics (1) in both theories. We assume that the field redefinitions leave the asymptotic properties of the black holes (e.g., the mass and angular momentum) unchanged in transforming between the two theories. Now in comparing different black holes, the extensive variations on the right hand side of the first law are the same in both theories. Therefore, since the surface gravities are the same, the variation of the entropy must be the same in the two theories for all variations. Therefore the entropies must be the same up to a constant, within any connected set of stationary black hole solutions. Therefore, the entropy of a black hole in the theory with action $I[g_{ab},\ldots]$ is given by the entropy in the theory with action $\bar{I}[\bar{g}_{ab},\ldots] \equiv I[g_{ab}(\bar{g},\ldots)]$, evaluated on the (common) Killing horizon, and reexpressed in terms of g_{ab} and the rest of the fields.

We now illustrate the procedure with an example. Consider the theory governed by the action

$$I = \int d^{D}x \sqrt{-g} \left[\frac{R}{16\pi G} + L_{m} + \lambda (a_{1}R^{ab}R_{ab} + a_{2}R^{2}) \right],$$
(16)

where L_m is a conventional matter Lagrangian (depending on no more than first derivatives of the matter fields). Now consider the field redefinition

$$g_{ab} = \bar{g}_{ab} + 16\pi G\lambda \bigg[a_1 R_{ab} - \frac{\bar{g}_{ab}}{D-2} (a_1 + 2a_2) R + 8\pi G a_1 T_{ab} - \frac{8\pi G \bar{g}_{ab}}{(D-2)^2} ((D-4)a_1 - 4a_2) T \bigg], \tag{17}$$

where the energy-momentum tensor has the usual definition

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g_{ab}} = 2 \frac{\delta L_m}{\delta g_{ab}} + g^{ab} L_m$$

and $T = g_{ab}T^{ab}$. In terms of the field \bar{g}_{ab} the action takes the form

$$\bar{I} = \int d^D x \sqrt{-\bar{g}} \left[\frac{\bar{R}}{16\pi G} + L_m(\bar{g}) + (8\pi G)^2 \lambda (a_1 T^{ab} T_{ab} + b_2 T^2) + O(\lambda^2) \right],$$

where $b_2 = [4a_2 - (D-4)a_1]/(D-2)^2$ [25]. The coefficients in the field redefinition (17) were chosen to eliminate the curvature squared interactions, and any curvature matter couplings such as $T^{ab}R_{ab}$ arising at order λ . The resulting action is not quite a conventional action for Einstein gravity coupled to matter fields, since the matter fields have some higher dimension interactions. As long as \tilde{L}_m contains only first order derivatives of fields, these extra interactions do as well. From the Noether charge method, we know that such terms do not lead to modifications of the black hole entropy (or the mass, angular momentum, or any parameters characterizing the black holes at infinity).

For the theory governed by $\bar{I}(\bar{g})$, the entropy is simply given by the standard formula to $O(\lambda^2)$:

$$\bar{S} = \frac{\bar{A}}{4G} + O(\lambda^2) = \frac{1}{4G} \oint_{\Sigma} d^{D-2}x \sqrt{\bar{h}} + O(\lambda^2), \tag{18}$$

where \bar{h}_{ab} is the induced metric on a cross section of the horizon Σ . Now expressing the entropy in terms of the original metric, one finds

$$S = \frac{1}{4G} \oint_{\Sigma} d^{D-2}x \sqrt{h} \left[1 + \frac{1}{2} h^{ab} \,\delta h_{ab} + O(\lambda^2) \right], \tag{19}$$

where δh_{ab} is the difference $\bar{h}_{ab} - h_{ab}$. The intrinsic metric may be written as $h_{ab} = g_{ab} - \chi_a \beta_b - \beta_a \chi_b$, where χ^a is the Killing field and β^a is vector field orthogonal Σ , satisfying $\beta^a \beta_a = 0$ and $\beta^a \chi_a = 1$ on the horizon. Thus to first order, $\delta h_{ab} = \delta g_{ab} - \delta g_{ac} \chi^c \beta_b - \chi_a \delta \beta_b - \delta \beta_a \chi_b - \beta_a \delta g_{bc} \chi^c$, and since $h^{ab} \chi_b = 0 = h^{ab} \beta_b$, one has $h^{ab} \delta h_{ab} = h^{ab} \delta g_{ab}$. Therefore,

$$S = \oint_{\Sigma} d^{D-2}x \sqrt{h} \left[\frac{1}{4G} + 2\pi\lambda \left((a_1 + 2a_2)R - a_1h^{ab}R_{ab} + \frac{8\pi G}{D-2} [(D-4)a_1 - 4a_2]T - 8\pi Ga_1h^{ab}T_{ab} \right) + O(\lambda^2) \right]$$

From the Noether charge method, we know that the entropy for the action (16) can be given entirely by metric expressions, independent of the matter fields. Using the leading order equations of motion, namely,

$$T_{ab} = \frac{1}{8\pi G}(R_{ab} - \frac{1}{2}g_{ab}R) + O(\lambda) \quad , \label{eq:Tab}$$

the contributions proportional to T_{ab} can be replaced by

TABLE I. Contributions to black hole entropy from higher derivative interactions.

$(Interaction)/\sqrt{-g}$	$({ m Entropy \ density})/(4\pi\sqrt{h})$	Derivation ^a
(1) R (2) $f(\phi)R_{\mu\nu}R^{\mu\nu}$ (3) $F(\phi,R)$ (4) $f(\phi)\tilde{L}_p(g)^{\rm b}$ (5) $\tilde{L}(\psi_m, \nabla_a\psi_m, g_{ab}, R_{abcd}, \nabla_e R_{abcd})$ (6) $\nabla^{2p}\phi \nabla^{2q}\phi$	$1 \\ f(\phi)(R - h^{\mu\nu}R_{\mu\nu}) = f(\phi)g_{\perp}^{\mu\nu}R_{\mu\nu} \\ \partial_R F(\phi, R) \\ p f(\phi)\widetilde{L}_{p-1}(h) \\ -\frac{1}{2} \left(Y^{abcd} - \nabla_e Z^{e:abcd}\right) \hat{\epsilon}_{ab}\hat{\epsilon}_{cd} \\ 0$	F, N F, H, N H, N N F, N

^aMethods used to derive the entropy density: [F] = field redefinition method; [H] = Hamiltonian method of Ref. [8]; [N] = Noether charge method of Ref. [6]. ^b $\tilde{L}_p(g) = \frac{1}{2^p} \delta_{c_1 d_1 \cdots c_p d_p}^{a_1 b_1 \cdots a_p b_p} R_{a_1 b_1}^{c_1 d_1} \cdots R_{a_p b_p}^{c_p d_p}(g), \tilde{L}_0 = 1.$

curvature quantities. This yields, up to terms of $O(\lambda^2)$,

$$S = \oint_{\Sigma} d^{D-2}x \sqrt{h} \left[\frac{1}{4G} + 4\pi\lambda [(a_1 + 2a_2)R - a_1h^{ab}R_{ab}] \right]$$
$$= \oint_{\Sigma} d^{D-2}x \sqrt{h} \left[\frac{1}{4G} + 4\pi\lambda \left(2a_2R + a_1g_{\perp}^{ab}R_{ab} \right) \right] .$$
(20)

where $g_{\perp}^{ab} = g^{ab} - h^{ab} = (\chi^a \beta^b + \beta^a \chi^b)$ is the metric in the subspace normal to the horizon.

In making the perturbative expansion, we have consistently ignored terms of order λ^2 . Recalling the Noether charge approach once again we see that, since the action (16) is linear in λ , the modifications to the entropy from higher curvature terms in the original action would only be linear in λ . Therefore the leading order result (20) is in fact the exact black hole entropy for the action (16). (For generic stress tensors, R and R_{ab} are zeroth order in λ , so there is no ambiguity in identifying the $O(\lambda)$ terms as we have done.) Let us add that Eq. (20) agrees with the Noether charge result in Eq. (13).

Note that if we had *not* accounted for the possible presence of matter fields, the above method would have led to a modification of the entropy with precisely one-half the coefficients given in Eq. (20). These results are not inconsistent, however, since for any asymptotically flat *vacuum* solution for the action (16), one has that $R_{ab} = O(\lambda)$, and so we cannot make the argument as above that the leading order result is exact. In general, in the presence (or absence) of matter or other higher curvature interactions, Eq. (20) gives the correct (exact) modification of the entropy induced by the interactions appearing in the action (16). Also note that the terms proportional to T_{ab} in the field redefinition (17) were required to eliminate interactions such as $R^{ab}T_{ab}$, which would arise at $O(\lambda)$, and which would make contributions to the black hole entropy. Thus, when using field redefinitions to reduce an action with higher curvature interactions to a theory for which the black hole entropy is known, it is important to include a matter Lagrangian L_m , and to ensure that extra matter interactions arising after the field redefinition make no contribution to the black hole entropy or make contributions one can evaluate.

The form of the higher curvature interactions for which the black hole entropy can be determined via field redefi-

nitions is not completely general. However, this approach provides a simple method to verify results derived via the more comprehensive methods now available [6, 7]. Note that field redefinitions of matter fields are also possible and easily show that many matter interactions do not modify the black hole entropy despite the fact that they involve higher derivatives. A simple example of such results is listed in the last line of Table I. This result is derived as follows: Beginning with Einstein gravity coupled to a scalar field with $\tilde{L} = -\frac{1}{2}[(\nabla \phi)^2 + m^2 \phi^2]$, a field redefinition $\phi \rightarrow \phi + \lambda \nabla^2 \phi$ may be used to show that an interaction $(\nabla^2 \phi)^2$ produces a vanishing entropy density. Similarly the field redefinition $\phi \to \phi + \lambda \nabla^4 \phi$ shows the entropy is unmodified by a combination of interactions, $(\nabla^2 \phi)^2$ and $\nabla^2 \phi \nabla^4 \phi$. Having shown that the first of these does not contribute, it must also be true that the entropy density vanishes for $\nabla^2 \phi \nabla^4 \phi$. Working iteratively in this way, it is easy to see that an arbitrary term $\nabla^{2p} \phi \nabla^{2q} \phi$ yields no contribution to the black hole entropy. One may similarly arrive at the same conclusion via the Noether charge technique as well.

IV. DISCUSSION

Several results have been established in this paper. These are the following.

(i) Ambiguities in the definition of the Noether charge Q associated with the horizon generating Killing field (normalized to unit surface gravity) have no effect when Q is pulled back to the horizon of a stationary black hole.

(ii) The entropy of a stationary black hole can be expressed as the integral $2\pi \oint Q$ over any cross section of the horizon, not just the bifurcation surface.

(iii) The pullback of Q to any cross section of the horizon can be expressed without reference to the Killing field, yielding the same expression found by Wald at the bifurcation surface.

(iv) The Killing horizon and surface gravity of a stationary black hole metric are invariant under field redefinitions of the metric of the form $\bar{g}_{ab} \equiv g_{ab} + \Delta_{ab}$, where Δ_{ab} is a tensor field constructed out of stationary fields.

(v) Using the preceding result, a new technique has been developed for evaluating the black hole entropy in a given theory in terms of that of another theory related by field redefinitions. Certain perturbative, first order,

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results obtained with this method are shown to be exact.

(vi) The entropy has been evaluated explicitly for black holes in a wide class of theories using both Wald's Noether charge approach and the field redefinition method developed in this paper. In Table I, we have compiled a list of explicit results for certain sample higher curvature interactions. The first line of the table gives the result for the Einstein-Hilbert action as a reference for our notation. The next two lines are simple extensions of the results in Eq. (20), to include potential couplings (i.e., no derivatives) of a scalar field to the curvatures. Note that in these cases, the variation of the entropy in the first law (1) includes variations of both the metric and the scalar field, on the horizon. In the third line, $F(\phi, R)$ also generalizes R^2 to an arbitrary polynomial in the Ricci scalar with scalar-field interactions. The black hole entropy of the latter interactions have also been verified by Hamiltonian methods [26]. The third line provides the generalization of the results of Ref. [5] to include scalar potential couplings to Lovelock curvature interactions. All of these results are encompassed by the general result in Eq. (14), which is listed in the fifth line.

The results of Refs. [6-8] and this paper establish that a first law of black hole mechanics holds for black holes with bifurcate Killing horizons in *all* generally covariant theories of gravity, with the entropy being a local geometrical quantity given by an integral over an arbitrary cross section of the black hole horizon. The generality of this result seems somewhat surprising. General covariance plays a crucial role in allowing the total energy and angular momentum to be expressible as surface integrals at infinity. The underlying reason for the local geometrical nature of the entropy seems less transparent.

It is not clear how strong a restriction it is to include only black holes with a bifurcate Killing horizon. The assumption that a regular bifurcation surface exists is not physically well motivated since, if a black hole forms from collapse, the bifurcation surface is not even in the physical space-time, but only in a virtual extension thereof. On the other hand, if the surface gravity is constant and nonvanishing on a patch of the horizon including a spacelike cross section, then the existence of a regular bifurcation surface (perhaps in an extension of the space-time) is guaranteed [11]. Thus the assumption of a bifurcate Killing horizon is in fact implied by the zeroth law (constancy of the surface gravity). In general relativity, with matter satisfying the dominant energy condition, the zeroth law can be established from the field equations. The validity of the zeroth law in other theories remains an open question that clearly deserves more attention, since the validity of black hole thermodynamics rests on it.

It is worth emphasizing that it was necessary for us to assume that not just the curvature but *all* the physical fields and their derivatives are regular at the bifurcation surface. This condition follows from stationarity for scalar fields [21], but may require additional assumptions for general matter fields.

Originally, the laws of black hole mechanics were a feature of classical general relativity [13], and their relation with thermodynamics was only by way of an analogy. With the discovery that black holes radiate quantum mechanically with a temperature $\hbar \kappa / (2\pi)$ [27], the interpretation of these laws [e.g., Eq. (1)] as true thermodynamic relations became entirely justified. Yet, the deep significance of the fact that classical general relativity already "knew about" Hawking radiation remains to be discovered. Can any insight into this mystery be gained by studying the way classical black hole thermodynamics generalizes to arbitrary generally covariant gravity theories?

Within the context of classical general relativity, Hawking's area theorem [9] implies that the "entropy" can never decrease. Whether or not the entropy in a general theory satisfies such a second law remains an open question, although some positive results do exist, and will be discussed in another paper [28]. Here we just remark that we have shown, via a field redefinition technique, that the second law holds in a particular class of theories in which the gravitational Lagrangian is built algebraically out of the Ricci scalar.

Although the entropy is always a local geometrical quantity at the horizon, it is in general not just dependent on the *intrinsic* geometry of a horizon cross section. General relativity and the Lovelock theories [29] are exceptional, in that the entropy is purely intrinsic. Should anything be made of this distinction? To address this question, it would seem to be necessary to address a more general question: can one understand the origin of the "corrections" to the area-equals-entropy law in more fundamental terms?

In this regard, it is interesting to consider the various approaches to deriving black hole entropy from statistical considerations. One method is to evaluate the entropy using a stationary point approximation to the formal path integral for the canonical partition function [30] or for the density of states [31] in quantum gravity. These manipulations yield the same black hole entropy as that defined by the first law, and this correspondence should continue to hold for arbitrary gravitational actions. This interpretation of black hole entropy thus seems to be robust.

Another approach [32] locates the entropy in the thermal bath of ambient quantum fields perceived by stationary observers under the stretched horizon. In this approach, the gravitational field equations play no role. It is simply argued that changes in this entropy satisfy the first law of black hole mechanics. Since the geometrically defined entropy also satisfies the first law, the two must coincide. This argument works for any gravitational action, provided the dynamics leads to a stable equilibrium state. On the other hand, this approach offers no insight into why the black hole entropy is expressed in a particular geometric fashion. Moreover, the total entropy of the bath is infinite.

If this divergence of the entropy is regulated by imposing a cutoff of some kind, one can obtain a definite result for the black hole entropy. This is what is done in various other approaches [33], closely related to the membrane viewpoint of [32]. In those methods, the black hole entropy is defined essentially by counting quantum field degrees of freedom either outside or inside the horizon. These regulated counting approaches all yield an entropy proportional to horizon area in units of the cutoff, basically because the dominant contribution comes from very short wavelength modes near the horizon. Choosing the cutoff equal to something of order the Planck length, one recovers the entropy inferred from the first law (together with the Hawking temperature) in Einstein gravity. How, therefore, can these counting approaches accommodate the corrections to the area-equals-entropy law that one finds from higher derivative terms in the action?

In response to this query we can offer the following observations. First, it seems likely that if one could carry out the counting more precisely, one would find corrections to the entropy of higher order in the cutoff (Planck) length. Moreover, the occurrence of curvature quantities in such corrections would not be too surprising. In a somewhat analogous context, one obtains divergent curvature-dependent terms in evaluating the Casimir energy of a curved conducting cavity, where the curvature refers to the geometry of the cavity boundary [34]. The presence of a cutoff then renders these terms finite, and they have a physically well-defined origin and value.

Since the counting argument introduces only one (cutoff) scale, the above plausibility argument seems to fail when the higher derivative terms in the Lagrangian have coefficients whose orders of magnitude are not all set by the same (Planck) scale. However, it is conceivable that dependence of the "counting entropy" on the adjustable coefficients in the Lagrangian might arise via the effect they have on the geometry of the black hole background in which the counting is done. It will be interesting to see whether or not this effect on the background modes has the right form to reproduce the entropy defined via the classical first law. If not, then either this counting interpretation of the black hole entropy is wrong, or it is correct and it determines the coefficients in a curvature expansion of the entropy, not leaving any adjustable freedom in the Lagrangian of the theory. That is, in a sense, the entropy functional would determine the theory. Indeed one can hope, more generally, that the quest for a statistical understanding of black hole entropy will lead us not just to a particular low energy effective Lagrangian, but to a more fundamental theory of gravity and matter.

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- [1] See, for example, N.D. Birrell and P.C.W. Davies, in *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
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- [3] In D dimensions, a cross section of the horizon is a (D-2)-dimensional space, and so the surface area A refers to the volume of such a slice.
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