Wilson loops in four-dimensional quantum gravity

Giovanni Modanese*
Center for Theoretical Physics, Laboratory for Nuclear Science, Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts, 02139
(Received 23 July 1993)

A Wilson loop is defined, in four-dimensional pure Einstein gravity, as the trace of the holonomy of the Christoffel connection or of the spin connection, and its invariance under the symmetry transformations of the action is shown (diffeomorphisms and local Lorentz transformations). We then compute the loop perturbatively, both on a flat background and in the presence of an external source; we also allow some modifications in the form of the action, and test the action of "stabilized" gravity. A geometrical analysis of the results in terms of the gauge group of the Euclidean theory, SO(4), leads us to the conclusion that the corresponding statistical system does not develop any configuration with localized curvature at low temperature. This "nonlocal" behavior of the quantized gravitational field strongly contrasts with that of usual gauge fields. Our results also provide an explanation for the absence of any invariant correlation of the curvature in the same approximation.

PACS number(s): 04.60.Ds, 04.20.Cv, 04.62.+v

I. INTRODUCTION

One open issue of fundamental interest in (3+1)-dimensional quantum gravity is the investigation of meaningful observable quantities.

If we regard quantum gravity either as a (not yet completely established) fundamental theory, or as an effective quantum field theory which has general relativity as its classical limit and goes to some more fundamental theory at short distances, the observable quantities are important in guiding the research.

Precisely because a complete quantum theory of gravity is still lacking, it is not possible to define in a rigorous way what an observable is. The task is particularly difficult also due to the huge invariance group of gravity, namely, the group of the diffeomorphisms. The most advanced steps in this direction have been made by the Hamiltonian theory in the Ashtekar variables [1]. In this paper we shall take a simpler view and agree to consider a quantity as "physically observable" if the corresponding classical quantity is a scalar under arbitrary transformations of the coordinates.

More specifically, the quantities we intend to study are the Wilson loops, or "holonomies," of the Christoffel connection or of the spin connection $\Gamma^{a}_{\mu b}$. In the mentioned canonical approach to quantum gravity they form the basis of the so-called "loop representation" of the quantum theory [2]. Also it is known that the quantum averages of the loop operators have to satisfy the analogues of the Migdal-Polyakov loop equations [3]. Some general features of these equations have been studied by Makeenko and Voronov [4], considering the Christoffel connection and the usual Einstein action in the metric formalism. What we shall do is simpler but more explicit. Keeping the local fields as the fundamental dynamical variables, we shall compute the loops in a few different cases, in order to learn about their behavior and their geometrical meaning. The latter turns out to be quite different from that of the holonomies of Yang-Mills fields.

Our calculations are based on the Einstein-Hilbert action. We shall see, however, that certain properties of the loops do not depend on the detailed form of the action.

Since we work essentially in perturbation theory, some problems such as the lower unboundedness of the Euclidean Einstein action do not strictly affect our results. Nevertheless, the formalism we develop will also lead us to consider, in the final section, a different "source" for the dynamics of the Euclidean gravitational field, namely, the "stabilized action" of Greensite.

The plan of the paper is the following. In Sec. II, we define geometrically in detail the Wilson loop of the Christoffel connection and of the spin connection (in the vector representation) and show their equivalence. In Sec. III classical and quantum dynamics are introduced. We also recall the well known fact that Einstein's action is locally invariant under SO(3,1), but not under ISO(3,1); so the invariant Wilson loops are just those of the Lorentz connection, and not, like in 2+1 gravity, those of a generalized connection which contains the generators of the translations. In Sec. IV we give one illustrative example of a classical holonomy, computing it along a circle of constant radius in a Schwarzschild metric. In Sec. V we consider the case of a weak gravitational field, quantized around a flat background. We briefly review the corresponding perturbation theory and prove that the leading contribution to the Wilson loop, proportional to $\hbar\kappa^2$, vanishes for quite general dimensional and symmetry reasons. In Sec. VI an expression is given for the contri-

---
*On leave from University of Pisa, Pisa, Italy.

bution of order $\hbar$ to the holonomies computed on a nonflat background. In general this contribution is not vanishing in that case, due to the lower symmetry of the background; however, it is only of order $\hbar\kappa^3$. In Sec. VII we work out in detail the geometrical meaning of the matrix $\mathcal{U}$ of the parallel transport in the Euclidean theory and conclude that its trace, that is, the loop $\mathcal{W}$, is the sum of the squares of two angles, describing an SO(4) rotation. So the vanishing of $\langle \mathcal{W} \rangle_0$ to order $\hbar$ implies that in the equivalent statistical system there are no excitations with localized curvature at low temperature. This quite unexpected physical picture also explains the absence of any invariant correlation of the curvature in this approximation [11]. Finally, in Sec. VIII we consider a recently proposed "stabilized" version of Euclidean quantum gravity [14] and show that the basic property of the holonomies found in Sec. V is maintained in this case. Section IX contains our conclusions.

II. DEFINITIONS

In the so-called "second order" (or metric) formalism, classical spacetime is described by a Lorentzian manifold M, whose geometry is encoded in a metric tensor $g_{\mu\nu}(x)$ of signature $(-1,1,1,1)$ (our conventions are those of Weinberg [5]).

There is a natural definition of parallel transport of tensors on M. For instance, the variation of a vector V^α by an infinitesimal displacement dx^μ is defined by

$$dV^\alpha = -\Gamma^\alpha_{\mu\beta}(x) V^\beta dx^\mu \ , \tag{1}$$

where $\Gamma^\alpha_{\mu\beta}$ is the Christoffel connection:

$$\Gamma^\alpha_{\mu\beta} = \tfrac{1}{2} g^{\alpha\gamma}(\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\mu\gamma} - \partial_\gamma g_{\mu\beta}) \ . \tag{2}$$

Integrating (1) we find that the parallel transport of V along a finite differentiable curve connecting the points x and x' is performed by the matrix

$$\mathcal{U}^\alpha_\beta(x,x') = P \exp\left[\int_x^{x'} dy^\mu \Gamma^\alpha_{\mu\beta}(y) \right] , \tag{3}$$

where the symbol P means that the matrices $(\Gamma_\mu)^\alpha_\beta = \Gamma^\alpha_{\mu\beta}$ are ordered along the path. The indices of $\mathcal{U}^\alpha_\beta(x,x')$ are lowered and raised by $g_{\alpha\gamma}(x)$ and $g^{\beta\gamma}(x')$, respectively.

When the manifold is curved, the matrix $\mathcal{U}$ depends not only on the end points x and x', but also on the path. In fact, if C is a smooth closed curve on M, we define the loop functional (or "holonomy") $\mathcal{W}(C)$ as

$$\begin{aligned} \mathcal{W}(C) &= -4 + \mathrm{Tr}\,\mathcal{U}(C) \\ &= -4 + \mathrm{Tr} P \exp\left[\oint_C dx^\mu \Gamma_\mu(x) \right] . \end{aligned} \tag{4}$$

We make the physical requirement that the curve C should be possibly defined in an intrinsic way (that is, independently of the coordinates), and that its form and size should be eventually specified by invariant distances and angles. We also recall that we are not interested here in self-intersecting loops, nontrivial topologies, or global effects, but only in "local" effects. Our attitude should be different, of course, in analyzing the (2+1)-dimensional theory, where there are no local degrees of freedom.

The term -4 in Eq. (4) sets the holonomy to zero in the case of a flat space, when the matrix $\mathcal{U}$ reduces to an identity matrix.

Under a coordinates transformation $x \to \zeta$, the matrix $\mathcal{U}$ transforms as

$$\mathcal{U}^\alpha_\beta(x,x') \to \mathcal{U}^\alpha_\beta(x,x') \left[\frac{\partial \zeta^\gamma}{\partial x^\alpha} \right]_x \left[\frac{\partial x^\beta}{\partial \zeta^\varepsilon} \right]_{x'} . \tag{5}$$

For a closed curve, this transformation, being of the form

$$\mathcal{U} \to \Omega \mathcal{U} \Omega^{-1} \ , \tag{6}$$

does not affect the trace of $\mathcal{U}$. So the loop $\mathcal{W}(C)$ is invariant with respect to coordinate transformations.

Instead of the metric formalism, it is also possible to use a "first order" formalism, by introducing the vierbein field $e^a_\mu(x)$ and its inverse $E^\mu_a(x)$ [6], which satisfy the relations

$$e^\mu_a(x) E^a_\nu(x) = \delta^\mu_\nu, \quad e^\mu_a(x) E^b_\mu(x) = \delta^b_a \ , \tag{7}$$

$$E^a_\mu(x) E^b_\nu(x) \eta_{ab} = g_{\mu\nu}(x) \ . \tag{8}$$

Any vector V^μ (or, more generally, any tensor) can be referred to the vierbein, with "anholonomic" components V^a given, in any point x, by

$$V^a = V^\mu e^a_\mu(x) \ . \tag{9}$$

The equivalent of (1) in terms of the anholonomic connection $\Gamma^a_{\mu b}$ is

$$dV^a = -\Gamma^a_{\mu b}(x) V^b dx^\mu \tag{10}$$

and the matrix $\mathcal{U}$ of the finite parallel transport has an expression which is formally the analogue of (3): namely,

$$\mathcal{U}^a_b(x,x') = P \exp\left[\int_x^{x'} dy^\mu \Gamma^a_{\mu b}(y) \right] . \tag{11}$$

Using (1), (10), and (7) it is straightforward to verify that the relation between the connections $\Gamma^\alpha_{\mu\beta}$ and $\Gamma^a_{\mu b}$ is

$$\Gamma^\alpha_{\mu\beta} = E^\alpha_a e^b_\mu \Gamma^a_{\beta b} + E^\alpha_a \partial_\beta e^a_\mu \tag{12}$$

and that the relation between the matrices $\mathcal{U}^\alpha_\beta$ and $\mathcal{U}^a_b$ is

$$\mathcal{U}^a_b(x,x') = e^a_\alpha(x) \mathcal{U}^\alpha_\beta(x,x') E^\beta_b(x') \ . \tag{13}$$

It is known that gravity in the vierbein formalism has a local Lorentz invariance, since Eq. (8) is insensitive to a Lorentz rotation of $E^a(x), E^b(x)$. The connection $\Gamma^a_{\mu b}$ is then completely analogous to a usual gauge connection, and its Wilson loop

$$\mathcal{W}(C) = -4 + \mathrm{Tr}(\mathcal{U}^a_b)(C) \tag{14}$$

is a natural invariant quantity of the theory. But from Eq. (13) we see that this loop is equal to that defined in (4). So the Christoffel connection $\Gamma^\alpha_{\mu\beta}$ and the anholonomic $\Gamma^a_{\mu b}$ connection have the same loop, denoted by $\mathcal{W}(C)$. In the computations we shall employ the connection $\Gamma^\alpha_{\mu\beta}$, which is usually simpler to deal with.

When the exponential in (4) is expanded, one obtains terms with $1,2,3,\ldots$ fields Γ. We introduce the notation to be used in the following:

$$\mathcal{U}=\mathbb{1}+\oint_C dx^\mu\Gamma_\mu(x)+\tfrac{1}{2}P\oint dx^\mu\oint dy^\nu\Gamma_\mu(x)\Gamma_\nu(y)+\cdots \tag{15}$$

$$=\mathbb{1}+\mathcal{U}^{(1)}+\tfrac{1}{2}\mathcal{U}^{(2)}+\cdots \tag{16}$$

and

$$\mathcal{W}=-4+\mathrm{Tr}\,\mathcal{U}=\mathrm{Tr}\,\mathcal{U}^{(1)}+\tfrac{1}{2}\mathrm{Tr}\,U^{(2)}+\cdots . \tag{17}$$

III. DYNAMICS

We shall assume that the dynamics of the gravitational field is given by the Einstein-Hilbert action

$$S=-\frac{1}{16\pi G}\int d^4x\sqrt{g(x)}R(x) . \tag{18}$$

In the vierbein formalism S is expressed as

$$S=-\frac{1}{16\pi G}\int d^4x\, R_{\mu\nu}^{ab}(x)e_\rho^c(x)e_\sigma^d(x)\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd} , \tag{19}$$

where $R_{\mu\nu}^{ab}$ is the usual gauge curvature of $\Gamma_{b\mu}^a$.

As is well known, Einstein's gravity written in the form (19) is a gauge theory of the Lorentz group (i.e., S is invariant under local Lorentz transformations), but not of the whole Poincaré group ISO(3,1). A gauge formulation can be obtained only introducing some auxiliary fields q^a [7].

So it is not possible to consider in $3+1$ dimensions, like in $2+1$ gravity [8], the holonomies of the Lie algebra valued connection,

$$\mathcal{A}_\mu(x)=e_\mu^a(x)P_a+\Gamma_\mu d^{ab}(x)\omega_{ab} , \tag{20}$$

where P_a and ω_{ab} are the generators of the translations and of the Lorentz transformations.

From the dynamical point of view, the holonomies of $\mathcal{A}_\mu$ have certainly more content than the holonomies of Γ_μ alone. For instance, it can be easily verified that the term

$$\mathrm{Tr}\oint dx^\mu\oint dy^\nu\langle e_\mu^a(x)P_a e_\nu^b(y)P_b\rangle_0 = -2\delta_{ab}\oint dx^\mu\oint dy^\nu\langle e_\mu^a(x)e_\nu^b(y)\rangle_0 \tag{21}$$

is not trivial to leading order, unlike the corresponding term containing the connection (see Sec. V). However, this term does not respect the invariance of the action. In conclusion, the loop $\mathcal{W}(C)$ defined in Eqs. (4) and (14) is the only invariant loop functional which we can define in Einstein's gravity.

In the quantum case, we assume the quantum averages to be given by the functional integral

$$z=\int d[g]\exp\left[\frac{i}{\hbar}\{S[g]\}\right] , \tag{22}$$

or by the analogous formula in the first order formalism.

A roman letter corresponding to a calligraphic one will denote the vacuum average of the classical quantity. For instance, we write

$$U=\langle\mathcal{U}\rangle_0=\mathbb{1}+\langle\mathcal{U}^{(1)}\rangle_0+\tfrac{1}{2}\langle\mathcal{U}^{(2)}\rangle_0+\cdots$$
$$=\mathbb{1}+U^{(1)}+\tfrac{1}{2}U^{(2)}+\cdots , \tag{23}$$

$$W=\langle\mathcal{W}\rangle_0=-4+\mathrm{Tr}\,U$$
$$=\mathrm{Tr}\,U^{(1)}+\tfrac{1}{2}\mathrm{Tr}\,U^{(2)}+\cdots$$
$$=W^{(1)}+\tfrac{1}{2}W^{(2)}+\cdots . \tag{24}$$

IV. CLASSICAL CASE

We just give one typical example of a classical holonomy, namely that of the Schwarzschild solution. Let us consider the Schwarzschild metric [5]

$$d\tau^2=B(r)dt^2-A(r)dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) , \tag{25}$$

where

$$B(r)=\left[1-\frac{2MG}{r}\right],\quad A(r)=\left[1-\frac{2MG}{r}\right]^{-1} . \tag{26}$$

The corresponding Christoffel connection has the nonvanishing components

$$\Gamma_{\phi\phi}^r=-\frac{r\sin^2\theta}{A(r)},\quad \Gamma_{\phi r}^\phi=\frac{1}{r} , \tag{27}$$

$$\Gamma_{\phi\phi}^\theta=-\sin\theta\cos\theta,\quad \Gamma_{\phi\theta}^\phi=\cot\theta . \tag{28}$$

Let us take as the curve C a circle of radius r_0, azimuth θ_0, and constant time t_0; that is, C is described by the function

$$x^\mu(\phi)=(t_0,r_0,\theta_0,\phi) . \tag{29}$$

The linear term $\mathcal{W}^{(1)}$ in the holonomy is given by (we omit, for brevity, the arguments of the field)

$$\mathcal{W}^{(1)}=\delta_\rho^\sigma\oint dx^\mu\Gamma_{\mu\sigma}^\rho$$
$$=\delta_\rho^\sigma\left\{\oint dt\,\Gamma_{t\sigma}^\rho+\oint dr\,\Gamma_{r\sigma}^\rho+\oint d\theta\,\Gamma_{\theta\sigma}^\rho+\oint d\phi\,\Gamma_{\phi\sigma}^\rho\right\}$$
$$=\oint d\phi\{\Gamma_{\phi t}^t+\Gamma_{\phi r}^r+\Gamma_{\phi\theta}^\theta+\Gamma_{\phi\phi}^\phi\}=0 . \tag{30}$$

The quadratic term is

$$\mathcal{W}^{(2)}=P\delta_\alpha^\gamma\oint dx^\mu\oint dy^\nu\Gamma_{\mu\beta}^\alpha\Gamma_{\nu\gamma}^\beta$$
$$=P\oint d\phi'\oint d\phi''\Gamma_{\phi'\beta}^\alpha\Gamma_{\phi''\alpha}^\beta$$
$$=2\int_0^{2\pi}d\phi'\int_0^{\phi'}d\phi''\{\Gamma_{\phi'\phi}^r\Gamma_{\phi''r}^\phi+\Gamma_{\phi'\phi}^\theta\Gamma_{\phi''\theta}^\phi+\Gamma_{\phi'r}^\phi\Gamma_{\phi''\phi}^r+\Gamma_{\phi'\theta}^\phi\Gamma_{\phi''\phi}^\theta\}$$
$$=-8\pi^2\left[\frac{\sin^2\theta_0}{A(r_0)}+\cos^2\theta_0\right] . \tag{31}$$

It is easy to check that the contribution $\mathcal{W}^{(3)}$, of the form

$$\oint d\phi' \oint d\phi'' \oint d\phi''' \Gamma^{\alpha}_{\phi'\beta}\Gamma^{\beta}_{\phi''\gamma}\Gamma^{\gamma}_{\phi'''\alpha} , \tag{32}$$

vanishes, and so do the following contributions. Thus the total holonomy is exactly given by

$$\mathcal{W} = -(2\pi)^2 \left[\frac{\sin^2\theta_0}{A(r_0)} + \cos^2\theta_0 \right]$$

$$\simeq -(2\pi)^2 \left[1 + \frac{2MG\sin^2\theta_0}{r_0} \right] . \tag{33}$$

We see that if we set $\theta_0 = 0$ ("equatorial" circle) the loop is constant and equal to $-(2\pi)^2$ (for symmetry reasons); if we set $\theta_0 \neq 0$, we have a small "precession angle" (see Sec. VII) which depends on r_0 and vanishes when $r_0 \to \infty$.

V. SMALL QUANTUM FLUCTUATIONS AROUND A FLAT BACKGROUND

In this case, following the usual approach, we decompose the metric $g_{\mu\nu}(x)$ as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \quad \kappa = \sqrt{16\pi G} , \tag{34}$$

and we interpret $\eta_{\mu\nu}$ as the classical background while $\kappa h_{\mu\nu}(x)$ is regarded as a small quantized perturbation, which represents gravitons propagating in the vacuum. The Einstein-Hilbert Lagrangian (18) is then split into a quadratic part, whose inverse gives the bare graviton propagator, and into interaction vertices. Because of the nonpolynomial character of the Lagrangian, there are infinitely many vertices; the first two ones, respectively proportional to κ and κ^2, connect 3 and 4 fields h. Hence the first few orders of perturbation theory are formally very similar to those of Yang-Mills theory.

The leading contribution to W, of order $\hbar\kappa^2$, is given by $W^{(2)}$ with one bare propagator: namely,

$$W^{(2)} = \oint_C dx^\mu \oint_C dy^\nu \langle \Gamma^{\beta}_{\mu\alpha}(x)\Gamma^{\alpha}_{\nu\beta}(y) \rangle . \tag{35}$$

Here the angular brackets denote the bare propagator of the Γ's, obtained using their definition (2), Eq. (34), and the propagator of $h_{\mu\nu}(x)$ [see Sec. VIII, Eq. (81)].

The following two contributions to W, of order $\hbar^2\kappa^4$, are given by the term $W^{(4)}$ with two bare propagators and by the term $W^{(3)}$ with three propagators and one κ vertex. Finally, the three contributions of order $\hbar^3\kappa^6$ are given by the term $W^{(6)}$ with three propagators, by the term $W^{(4)}$ with four propagators and one κ^2 vertex, and by the term $W^{(2)}$ with the radiatively corrected propagator.

What is remarkable, and easily shown [9], is that the leading term (35), of order $\hbar$, vanishes in Einstein's theory. (This opened the problem of finding a gauge invariant expression for the static gravitational potential; see [10].) In the remainder of this section, we would like to show that this vanishing, in fact, does not depend on the dynamic content of Einstein's action, but is only due to the symmetries of the propagator, to the Poincaré invariance of the background, and to the absence of a dimensional coupling (apart from the overall factor κ^{-2}) in the action (18).

Let us write the most general form of the propagator $\langle h_{\mu\nu}(x) h_{\alpha\beta}(y) \rangle$ which is compatible with the symmetries in the indices and with Poincaré invariance (in any dimension N). We have

$$\langle h_{\mu\nu}(x) h_{\alpha\beta}(y) \rangle = a \frac{\Delta_{\mu\nu\alpha\beta}}{X^{(N-2)}} + b \frac{\eta_{\mu\nu}\eta_{\alpha\beta}}{X^{(N-2)}} + c \frac{s_{\mu\nu\alpha\beta}}{X^N} + d \frac{S_{\mu\nu\alpha\beta}}{X^N} + e \frac{X_\mu X_\nu X_\alpha X_\beta}{X^{(N+2)}} , \tag{36}$$

where

$$X_\mu = x_\mu - y_\mu, \quad X^N = [(\mathbf{x}-\mathbf{y})^2 - (x^0 - y^0)^2 - i\epsilon]^{N/2} , \tag{37}$$

$$\Delta_{\mu\nu\alpha\beta} = \tfrac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) , \tag{38}$$

$$s_{\mu\nu\alpha\beta} = (\eta_{\mu\nu} X_\alpha X_\beta + \eta_{\alpha\beta} X_\mu X_\nu) , \tag{39}$$

$$S_{\mu\nu\alpha\beta} = (\eta_{\mu\alpha} X_\nu X_\beta + \eta_{\mu\beta} X_\nu X_\alpha + \eta_{\nu\alpha} X_\mu X_\beta + \eta_{\nu\beta} X_\mu X_\alpha) . \tag{40}$$

The tensor $\Delta_{\mu\nu\alpha\beta}$ is the generalization of $\eta_{\mu\nu}$ to tensors with a symmetric couple of symmetric indices, and also the tensors $s_{\mu\nu\alpha\beta}$ and $S_{\mu\nu\alpha\beta}$ are defined in such a way that the decomposition (36) is left invariant by the exchange of the pair $(\alpha\beta)$ with $(\mu\nu)$ and of the indices inside each pair.

In (36), a,b,c,d,e are numerical constants. No other terms can be present, since there are no other dimensional parameters in the linearized action. The contribution of order $\hbar\kappa^2$ to the holonomy is

$$W^{(2)} = \oint dx^\mu \oint dy^\nu \langle \Gamma^{\alpha}_{\mu\beta}(x)\Gamma^{\beta}_{\nu\alpha}(y) \rangle$$
$$= \tfrac{1}{4} \oint dx^\mu \oint dy^\nu \langle \{ \partial_\beta h^{\alpha}_{\mu}(x) - \partial^\alpha h_{\mu\beta}(x) \} \times \{ \partial_\alpha h^{\beta}_{\nu}(y) - \partial^\beta h_{\nu\alpha}(y) \} \rangle \tag{41}$$
$$= \tfrac{1}{2} \oint dx^\mu \oint dy^\nu \{ \eta^{\alpha\beta} \Box \langle h_{\mu\alpha}(x) h_{\nu\beta}(y) \rangle - \partial^\alpha \partial^\beta \langle h_{\mu\alpha}(x) h_{\nu\beta}(y) \rangle \} . \tag{42}$$

It is straightforward to verify that the substitution of (36) into (42) gives rise only to terms which are either gradients, or ultralocal terms [that is, containing $\delta^4(x-y)$], or finally are proportional to the functions

$$\eta_{\mu\nu} \partial^\alpha \partial^\beta \frac{X_\alpha X_\beta}{X^N}, \quad \partial^\alpha \partial^\beta \frac{X_\mu X_\nu X_\alpha X_\beta}{X^{(N+2)}} , \tag{43}$$

$$\partial^\alpha \frac{X_\mu X_\nu X_\alpha}{X^{(N+2)}} , \tag{44}$$

which vanish by homogeneity. It is easy to check that the derivation above also holds in the Euclidean case (compare also Sec. VIII).

As was pointed out in [11], if we admit dimensional couplings in the action, like in $(R + R^2)$ gravity, some nonvanishing contribution to $W^{(2)}$ may arise.

Finally, we would like to justify our omission of higher order calculations by observing that the vanishing of the leading term has a geometrical interpretation which strongly affects the physical significance of the holonomies (see Sec. VII). Furthermore, higher order calculations in quantum gravity are very complicated, and give rise to nonrenormalizable infinities, which require the introduction of some effective cutoff. What would thus seem more appropriate to us, and is in progress now, is to apply higher order perturbation technique to the formula for the static potential [10].

In the next section, instead, we shall give a kind of semiclassical expression for the Wilson loops.

VI. NONFLAT BACKGROUND

The discussion of the preceding section suggests that a contribution to the holonomy proportional to $\hbar$ could arise on a nonflat background. In order to illustrate this point, let us suppose that a weak external source J for the gravitational field is present. The field produced by this source, as given by the Einstein equations, will be denoted, in the variable h defined in (34), by $h_{0,\mu\nu}(x)$. The functional integral (22) will now depend on J: omitting the indices of the field, it is given by

$$z[J]=\int d[h]\exp\left[\frac{i}{\hbar}\left\{S[h]+\int dx\, h(x)J(x)\right\}\right]. \tag{45}$$

If we expand the action $S[h]$ around the classical solution h_0, we find

$$S[h]=S[h_0]+\int dx\left[\frac{\delta S}{\delta h(x)}\right]_{h=h_0}\hat{h}(x)+\frac{1}{2}\int dx\int dy\left[\frac{\delta^2 S}{\delta h(x)\delta h(y)}\right]_{h=h_0}\hat{h}(x)\hat{h}(y)+S_3, \tag{46}$$

where

$$h=h_0+\hat{h}\ . \tag{47}$$

Since by definition we have

$$\left[\frac{\delta S}{\delta h(x)}\right]_{h=h_0}=-J(x)\ , \tag{48}$$

we are left with [12]

$$z[J]=\exp\left[\frac{i}{\hbar}\left\{S[h_0]+\int dx\, h_0(x)J(x)\right\}\right]\int d[\hat{h}]\exp\left[\frac{i}{2\hbar}\left\{\int dx\int dy\left[\frac{\delta^2 S}{\delta h(x)\delta h(y)}\right]_{h=h_0}\hat{h}(x)\hat{h}(y)+S_3\right\}\right]. \tag{49}$$

The operator

$$G(x,y)=\left\{\left[\frac{\delta^2 S}{\delta h(x)\delta h(y)}\right]_{h=h_0}\right\}^{-1} \tag{50}$$

is the graviton propagator in the background h_0. If we write symbolically the Einstein action as a quadratic part Q plus the interaction vertices $V^{(3)}$ and $V^{(4)}$ as

$$S[h]=\tfrac{1}{2}Qh^2+\tfrac{1}{6}\kappa V^{(3)}h^3+\tfrac{1}{12}\kappa^2V^{(4)}h^4+O(\kappa^3)\ , \tag{51}$$

we have

$$\left[\frac{\delta^2 S}{\delta h^2}\right]_{h=h_0}=Q+\kappa V^{(3)}h_0+\kappa^2V^{(4)}h_0^2+O(\kappa^3)\ , \tag{52}$$

whose inverse is

$$G=Q^{-1}-\kappa V^{(3)}h_0+O(\kappa^2)\ , \tag{53}$$

where Q^{-1} is the usual propagator of the graviton on a flat background. When evaluating the holonomy, we have to compute the expectation value

$$W=\frac{\int d[\hat{h}]\exp\left[\frac{i}{2\hbar}\left\{\int dx\int dy G^{-1}(x,y)\hat{h}(x)\hat{h}(y)+S_3\right\}\right]\mathcal{W}[h_0+\hat{h}]}{\int d[\hat{h}]\exp\left[\frac{i}{2\hbar}\left\{\int dx\int dy G^{-1}(x,y)\hat{h}(x)\hat{h}(y)+S_3\right\}\right]}\ . \tag{54}$$

It is known that S_3 is of higher order in $\hbar$; thus the contribution of order $\hbar$ to W is still given by Eq. (35), where the propagator is now given by (53). The term with Q^{-1} vanishes, as we saw in the preceding section; the other term in general does not vanish, and gives a contribution to the holonomy proportional to $\hbar\kappa^3$.

Thus we have seen that by breaking the Poincaré invariance with a small source term which produces a nonflat background we may obtain a contribution to the quantum holonomies proportional to $\hbar$, while there is no such contribution on a flat background. Nevertheless, this is a small effect, being proportional to κ^3.

VII. GEOMETRICAL AND PHYSICAL INTERPRETATION

It is interesting at this point to do a sharper analysis of the properties of the matrix $\mathcal{U}(C)$ of the parallel transport defined in Sec. II. We shall see that in the Euclidean theory the vanishing of its trace amounts to a very strong geometrical statement.

Let us first consider, for illustrative purposes, the case of a Yang-Mills theory of the group SO(3). The gauge connection has the form

$$A_\mu(x)=A^i_\mu(x)L_i, \quad i=1,2,3 \ , \tag{55}$$

where the matrices L_i constitute a representation of the Lie algebra of the group. In particular, to fix the ideas, let us choose the adjoint representation; in this case the matrices L_i have elements $(L_i)^m_n$ $(m,n=1,2,3)$, which are related to the structure constants ε_{imn} of the group. The connection $A_\mu(x)$ performs the parallel transport of a three-dimensional vector V^n in the "internal" space according to the formula [compare Eq. (10)]

$$dV^m=A^i_\mu(x)(L_i)^m_n V^n dx^\mu \ . \tag{56}$$

The vector is rotated during the transport, but its length remains unchanged. Let us consider the matrix $\mathcal{O}(C)$ which describes the parallel transport along a closed curve C. $\mathcal{O}(C)$ is defined by a P exponential, through a formula similar to Eq. (4). Suppose that we take a vector V in a point P of C, and parallel transport it along C, returning to P; let us denote by V' the new vector we obtain in this way. The vectors V and V' have the same length, that is,

$$\delta_{mn}V^m V^n=\delta_{mn}V'^m V'^n \ , \tag{57}$$

but they differ by an angle θ, which is related to the trace of $\mathcal{O}(C)$. For small angles, we have, by a proper choice of the coordinate axes in the internal space,

$$\mathcal{O}(C)=\begin{bmatrix} 1-\frac{1}{2}\theta^2 & \theta & 0 \\ -\theta & 1-\frac{1}{2}\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{58}$$

that means

$$\mathrm{Tr}\mathcal{O}(C)=3-\theta^2 \ . \tag{59}$$

More generally, we recall that the Lie algebra of SO(3) has just one Casimir invariant: namely, the operator

$$L^2=L_1^2+L_2^2+L_3^2 \ . \tag{60}$$

This operator commutes with each of the L_i's, so we can in general rotate our coordinate system as to have $L^2=L_3^2$, and the rotation matrix takes in this case the form (58); i.e., we have

$$\mathcal{O}(C)=\mathbb{1}+\theta L_3+\tfrac{1}{2}\theta^2 L_3^2+\cdots \ . \tag{61}$$

Taking the trace of (61), remembering that $\mathrm{Tr}L_i=0$, and using the normalization condition of the Lie generators

$$\mathrm{Tr}L_iL_j=-2\delta_{ij} \ , \tag{62}$$

we find that θ^2 is the coefficient of the Casimir invariant in the expansion of the exponential.

Next we come to consider the group SO(4). Intuitively, adding a new dimension we can make an independent rotation. Multiplying two four-dimensional matrices similar to (58), the first representing a rotation by an angle $\theta_{\rm I}$ perpendicular to one plane and the second a rotation by another angle $\theta_{\rm II}$ perpendicular to another plane, we find that

$$\mathrm{Tr}\mathcal{O}(C)=4-(\theta_{\rm I}^2+\theta_{\rm II}^2) \ . \tag{63}$$

Also we know that SO(4) = $[\mathrm{SO(3)}]_{\rm I}\times[\mathrm{SO(3)}]_{\rm II}$ and that we have two Casimirs now [13], corresponding to $(L_{\rm I}^2+L_{\rm II}^2)$, whose "eigenvalue" appears in (63), and $(L_{\rm I}^2-L_{\rm II}^2)$, which is not of interest in this case.

The group SO(4) is the relevant one for Euclidean quantum gravity. In fact, the geometrical interpretation of the matrix $\mathcal{U}(C)$ is the following. During the parallel transport of a vector V in spacetime, its length, given by

$$|V|^2=V^aV^b\delta_{ab}=V^\mu V^\nu g_{\mu\nu}(x) \ , \tag{64}$$

does not change. If we transport V along a closed curve C, returning to the starting point, we obtain another vector V', which has the same length of V, and differs from it only in the orientation. Hence we have, for any vector,

$$V^aV^b\delta_{ab}=V'^aV'^b\delta_{ab}=\mathcal{U}^a_c(C)V^c\mathcal{U}^b_d(C)V^d\delta_{ab} \ , \tag{65}$$

or, in matrix notation,

$$\mathcal{U}^T(C)\mathcal{U}(C)=\mathbb{1} \ . \tag{66}$$

The matrix $\mathcal{U}$ belongs then to SO(4) and its trace has the form (63).

If the variance of the angles $\theta_{\rm I}$ and $\theta_{\rm II}$ is zero to order $\hbar$ (because $W^{(2)}$ vanishes), the angles themselves have to vanish identically in any configuration, that is,

$$\mathcal{U}(C)=\mathbb{1} \quad \text{for any } C \ . \tag{67}$$

This is a very strong geometrical statement, as it implies that, still to order $\hbar$, all the weak field configurations which effectively enter the functional integral

$$z=\int d[h]\exp\{-\hbar^{-1}S[h]\} \tag{68}$$

have no curvature. In other words, the curved configurations, which possibly dominate in other regimes, are in this approximation totally suppressed.

This unexpected situation should be compared with what happens, for instance, in an ordinary SO(3) or SO(4) gauge theory. In this case the leading term $W^{(2)}(C)$ does not vanish and the variance of the rotation angles is not zero to order $\hbar$. For instance, if the curve C has the form of a rectangle of sides L and T, with $L \ll T$, the quantity $-(\hbar T)^{-1}\ln\langle\theta^2\rangle_0$ is the potential energy of two non-Abelian charges kept at rest at a distance L each from the other.

So the matrices of the parallel transport in the "internal" gauge manifold, considered configuration by configuration, are not equal to the identity matrix. Interpreting $\hbar$ as the temperature Θ of an equivalent statistical system, we see that when Θ grows from zero to some small value, such that we may disregard Θ^2 or higher orders, the Yang-Mills fields develop "localized excitations," i.e., regions of various sizes where the Yang-Mills curvature is not vanishing.

All this does not happen for the gravitational field, which remains essentially in a "flat" state. Such a picture also explains the absence in this approximation of any invariant correlation of the curvature [11].

We also have seen that the introduction of a small external source in the functional integral (68), breaking the Poincaré invariance of the background, gives rise in general to a nonvanishing contribution to the loop proportional to $\hbar$. In this case we may have excitations with localized curvature, but they are very small, since their variance is proportional to κ^3 instead of κ^2 (they are in fact originated by the nonlinear interaction of gravitons).

VIII. STABILIZED GRAVITY

In a series of papers [14], Greensite has recently proposed a new "stabilized" action for Euclidean quantum gravity. It is known [15] that the Euclidean action obtained from Einstein's action by *naive* analytical continuation is not bounded from below, due to the "wrong sign" of the conformal term in the kinetic operator. On the other hand, it is not obvious in quantum gravity that the simple analytical continuation is a correct procedure.

Both field-theoretical work [16] and a suitably modified stochastic quantization procedure [14] suggest that in the "right" Euclidean action the sign of the conformal factor is flipped to lowest order, while to higher orders the action itself becomes nonlocal.

In the remainder of this section, also in view of future applications, we shall find the propagator of stabilized gravity and verify that it gives the expected result for the holonomies to leading order.

According to the notation of Ref. [14], the linearized Euclidean gravitational action is written as

$$S^0=\int \frac{d^4p}{(2\pi)^4}\tilde{h}_{\mu\nu}(p)p^2\tilde{K}_{\mu\nu\alpha\beta}(p)\tilde{h}_{\alpha\beta}(p) \; . \tag{69}$$

In the usual Einstein theory the kinetic operator $\tilde{K}$ is given by

$$\tilde{K}^{\text{Ei}}_{\mu\nu\alpha\beta}=P^{(2)}_{\mu\nu\alpha\beta}-2P^{(0-s)}_{\mu\nu\alpha\beta} \; , \tag{70}$$

where $P^{(2)}$ and $P^{(0-s)}$ are the projection operators

$$P^{(2)}_{\mu\nu\alpha\beta}=\tfrac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta}+\theta_{\mu\beta}\theta_{\nu\alpha})-\tfrac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta} \; , \tag{71}$$

$$P^{(0-s)}_{\mu\nu\alpha\beta}=\tfrac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta} \; , \tag{72}$$

$$\theta_{\mu\nu}=\delta_{\mu\nu}-\frac{1}{p^2}p_\mu p_\nu \; . \tag{73}$$

The kinetic operator of the linearized effective action of stabilized gravity is simply obtained by changing the sign in (70):

$$\tilde{K}^{\text{st}}_{\mu\nu\alpha\beta}=P^{(2)}_{\mu\nu\alpha\beta}+2P^{(0-s)}_{\mu\nu\alpha\beta} \; . \tag{74}$$

Explicit evaluation of $\tilde{K}^{\text{Ei}}$ leads to the expression

$$\tilde{K}^{\text{Ei}}_{\mu\nu\alpha\beta}=\Delta_{\mu\nu\alpha\beta}-\delta_{\mu\nu}\delta_{\alpha\beta}+\frac{1}{p^2}\tilde{s}_{\mu\nu\alpha\beta}-\frac{1}{2p^2}\tilde{S}_{\mu\nu\alpha\beta} \; , \tag{75}$$

where the tensors $\Delta_{\mu\nu\alpha\beta}$, $\tilde{s}_{\mu\nu\alpha\beta}$, and $\tilde{S}_{\mu\nu\alpha\beta}$ are the analogues in p space of those defined in Eqs. (37)–(40).

For $\tilde{K}^{\text{st}}$ we have instead, expanding (74),

$$\tilde{K}^{\text{st}}_{\mu\nu\alpha\beta}=\Delta_{\mu\nu\alpha\beta}+\frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta}-\frac{1}{3p^2}\tilde{s}_{\mu\nu\alpha\beta}-\frac{1}{2p^2}\tilde{S}_{\mu\nu\alpha\beta}+\frac{4}{3p^4}p_\mu p_\nu p_\alpha p_\beta \; . \tag{76}$$

The kinetic operators above are not invertible. In order to find the corresponding propagators, we must add to them a gauge-fixing term, usually the harmonic gauge fixing

$$\tilde{K}^{\text{harmonic}}_{\mu\nu\alpha\beta}=\frac{1}{2}\delta_{\mu\nu}\delta_{\alpha\beta}-\frac{1}{p^2}\tilde{s}_{\mu\nu\alpha\beta}+\frac{1}{2p^2}\tilde{S}_{\mu\nu\alpha\beta} \; . \tag{77}$$

Then we consider the propagator equation

$$p^2[\tilde{K}_{\mu\nu\alpha\beta}(p)+\tilde{K}^{\text{harmonic}}_{\mu\nu\alpha\beta}(p)]\tilde{G}_{\alpha\beta\rho\sigma}(p)=-\Delta_{\mu\nu\rho\sigma} \tag{78}$$

and look for a solution of the general form

$$\tilde{G}_{\alpha\beta\rho\sigma}=\frac{a}{p^2}\Delta_{\alpha\beta\rho\sigma}+\frac{b}{p^2}\delta_{\alpha\beta}\delta_{\rho\sigma}+\frac{c}{p^4}\tilde{s}_{\alpha\beta\rho\sigma}+\frac{d}{p^4}\tilde{S}_{\alpha\beta\rho\sigma}+\frac{e}{p^6}p_\alpha p_\beta p_\rho p_\sigma \; , \tag{79}$$

where a,b,c,d,e are numerical constants.

In the case of Einstein's theory we find

$$(a^{\text{Ei}}=-1, b^{\text{Ei}}=\tfrac{1}{2}, c^{\text{Ei}}=0, d^{\text{Ei}}=0, e^{\text{Ei}}=0) \; , \tag{80}$$

which corresponds in the x space to the familiar Feynman-DeWitt propagator [17]

$$\langle h_{\mu\nu}(x)h_{\rho\sigma}(y)\rangle^{\text{Ei}}=-\frac{\hbar}{8\pi^2}\frac{\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\rho\sigma}}{(x-y)^2} \; . \tag{81}$$

In the case of stabilized gravity the solution is

$$(a^{\text{st}}=-1, b^{\text{st}}=\tfrac{1}{6}, c^{\text{st}}=-\tfrac{2}{3}, d^{\text{st}}=0, e^{\text{st}}=-\tfrac{4}{3}) \; . \tag{82}$$

In order to write the corresponding x space propagator, we must compute the Fourier transforms of the nonstandard terms of the form $p^{-4}p_\alpha p_\beta$ and $p^{-6}p_\alpha p_\beta p_\rho p_\sigma$. This computation is done in detail in the Appendix. The result is rather simple:

$$\int \frac{d^4p}{(2\pi)^4}\frac{p_\alpha p_\beta}{p^4}e^{ipx}=\frac{1}{(2\pi)^2}\frac{x_\alpha x_\beta}{x^4} \; , \tag{83}$$

$$\int \frac{d^4p}{(2\pi)^4}\frac{p_\alpha p_\beta p_\rho p_\sigma}{p^6}e^{ipx}=\frac{1}{(2\pi)^2}\frac{x_\alpha x_\beta x_\rho x_\sigma}{x^6} \; . \tag{84}$$

So the propagator is (with $X=x-y$)

$$\langle h_{\mu\nu}(x)h_{\rho\sigma}(y)\rangle^{\text{st}}=-\frac{1}{(2\pi)^2X^2}\Delta_{\mu\nu\rho\sigma}+\frac{1}{6(2\pi)^2X^2}\delta_{\mu\nu}\delta_{\rho\sigma}-\frac{2}{3(2\pi)^2X^4}s_{\mu\nu\rho\sigma}-\frac{4}{3(2\pi)^2X^6}X_\mu X_\nu X_\rho X_\sigma \; . \tag{85}$$

Being of the form (36) it gives no contribution to the leading term of the holonomies (see Sec. V).

IX. CONCLUDING REMARKS

In this work the behavior of quantum and semiclassical Wilson loops has been studied perturbatively in four-dimensional pure Einstein gravity. The main results comprise the vanishing of the leading perturbative contribution to the loops and a geometrical interpretation of this vanishing in terms of the structure of the vacuum state. We also have treated the case of a nonflat background and that of stabilized gravity.

The most interesting "discovery" of our analysis, from the physical point of view, is that the vacuum state of quantum gravity does not show, to order $\hbar$, any field configuration with localized curvature. This behavior is very different from that of other gauge fields.

But if the Wilson loops vanish and if there is no localized curvature, how can we express in an invariant way the interaction energy of two masses, and how can we think of the "mechanism" of their gravitational interaction?

The first question has a definite formal answer, in terms of an essentially nonlocal formula [10]. The second question is more subtle, also in view of the difficulties encountered already at the classical level for the definition of a localized gravitational energy (see [10]). All we can say at the present stage is that the mechanism seems to be not strictly local. It could be possible to find some analogue of this behavior in other field models; work is in progress in this direction.

One limit of our analysis resides in its perturbative nature. Of course, nonperturbative analyses of quantum gravity are a major challenge. Nevertheless, all the considerations above do not involve particularly short distances, where gravity is thought to develop highly nonperturbative features. As we pointed out in the Introduction, perturbative quantum gravity may be regarded in our case just as an effective field theory.

ACKNOWLEDGMENTS

I would like to thank Professor Roman Jackiw for his kind hospitality at MIT and for very invaluable discussions. I have also benefited from helpful discussions with D. Cangemi, D. Z. Freedman, and C. Lucchesi and from an earlier conversation with M. Toller. The author was supported by the Foundation "A. Della Riccia" of Florence, Italy.

APPENDIX

We want to prove Eqs. (83) and (84), namely,

$$\int \frac{d^4p}{(2\pi)^2} \frac{p_\alpha p_\beta}{p^4} e^{ipx} = \frac{x_\alpha x_\beta}{x^4} , \tag{A1}$$

$$\int \frac{d^4p}{(2\pi)^2} \frac{p_\alpha p_\beta p_\rho p_\sigma}{p^6} e^{ipx} = \frac{x_\alpha x_\beta x_\rho x_\sigma}{x^6} , \tag{A2}$$

starting from the known result

$$\int \frac{d^4p}{(2\pi)^2} \frac{e^{ipx}}{p^2} = \frac{1}{x^2} . \tag{A3}$$

Let us first consider (A1). By Euclidean invariance we will have

$$\int \frac{d^4p}{(2\pi)^2} \frac{p_\alpha p_\beta}{p^4} e^{ipx} = A\frac{\delta_{\alpha\beta}}{x^2} + B\frac{x_\alpha x_\beta}{x^4} , \tag{A4}$$

where A and B are two unknown coefficients, which we may determine imposing the conditions

$$\delta^{\alpha\beta}\left[A\frac{\delta_{\alpha\beta}}{x^2} + B\frac{x_\alpha x_\beta}{x^4} \right] = \frac{1}{x^2} , \tag{A5}$$

$$\int \frac{d^4x}{(2\pi)^2} \left[A\frac{\delta_{\alpha\beta}}{x^2} + B\frac{x_\alpha x_\beta}{x^4} \right] e^{-ipx} = \frac{p_\alpha p_\beta}{p^4} . \tag{A6}$$

Equation (A6) expresses the invertibility of the Fourier transform. We obtain two possible solutions:

$$(A = \tfrac{1}{2}, B = -1) \text{ and } (A = 0, B = 1) . \tag{A7}$$

Then we write

$$\int \frac{d^4p}{(2\pi)^2} \frac{p_\alpha p_\beta p_\rho p_\sigma}{p^6} e^{ipx} = \frac{a}{x^2}(\delta_{\alpha\beta}\delta_{\rho\sigma} + \cdots) + \frac{b}{x^4}(\delta_{\alpha\beta}x_\rho x_\sigma + \cdots) + \frac{c}{x^6} x_\alpha x_\beta x_\rho x_\sigma , \tag{A8}$$

where the ellipsis denotes all the possible symmetrizations, and impose the conditions analogous to (A5) and (A6). In this way we find that a solution for (a,b,c) exists only if $(A = 0, B = 1)$ and in this case we have

$$(a = 0, b = 0, c = 1) . \tag{A9}$$

This work was supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under Contract No. DE-AC02-76ER03069.

[1] *Conceptual Problems of Quantum Gravity*, edited by A. Ashtekar and J. Stachel (Birkhauser, Boston, 1991); C. Rovelli, Class. Quantum Grav. **8**, 297 (1991); **8**, 317 (1991); C. Misner, in *Magic Without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972); K. V. Kuchar and C. G. Torre, Phys. Rev. D **43**, 419 (1991).

[2] A. Ashtekar, *Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991); C. Rovelli, Class. Quantum Grav. **8**, 1613 (1991); L. Smolin, *Quantum Gravity and*

Cosmology (World Scientific, Singapore, 1993).
[3] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic, Chur, Switzerland, 1987).
[4] Yu. M. Makeenko and N. A. Voronov, Yad. Fiz. **36**, 758 (1982) [Sov. J. Nucl. Phys. **36**, **444** (1982)].
[5] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
[6] F. W. Hehl *et al.*, Rev. Mod. Phys. **48**, 393 (1976).
[7] G. Grignani and G. Nardelli, Phys. Rev. D **45**, 2719 (1992); Nucl. Phys. **B370**, 491 (1992).
[8] E. Witten, Nucl. Phys. **B311**, 46 (1988/89).
[9] G. Modanese, Phys. Rev. D **47**, 502 (1993).
[10] G. Modanese, "A formula for the static potential energy in quantum gravity," Report CTP No. 2217, hep-th/9306078 (unpublished).
[11] G. Modanese, Phys. Lett. B **288**, 69 (1992).
[12] R. Jackiw, Phys. Rev. D **9**, 1686 (1974).
[13] B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).
[14] J. Greensite, Nucl. Phys. **B361**, 729 (1991); Phys. Lett. B **291**, 405 (1992); Nucl. Phys. **B390**, 439 (1993).
[15] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[16] P. Mazur and E. Mottola, Nucl. Phys. **B341**, 187 (1990), and references therein.
[17] M. J. G. Veltman, in *Methods in Field Theory,* Proceedings of the Les Houches Summer School, Les Houches, France, 1975, edited by R. Balian and J. Zinn-Justin, Les Houches Summer School Proceedings Vol. 28 (North-Holland, Amsterdam, 1977).