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Lorentz group and spherical impulsive gravity waves

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The method for constructing a spherical impulsive gravitational wave solution of Einstein's vacuum field equations has been described by Penrose. Recently the author gave an explicit transformation leading to coordinates in which the metric tensor is continuous across the history of the wave front in Minkowskian space-time. In this paper we exhibit the relationship between this transformation and proper, orthochronous Lorentz transformations. This provides a novel viewpoint on spherical impulsive gravity waves which facilitates a description of them in terms of new continuous coordinate systems.

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I. INTRODUCTION

The technique for constructing impulsive gravitational wave solutions of Einstein's field equations has been described in a classic paper by Penrose [1]. When applied to an impulsive wave with a spherical front propagating in a vacuum, the procedure given by Penrose is as follows: The history of the wave front is, in this case, a future null cone in Minkowskian space-time M. Take the flat space-time line element in the form

$$ds^2=2u^2d\zeta\, d\bar{\zeta}+2du\, dv \tag{1.1}$$

(here ζ is a complex coordinate with complex conjugate $\bar{\zeta}$ and u,v are real coordinates). The hypersurfaces $v=\text{const}$ are future null cones with vertices on the null geodesic $u=0$, and v is an affine parameter along this null geodesic. Subdivide M into two halves $M^+(v>0)$ and $M^-(v<0)$ and reattach the halves on $v=0$ with the identification or "warp"

$$(\zeta,\bar{\zeta},u,v=0)_{M^-}=\left[h(\zeta),\bar{h}(\bar{\zeta}),\frac{u}{|h'(\zeta)|},v=0\right]_{M^+}, \tag{1.2}$$

where h is an arbitrary analytic function of ζ and $h'=dh/d\zeta$. The identification (1.2) ensures that the metric induced on $v=0$ from M^- coincides with the metric induced on $v=0$ from M^+. It now follows from Penrose's theory that $v=0$ is the history of a spherical impulsive gravitational wave, the space-time has vanishing Ricci tensor, and the curvature tensor is Petrov type N with Dirac δ-function dependence on v, singular at $v=0$. Recently a coordinate transformation has been given [2] that puts the line element of the space-time described above in the form

$$ds^2=2U^2\left|dZ+\frac{V\vartheta(V)}{2U}\bar{H}\,d\bar{Z}\right|^2+2dU\,dV\ , \tag{1.3}$$

which is continuous across the history of the impulsive wave $V=0$ (corresponding to $v=0$ above). Here Z is a complex coordinate with complex conjugate $\bar{Z}$ and U,V are real coordinates. Also H is an arbitrary analytic function of Z only, and $\vartheta(V)$ is the Heaviside step function which is equal to unity if $V>0$ and equal to zero if $V<0$. For $V<0$ the transformation from (1.1) to (1.3) is the identity transformation

$$\zeta=Z\ ,\quad \bar{\zeta}=\bar{Z}\ ,\quad u=U\ ,\quad v=V\ . \tag{1.4}$$

For $V>0$ the transformation from (1.1) to (1.3) is [2]

$$\zeta=h(Z)+\frac{V}{2U}\frac{h'\bar{h}''}{\bar{h}'}\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]^{-1}, \tag{1.5a}$$

$$\bar{\zeta}=\bar{h}(\bar{Z})+\frac{V}{2U}\frac{\bar{h}'h''}{h'}\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]^{-1}, \tag{1.5b}$$

$$u=\frac{U}{|h'|}\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]. \tag{1.5c}$$

$$v=V|h'|\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]^{-1}. \tag{1.5d}$$

Here h is an arbitrary analytic function of $Z, h'=dh/dZ$, $h''=d^2h/dZ^2$, and H in (1.3) is derived from h as follows:

$$H=\frac{h'''}{h'}-\frac{3}{2}\left[\frac{h''}{h'}\right]^2. \tag{1.6}$$

The Ricci tensor calculated with the metric tensor given via (1.3) vanishes for all values of V, and the only nonidentically vanishing Newman-Penrose component of the curvature tensor is

$$\Psi_4=\frac{1}{2U}H(Z)\delta(V)\ , \tag{1.7}$$

indicating a Petrov type N curvature with degenerate principal null direction given by the vector field $\partial/\partial U$ evaluated on $V=0$. Further aspects of the coordinate system $(Z,\bar{Z},U,V)$ can be found in [2], and an extension of the result to a spherical impulsive wave propagating through the de Sitter universe is given in [3]. Equations (1.3) and (1.6) [without (1.4) and (1.5)] have been independently presented in [4]. Spherical shock waves [for which the curvature tensor undergoes a finite discontinuity across the history of the shock in contradistinction to the delta function behavior visible in (1.7)] have also been constructed using the Penrose procedure in [5].

It is easy to see that if $h(Z)$ is a fractional linear function of Z then, by (1.6), $H=0$, and there is no spherical wave. This corresponds, in the terminology of the opening paragraph above, to a Lorentz warp, and the resulting obliteration of the wave is thus not surprising. In this case the transformation (1.5) is a proper, orthochronous Lorentz transformation. We will demonstrate this explicitly in Sec. II. This demonstration will suggest a way of constructing the spherical impulsive gravitational wave solution of the vacuum field equations in new continuous coordinate systems. We will give an example of this approach in Sec. III starting with the Minkowskian line element in the form

$$ds^2=2u^2(1+\tfrac{1}{2}\zeta\bar{\zeta})^{-2}d\zeta\, d\bar{\zeta}+2du\, dv-dv^2\ , \tag{1.8}$$

instead of the form (1.1). Here the hypersurfaces $v=$const are future null cones with vertices on the *timelike* geodesic $u=0$.

II. THE LORENTZ GROUP AND THE TRANSFORMATION TO CONTINUOUS COORDINATES

The line element (1.1) can be written

$$ds^2=dx^2+dy^2+dz^2-dt^2\ , \tag{2.1}$$

using the coordinate transformation

$$x+iy=\sqrt{2}u\zeta\ , \tag{2.2a}$$

$$x-iy=\sqrt{2}u\bar{\zeta}\ , \tag{2.2b}$$

$$z=-v+u\zeta\bar{\zeta}-\tfrac{1}{2}u\ , \tag{2.2c}$$

$$t=-v+u\zeta\bar{\zeta}+\tfrac{1}{2}u\ . \tag{2.2d}$$

To discuss proper, orthochronous Lorentz transformations we proceed in a standard way [6] by introducing the 2×2 Hermitian matrix

$$A=\begin{bmatrix}\frac{1}{\sqrt{2}}(z-t) & \frac{1}{\sqrt{2}}(x-iy)\\ \frac{1}{\sqrt{2}}(x+iy) & \frac{1}{\sqrt{2}}(-z-t)\end{bmatrix}\ . \tag{2.3}$$

Let $\mathcal{U}\in\mathrm{SL}(2,C)$, then every proper, orthochronous Lorentz transformation can be written as

$$A\to\mathcal{U}A\mathcal{U}^{\dagger}\ , \tag{2.4}$$

where $\mathcal{U}^{\dagger}$ is the Hermitian conjugate of $\mathcal{U}$. Writing

$$\mathcal{U}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\ , \tag{2.5}$$

where $\alpha,\beta,\gamma,\delta$ are complex numbers satisfying

$$\alpha\delta-\beta\gamma=1\ , \tag{2.6}$$

we utilize (2.2) to express (2.4) as a coordinate transformation from $(\zeta,\bar{\zeta},u,v)$ to $(Z,\bar{Z},U,V)$ or vice versa. It is convenient to write (2.4) is

$$u=(\alpha\bar{\alpha}-\sqrt{2}\alpha\bar{\beta}\bar{Z}-\sqrt{2}\bar{\alpha}\beta Z+2\beta\bar{\beta}Z\bar{Z})U-2\beta\bar{\beta}V\ , \tag{2.7a}$$

$$\zeta=\frac{(-\bar{\alpha}\gamma+\sqrt{2}\bar{\alpha}\delta Z+\sqrt{2}\bar{\beta}\gamma\bar{Z}-2\bar{\beta}\delta Z\bar{Z})U+2\bar{\beta}\delta V}{\sqrt{2}(\alpha\bar{\alpha}-\sqrt{2}\bar{\beta}\bar{Z}-\sqrt{2}\bar{\alpha}\beta Z+2\beta\bar{\beta}Z\bar{Z})U-2\sqrt{2}\beta\bar{\beta}V}\ , \tag{2.7b}$$

and v is given by

$$v-u\zeta\bar{\zeta}=\Big[-\tfrac{1}{2}\gamma\bar{\gamma}+\frac{1}{\sqrt{2}}\gamma\bar{\delta}\bar{Z}+\frac{1}{\sqrt{2}}\delta\bar{\gamma}Z-\delta\bar{\delta}Z\bar{Z}\Big]U+\delta\bar{\delta}V\ . \tag{2.7c}$$

Now let

$$h(Z)=\frac{1}{\sqrt{2}}\frac{\gamma-\sqrt{2}\delta Z}{-\alpha+\sqrt{2}\beta Z}\ . \tag{2.8}$$

Noting that

$$|h'|^{-1}=|-\alpha+\sqrt{2}\beta Z|^2\ , \tag{2.9}$$

and

$$\frac{h''}{h'}=\frac{-2\sqrt{2}\beta}{-\alpha+\sqrt{2}\beta Z}\ , \tag{2.10}$$

we easily see that (2.7a) may be written

$$u=\frac{U}{|h'|}\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]\ . \tag{2.11}$$

Rearranging terms in (2.7b) we find that we can write it as

$$\zeta=h(Z)+\frac{V(2\beta\bar{\beta}h(Z)+\sqrt{2}\bar{\beta}\delta)}{|-\alpha+\sqrt{2}\beta Z|^2U-2\beta\bar{\beta}V}\ . \tag{2.12}$$

By (2.9) and (2.10) this may be written

$$\zeta=h(Z)+\frac{V}{U}\left\{\frac{2\beta\bar{\beta}h(Z)+\sqrt{2}\bar{\beta}\delta}{|-\alpha+\sqrt{2}\beta Z|^2}\right\}\times\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]^{-1}\ . \tag{2.13}$$

We also have

$$\frac{2\beta\bar{\beta}h(Z)+\sqrt{2}\bar{\beta}\delta}{|-\alpha+\sqrt{2}\beta Z|^2}=\frac{1}{2}\frac{h'\bar{h}''}{\bar{h}'}\ , \tag{2.14}$$

and so (2.13) takes the form

$$\zeta=h(Z)+\frac{V}{2U}\frac{h'\bar{h}''}{\bar{h}'}\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]^{-1}\ . \tag{2.15}$$

Finally we may now solve (2.7c) for v to obtain

$$v=\frac{UV}{|-\alpha+\sqrt{2}\beta Z|^2U-2|\beta|^2V}\ . \tag{2.16}$$

Using (2.9) and (2.10) again, this can be put in the form

$$v=V|h'|\left[1-\frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right]^{-1}\ . \tag{2.17}$$

We see that (2.11), (2.15), and (2.17) have the same form as (1.5), and we conclude that (1.5) with $h(Z)$ given by (2.8) is the Lorentz transformation corresponding to the SL(2,C) element (2.5) with (2.6). We can thus obtain the transformation (1.5) by writing the Lorentz transformation in the form (2.11), (2.15), and (2.17) and *generalizing these formulas by allowing $h(Z)$ in them to be an arbitrary analytic function of Z.*

With the results to be described in the next section in mind it is helpful to make the following observations: In flat space-time with line element (1.3) and $\vartheta(V)$ replaced by unity the hypersurfaces

$$u(Z,\bar{Z},U,V)=\text{const} , \tag{2.18}$$

with u given by (1.5c), are null and can intersect the nullcone $V=0$. If $X^i=(Z,\bar{Z},U,V)$ then

$$u_{,i}=\frac{1}{|h'|}\left[-\frac{1}{2}\frac{h''}{h'}U-\frac{1}{4}V\frac{\bar{h}''}{\bar{h}'}H,-\tfrac{1}{2}\frac{\bar{h}''}{\bar{h}'}U-\frac{1}{4}V\frac{h''}{h'}\bar{H},1,-\frac{1}{4}\left|\frac{h''}{h'}\right|^2\right] , \tag{2.19}$$

and

$$g^{ij}u_{,i}u_{,j}=0 . \tag{2.20}$$

The *expansion* of the null geodesic generators of (2.18) is

$$\vartheta=-\frac{V}{4U^2}\frac{|H|^2}{|h'|} , \tag{2.21}$$

and the *complex shear* of these generators is

$$\sigma=-\frac{\bar{H}}{2U|h'|}+\frac{V}{4U^2|h'|}\left[\frac{1}{2}\left|\frac{h''}{h'}\right|^2\bar{H}-\frac{h''}{h'}\bar{H}'\right] . \tag{2.22}$$

We thus see that if, as indicated by (1.3), we effectively have $H=0$ for $V<0$ and $H\neq 0$ for $V>0$ then ϑ is continuous across $V=0$ while there is a jump in σ across $V=0$. It is a consequence of Penrose's theory that for the field equations to be satisfied ϑ should be continuous across the history of the impulsive wave and for a δ function in the curvature with a nonzero coefficient in front of it there must be a finite jump in σ across the history of the wave.

III. THE LORENTZ GROUP AND NEW CONTINUOUS COORDINATES

We now consider the construction of the spherical impulsive wave starting with the Minkowskian line element (1.8). The Penrose construction involves dividing the space-time into two halves $M^+(v>0)$ and $M^-(v<0)$ and reattaching them on the null cone $v=0$ with the identification

$$(\zeta,\bar{\zeta},u,v=0)_{M^-}=\left[h(\zeta),\bar{h}(\bar{\zeta}),\frac{u}{|h'|}\left[\frac{1+\frac{1}{2}|h|^2}{1+\frac{1}{2}|\zeta|^2}\right],v=0\right]_{M^+} , \tag{3.1}$$

with $h(\zeta)$ an arbitrary analytic function of ζ as before. This results in $v=0$ being the history of a spherical impulse gravity wave. The problem we consider here is to find new coordinates $(Z,\bar{Z},U,V)$ (there will be no danger of confusion with coordinates described in Sec. II using the same symbols) in terms of which the metric tensor components are continuous. In the light of the observations we have made in Sec. II we begin by looking at proper, orthochronous Lorentz transformations starting with the line element (1.8).

We can write (1.8) in the form (2.1) using the coordinate transformation

$$x+iy=\frac{\sqrt{2}u\zeta}{1+\frac{1}{2}\zeta\bar{\zeta}} , \tag{3.2a}$$

$$x-iy=\frac{\sqrt{2}u\bar{\zeta}}{1+\frac{1}{2}\zeta\bar{\zeta}} , \tag{3.2b}$$

$$z=\frac{u(1-\frac{1}{2}\zeta\bar{\zeta})}{1+\frac{1}{2}\zeta\bar{\zeta}} , \tag{3.2c}$$

$$t=v-u . \tag{3.2d}$$

We write the transformation (2.4) now as

$$\frac{2u}{1+\frac{1}{2}\zeta\bar{\zeta}}-v=\frac{2U}{1+\frac{1}{2}Z\bar{Z}}\left|\alpha+\frac{1}{\sqrt{2}}\beta Z\right|^2-V(|\alpha|^2+|\beta|^2) , \tag{3.3}$$

$$\frac{u\zeta}{1+\frac{1}{2}\zeta\bar{\zeta}}=\frac{U\sqrt{2}(\gamma+1/\sqrt{2}\delta Z)(\bar{\alpha}+1/\sqrt{2}\bar{\beta}\bar{Z})}{1+\frac{1}{2}Z\bar{Z}}-\frac{V}{\sqrt{2}}(\gamma\bar{\alpha}+\delta\bar{\beta}) , \tag{3.4}$$

$$\frac{u\zeta\bar{\zeta}}{1+\frac{1}{2}\zeta\bar{\zeta}}-v=\frac{2U}{1+\frac{1}{2}Z\bar{Z}}\left|\gamma+\frac{1}{\sqrt{2}}\delta Z\right|^2-V(|\gamma|^2+|\delta|^2) . \tag{3.5}$$

We can solve these equations for $(\zeta,\bar{\zeta},u,v)$ in terms of $(Z,\bar{Z},U,V)$, but this will not be necessary. Here $\alpha,\beta,\gamma,\delta$ satisfy (2.6) as before. If in place of (2.8), however, we take

$$h(Z)=\frac{\sqrt{2}(\gamma+1/\sqrt{2}\delta Z)}{\alpha+1/\sqrt{2}\beta Z} , \tag{3.6}$$

we can calculate the useful formulas

$$\frac{\beta}{\gamma+1/\sqrt{2}\delta Z}=-\frac{h''}{hh'} , \tag{3.7a}$$

$$\frac{\alpha}{\alpha+1/\sqrt{2}\beta Z}=1+\frac{1}{2}Z\frac{h''}{h'} , \tag{3.7b}$$

$$\frac{\delta}{\alpha+1/\sqrt{2}\beta Z}=h'-\frac{1}{2}\frac{hh''}{h'} \tag{3.7c}$$

$$\frac{\gamma}{\gamma+1/\sqrt{2}\delta Z}=1+\frac{1}{2}Z\frac{h''}{h'}-Z\frac{h'}{h} , \tag{3.7d}$$

$$|h'|=\left|\alpha+\frac{1}{\sqrt{2}}\beta Z\right|^{-2} , \tag{3.7e}$$

$$\frac{|h'|}{|h|^2}=\frac{1}{2}\left|\gamma+\frac{1}{\sqrt{2}}\delta Z\right|^{-2} . \tag{3.7f}$$

Substituting these in (3.3)–(3.5) we obtain

$$\frac{2u}{1+\frac{1}{2}\zeta\bar{\zeta}}-v=\frac{2U}{|h'|(1+\frac{1}{2}Z\bar{Z})}-VS_1\ , \tag{3.8a}$$

$$\frac{u\zeta}{1+\frac{1}{2}\zeta\bar{\zeta}}=\frac{h}{|h'|}\frac{U}{(1+\frac{1}{2}Z\bar{Z})}-VS\ , \tag{3.8b}$$

$$\frac{u\zeta\bar{\zeta}}{1+\frac{1}{2}\zeta\bar{\zeta}}-v=\frac{|h|^2}{|h'|}\frac{U}{(1+\frac{1}{2}Z\bar{Z})}-VS_2\ , \tag{3.8c}$$

with

$$S_1=\frac{1}{|h'|}\left\{\left|1+\frac{1}{2}Z\frac{h''}{h'}\right|^2+\frac{1}{2}\left|\frac{h''}{h'}\right|^2\right\}, \tag{3.9a}$$

$$S_2=\frac{|h|^2}{2|h'|}\left\{\frac{1}{2}\left|\frac{h''}{h'}-2\frac{h'}{h}\right|^2+\left|1+\frac{1}{2}Z\frac{h''}{h'}-Z\frac{h'}{h}\right|^2\right\}, \tag{3.9b}$$

$$S=\frac{h}{2|h'|}\left\{\left[1+\frac{1}{2}Z\frac{h''}{h'}-Z\frac{h'}{h}\right]\left[1+\frac{1}{2}\bar{Z}\frac{\bar{h}''}{\bar{h}'}\right]-\frac{1}{2}\left[2\frac{h'}{h}-\frac{h''}{h'}\right]\frac{\bar{h}''}{\bar{h}'}\right\} \tag{3.9c}$$

We now generalize the transformation given by (3.8) and (3.9) by taking $h(Z)$ in it to be an arbitrary analytic function.

We note from (3.8) that if $V=0$ then (see [7]) $v=0$ or $v=2u$. In the former case

$$\zeta=h(Z)\ ,\quad u=\left[\frac{1+\frac{1}{2}|h|^2}{1+\frac{1}{2}|Z|^2}\right]\frac{U}{|h'|}\ . \tag{3.10}$$

The hypersurfaces

$$u(Z,\bar{Z},U,V)-\tfrac{1}{2}v(Z,\bar{Z},U,V)=\text{const}\ , \tag{3.11}$$

are null and can intersect the null cone $V=0$. The *expansion* of the null geodesic generators of (3.11) calculated on $V=0$ is

$$\vartheta_{V=0}=-\left[\frac{1+\frac{1}{2}|Z|^2}{1+\frac{1}{2}|h|^2}\right]\frac{|h'|}{2U}=-\frac{1}{2u}\ , \tag{3.12}$$

with the last equality here following from (3.10). The *complex shear* of the generators of (3.11) calculated on $V=0$ turns out to be

$$\sigma_{V=0}=-\frac{(1+\frac{1}{2}|Z|^2)(1+\frac{1}{2}|h|^2)}{2U|h'|}\bar{H}\ , \tag{3.13}$$

with

$$H(Z)=\frac{h'''}{h'}-\frac{3}{2}\left[\frac{h''}{h'}\right]^2 . \tag{3.14}$$

There is an analogy between the conclusions to be drawn from the results (3.12)–(3.14) and the discussion at the end of the previous section. Clearly if we effectively allow $H\neq 0$ for $V>0$ and $H=0$ for $V<0$, there will be a finite jump in the shear of the generators of (3.11) across $V=0$ while the expansion is continuous across $V=0$.

We next use (3.8) and (3.9) [with now $h(Z)$ arbitrary, of course] to calculate the line element (1.8) in the coordinates $(Z,\bar{Z},U,V)$. It is very helpful to note in this regard that (1.8) may be written as

$$ds^2=2\left|d\left[\frac{u\zeta}{1+\frac{1}{2}\zeta\bar{\zeta}}\right]\right|^2-d\left[\frac{2u}{1+\frac{1}{2}\zeta\bar{\zeta}}-v\right]\times d\left[\frac{u\zeta\bar{\zeta}}{1+\frac{1}{2}\zeta\bar{\zeta}}-v\right]\ . \tag{3.15}$$

A rather lengthy but straightforward calculation results in (3.15) taking the form

$$ds^2=2U^2p^{-2}\left|dZ+\frac{V}{2U}\bar{H}p^2d\bar{Z}\right|^2+2dU\,dV-dV^2\ . \tag{3.16}$$

Here $p=1+\frac{1}{2}Z\bar{Z}$ and $H(Z)$ is given by (3.14).

Following from the results above we can now conclude that the line element of the space-time describing a spherical gravitational wave, with history $V=0$, propagating through flat space-time is given in a new coordinate system in which the metric is continuous across $V=0$ by

$$ds^2=2U^2p^{-2}\left|dZ+\frac{V\vartheta(V)}{2U}\bar{H}p^2d\bar{Z}\right|^2+2dU\,dV-dV^2\ , \tag{3.17}$$

where $\vartheta(V)$ is the Heaviside step function. When $V<0$ we transform this line element to the form (1.8) by the identity transformation while (3.17) for $V>0$ is transformed into (1.8) by (3.8) with (3.9), which reduces to (3.10) when $V=0$. This means that this pair of coordinate transformations incorporates the Penrose warp (3.1). With the metric tensor given via the line element (3.17) a calculation of the Ricci tensor reveals that it vanishes for all V. The Riemann curvature has only the one nonidentically vanishing Newman-Penrose component: namely,

$$\Psi_4=\frac{1}{2U}p^2H(Z)\delta(V)\ . \tag{3.18}$$

It follows from (3.17) that the hypersurfaces $V=0$ are null and are generated by the geodesic integral curves of the null vector field $\partial/\partial U$ having U as an affine parameter along them. This vector field restricted to $V=0$ is the degenerate principal null direction of the Petrov type N curvature tensor above. The complex shear of the integral curves of $\partial/\partial U$ is

$$\sigma=-\tfrac{1}{2}\phi^{-1}V\vartheta(V)H\ , \tag{3.19}$$

and their real expansion is

$$\vartheta=\phi^{-1}p^{-2}U\ , \tag{3.20}$$

where

$$\phi=U^2p^{-2}-\tfrac{1}{4}V^2\vartheta(V)|H|^2p^2\ . \tag{3.21}$$

Thus it is clear that $V=0$ is a null cone (as are all $V=\text{const}<0$) since on this null hypersurface the generators are shear-free and have expansion U^{-1}.

IV. DISCUSSION

We can combine (1.3) and (3.17) in a single formula:

$$ds^2=2U^2p^{-2}\left|dZ+\frac{V\vartheta(V)}{2U}\bar{H}p^2d\bar{Z}\right|^2+2dU\,dV-K\,dV^2\ , \qquad (4.1)$$

where now

$$p=1+\frac{K}{2}Z\bar{Z}\ , \qquad (4.2)$$

and $K=0$ or 1. The case $K=-1$ is also a further possibility. In this case the Ricci tensor vanishes for all V, and the formulas (3.18)–(3.21) continue to hold with p given by (4.2) with $K=-1$. This corresponds to starting with a form of the Minkowskian line element in coordinates based upon a family of future null cones with vertices on a *spacelike* geodesic rather than on a null geodesic as in the case of (1.1) or on a timelike geodesic as in the case of (1.8) (see [8]).

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[6] See, for example, R. Penrose and W. Rindler, in *Spinors and Space-Time* (Cambridge University Press, Cambridge, England, 1986), Vol. 1, Chap. 1.

[7] This is because, under (3.8), $V^2-2UV=v^2-2uv$.

[8] This family of three versions of the Minkowskian line element (in particular the cases $K=0$ and $K=-1$) seem first to have appeared together as a by-product of the work of I. Robinson and A. Trautman, Phys. Rev. Lett. **4**, 431 (1960); Proc. R. Soc. London **A265**, 463 (1962).