Weak gravitation waves in vacuum and in media: Taking nonlinearity into account

Michael Efroimsky*
Mathematical Institute, University of Oxford, Oxford OX1 3LB, United Kingdom
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The relevance of the nonlinear nature of gravity waves has been pointed out, in the astrophysical context, in recent publications by Christodoulou and by Thorne, who studied the nonlinear contribution to the memory effect. In the cosmological context, the role of nonlinearity has been discovered by Salopek, who studied the evolution of the primordial cosmological perturbations. In this article, we use the weak-field approximation to derive, in the perturbative approach, the wave equation for pure gravity waves with nonlinear correction terms.

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I. INTRODUCTION

An extensive literature on gravity waves and their role in cosmology and astrophysics exists, and it seems that all such papers can be divided into two types: the authors either consider the waves to be weak and describe the appropriate effects in the linear approximation only, or they consider analytically (sometimes numerically) very special toy models with nonlinearity, which are still very far from being used in current astrophysics. As regards the third possible way, i.e., extending the weak-wave approximation by taking into account the nonlinear correction terms in the wave equation, this is something most authors would rather do without, though on general grounds it is very natural to turn to the next order of approximation after the linear approach is exhausted. One of the strengths of such a description could be the light it may shed on the problem of the evolution of the power spectrum of the relic gravitons and thus on interpretation of Cosmic Background Explorer (COBE) data [1,2].

It is widely accepted that since the primordial gravity waves decoupled from matter a long time ago and since they interact with the fields of matter very weakly then their power spectrum has preserved its form completely. In fact, this pivotal point is doubtful because of the nonlinear nature of the phenomena: should one keep the higher terms in the wave equation one may face self-interaction effects known in the nonlinear differential equations theory as the "energy cascade." Thus a whole set of problems is called into being: to clarify the history of the primordial gravity-wave spectrum, to study the development of the relic adiabatic scalar perturbations, with respect to the nonlinearity, and of course to explore the role of the unavoidable nonlinear interaction between the excitations of these two types.

II. ARE THE NONLINEAR CORRECTIONS RELEVANT?

The relevance of the nonlinear nature of gravity waves was pointed out for the first time in the context of the so-called memory effect: as was independently discovered by Blanchet and Damour [3] and by Christodoulou [4], there is a contribution to this "memory" that arises from nonlinearities in the Einstein equation. As exhibited recently by Thorne [5], for the coalescence of a binary system made of two black holes, this contribution will produce a significant share of the total memory effect.

The nonlinear effects were further pointed out by Bond and Salopek [1,2], in the context of a cosmological problem: it appears that nonlinearity may be responsible for certain indirectly observable (via COBE-type experiments) changes in the power spectrum of the cosmological perturbations and the primordial gravitational waves.

In Ref. 1, Salopek aims at the analysis of the fully nonlinear solutions, via the Hamiltonian approach. However, we believe that since the amplitudes of the cosmological perturbation, in the framework of inflationary scenarios, are very small, this analysis may be executed in the weak-field approximation with the aid of the perturbative approach, provided the nonlinear correction terms are taken into account.

The purpose of this paper is to derive the quadratic and cubic terms in the wave equation for the purely gravitational waves. The main results are Eqs. (4.19) and (5.3). Before deriving the wave equation with nonlinear terms, we shall recall briefly the current approach in Sec. III.

III. THE STANDARD FORMALISM

It is fair to say that the problem of nonlinear corrections has not dropped from the sight of the preceding authors completely: the issue is addressed in the papers [6–9], but unfortunately none of them addressed the problem of the back reaction and the role of the observer. Ignoring these essential aspects led some of these authors to quite extravagant "results": for example, the authors of [6] "proved" that in Einstein's relativity the gravity waves do not transport energy, and it made these authors even doubt relativity. Although we shall not go to such lengths, it would be useful to recall some of their arguments and to point out their mistake. In the weak-field approximation to general relativity, one writes

*Electronic address: efroim@maths.ox.ac.uk

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$$g_{\mu\nu}=\gamma_{\mu\nu}+h_{\mu\nu} \tag{2.1}$$

and treats $h_{\mu\nu}$ as the new variables on the Ricci-flat background $\gamma_{\mu\nu}$. Of course, this decomposition is always legal and does not influence the physical nature of the theory. Now, if the theory is considered as a nonlinear theory of a tensor field $h_{\mu\nu}$ on the background geometry $\gamma_{\mu\nu}$, then the wave equation of this theory is

$$\delta(R\sqrt{-g}\,)/\delta h_{\mu\nu}=0 \ ,$$

and the symmetric stress-energy tensor density is

$$t^{\mu\nu}\equiv -2\delta(R\sqrt{-g}\,)/\delta\gamma_{\mu\nu} \ .$$

The authors of [6] argue that since the aforementioned decomposition is symmetrical with respect to the quantities γ and h and as the Lagrangian depends only on their sum, then for a free gravitation field the following equalities will be precise:

$$\delta(R\sqrt{-g}\,)/\delta h_{\mu\nu}=\delta(R\sqrt{-g}\,)/\delta\gamma_{\mu\nu} \ , \tag{2.2}$$

and thus, they conclude, the wave equation is incompatible with the nonvanishing of the stress tensor. If the background γ is fixed then Eq. (2.1) is valid in any approximation, not only in the weak-field one. In the weak-field approximation ($h \ll \gamma$),

$$-2\delta(R\sqrt{-g}\,)/\delta h_{\mu\nu}=-2\delta(R\sqrt{-g}\,)/\delta\gamma_{\mu\nu} =t^{(1)\mu\nu}+t^{(2)\mu\nu}+\cdots \ , \tag{2.3}$$

and the sum of all these terms is zero. The linear approximation of the wave equation is

$$-2\delta(R\sqrt{-g})/\delta h_{\mu\nu}\simeq t^{(1)\mu\nu}\simeq 0 \ , \tag{2.4}$$

whereas the first approximation for the stress energy will read

$$t^{\mu\nu}\equiv -2\delta(R\sqrt{-g}\,)/\delta\gamma_{\mu\nu}\simeq t^{(2)\mu\nu} \ . \tag{2.5}$$

The authors of [6] believe that since [according to (2.3)] $t^{(1)\mu\nu}\simeq t^{(2)\mu\nu}$ up to $O(h^3)$, then the second-order terms in the expression for $t^{\mu\nu}$ are completely compensated by the first-order term [up to $O(h^3)$]. On these grounds, they insist in [6] once more that the stress-energy tensor is zero.

Actually, there is nothing strange in such a conclusion: should one ignore the back reaction of the waves to the geometry, he must get than any formula such as $G_{\alpha\beta}(\gamma)=\tau_{\alpha\beta}(\gamma,h)$ indisputably means $\tau_{\alpha\beta}(\gamma,h)=0$ if $\gamma_{\mu\nu}$ stands for a vacuum. To avoid the absurdity one should keep in mind that in vacuum the gravity waves themselves create some average background $(\gamma_{\mu\nu}+\eta_{\mu\nu})$ in which

$$G_{\alpha\beta}(\gamma_{\mu\nu}+\eta_{\mu\nu})=\tau_{\alpha\beta}(\gamma_{\mu\nu}+\eta_{\mu\nu}\,;h_{\mu\nu}) \ , \quad \eta\simeq h^2 \ . \tag{2.6}$$

If the background shift η were taken into account in the calculations (2.1)–(2.5), they would not lead to the vanishing of $\tau_{\alpha\beta}$. (For more details, see Tsygan [10].) The idea that the effective energy of the gravity waves can affect the background curvature was first set out long ago by Brill and Hartle [11] and by Isaacson [12]. The background shift has also been mentioned by the authors of [13] (who called it a "nonlinear correction" and mistakenly considered that it becomes zero after averaging). Still, most authors traditionally consider the background to be $\gamma_{\mu\nu}$, not $(\gamma_{\mu\nu}+\eta_{\mu\nu})$. In this approach, after the new physical, metric

$$g_{\mu\nu}\equiv\gamma_{\mu\nu}+h_{\mu\nu} \tag{2.7}$$

is introduced, one has to establish, for the contravariant components, that

$$g^{\mu\nu}=\gamma^{\mu\nu}-h^{\mu\nu}+O(h^2) \tag{2.8}$$

in order to obey the condition

$$g_{\mu\nu}g^{\nu\rho}=\delta^\rho_\mu \ ,$$

whereupon it is usually affirmed that the subscripts and superscripts will be moved up and down by the initial metric tensor $\gamma_{\mu\nu}$. This puts forward a question: which Riemann space does the equality (2.8) relate to (if it relates to any)? If the quantities $g^{\mu\nu},\gamma^{\mu\nu},h^{\mu\nu}$ were defined as contravariant components of tensors in the Riemann space related to the metric $\gamma_{\mu\nu}$, then we have a discrepancy with the definition (2.7): using $\gamma_{\mu\nu}$ to move the scripts up, one will not get (2.8) but $g^{\mu\nu}=\gamma^{\mu\nu}+h^{\mu\nu}$, just according to the basic rules of how to treat tensors belonging to one and the same Riemann space. The answer to the question is very simple: there will be no contradiction between the formulas (2.7) and (2.8), and both will be acceptable simultaneously, in case we take that $g^{\mu\nu}$ and $\gamma^{\mu\nu}$ stand for components of contravariant tensors belonging to *different Riemann spaces*: $g^{\mu\nu}$ belongs to the Riemann space of contravariant tensors defined by the metric $g_{\mu\nu}$, and $\gamma^{\mu\nu}$ belongs to that defined by the metric $\gamma_{\mu\nu}$. (As for $h^{\mu\nu}$, in the linear approximation it appears to be irrelevant which of these two spaces it belongs to.)

The above statement needs clarification. As is well known, not only covariant but also contravariant tensors may be defined on the manifold, *independent from the metric*, and thus all contravariant tensors may be considered as elements of one space. However, this is not the case for the contravariant tensor *fields* whose parallel transport is defined via a metrical connection rigidly connected to this or that metric. For example, in any particular point of the manifold one may take the sum of two contravariant tensors $T_{\alpha\beta}\gamma^{\alpha\mu}\gamma^{\beta\nu}$ and $T_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}$ (where g and γ are different metrics) and denote this sum as $T_{\alpha\beta}q^{\alpha\mu}q^{\beta\nu}$ where q will be some third metric. There would be nothing wrong in such summarizing of two-forms, and the result will be a form as well. But this procedure will become illegitimate should one deal with covariant and contravariant tensor *fields* defined on the entire manifold, i.e., should one propose that the parallel transport of these fields must be rigidly defined by a certain metric via the appropriate metrical connection. In this case, one may not deal with $T_{\alpha\beta}\gamma^{\alpha\mu}\gamma^{\beta\mu}$ and $T_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}$ as with elements of one space, because they will be parallel transported with the aid of different connections. This is the rigorous formulation of the fact usually expressed in the following, rather blurred, form: "one must be explicit about which metric was used to raise indices." So, dealing with tensor *fields*, one must consider the contravariant tensor fields, with indices raised by different metrics, to be objects of different spaces.

In the most rough approximation we ignore this difference; i.e., we neglect the fact that the real observer finds himself not in the initial background $\gamma_{\mu\nu}$ (which had existed before the wave train had come) but in some effective average background $q_{\mu\nu}\equiv\gamma_{\mu\nu}+\eta_{\mu\nu}$, $\eta\simeq h^2$, which has been produced due to the back reaction. In other words, on this level of approximation one does not recognize the variations $(g_{\mu\nu}-\gamma_{\mu\nu})$ and $[g_{\mu\nu}-(\gamma_{\mu\nu}+\eta_{\mu\nu})]$ from one another: both are called $h_{\mu\nu}$. Although this approach leads to the proper linear approximation to the wave equation, it portends contradictions (as exhibited above) whenever one tries to deal with quadratic forms such as $\tau_{\mu\nu}$. The obvious equality (2.6) is already a step out of this approach. To correctly extend the formalism to the high-order terms, one has to establish that the back influence of the oscillations upon the background geometry effectively leads to producing such an average background $q_{\mu\nu}\equiv\gamma_{\mu\nu}+\eta_{\mu\nu}$, $\eta\simeq h^2$ that the interferometer of size L will feel all the modes with wavelengths shorter than L as waves, and all the longer modes will not exist for him as waves—they will be included into the effective background. To calculate the observables measurable by this device, one should use the average metric $q_{\mu\nu}$.

Thus, the very observer himself divides the physical metric $g_{\mu\nu}$ into the background and the waves. This procedure, called "the natural low-frequencies cut-off," has been introduced in [14]. It is on purpose that in the above statement we prefer to use a blurred term such as "size of interferometer" rather than a more definite term such as "base." The thing is that one can, at least in theory, measure waves of about a kilometer length with the aid of a meter-length bar. This will come off, exactly as in the radio techniques, if the duration of measurement extends the period of the longest mode observed. It comes off also due to the fact that the bar is installed on a long rigid body, the Earth. Taking this circumstance into account, we would point that the effective length of the total device is much longer than the length of the bar.

The aforementioned natural low-frequency cut-off gives the framework in which the field of gravity waves may be interpreted (for the particular observer) as a physical field that can be endowed with effectively observable energy and momentum densities. So one can introduce the effective stress-energy tensor whose integral over a certain three-space region will correspond to the result of some measurement.[1] In the proposed framework, one can also derive the wave equation with the nonlinear correction terms. We are going to carry out this derivation for the vacuum case (Sec. IV) and for the case of an ideal-fluid-filled space (Sec. V). These terms will depend on $h_{\mu\nu}$ as well as on $\eta_{\mu\nu}$. In Sec. VI, we shall present the direct expression of $\eta_{\mu\nu}$ via $h_{\alpha\beta}$ for a particular case of relic gravity waves in a Friedmann-Robertson-Walker model.

IV. WAVE-EQUATION FOR THE GRAVITY WAVES IN VACUUM

Let some smooth, nondegenerate, symmetrical pseudo-Riemannian metric γ be determined on a four-dimensional differential manifold M; then the functions

$$\begin{aligned} R_{\mu\nu}(\gamma)\equiv R^{\tau}_{\mu\tau\nu}=\partial_\chi\left[\frac{\gamma^{\chi\rho}}{2}(\partial_\mu\gamma_{\rho\nu}+\partial_\nu\gamma_{\rho\mu}-\partial_\rho\gamma_{\mu\nu})\right]-\partial_\nu\left[\frac{\gamma^{\chi\rho}}{2}(\partial_\mu\gamma_{\rho\chi}+\partial_\chi\gamma_{\rho\mu}-\partial_\rho\gamma_{\mu\chi})\right]\\ +\left[\frac{\gamma^{\chi\delta}}{2}(\partial_\chi\gamma_{\rho\delta}+\partial_\delta\gamma_{\rho\chi}-\partial_\rho\gamma_{\chi\delta})\right]\left[\frac{\gamma^{\delta\rho}}{2}(\partial_\mu\gamma_{\rho\nu}+\partial_\nu\gamma_{\rho\mu}-\partial_\rho\gamma_{\mu\nu})\right]\\ -\left[\frac{\gamma^{\chi\rho}}{2}(\partial_\nu\gamma_{\rho\delta}+\partial_\delta\gamma_{\rho\nu}-\partial_\rho\gamma_{\nu\delta})\right]\left[\frac{\gamma^{\delta\rho}}{2}(\partial_\mu\gamma_{\rho\chi}+\partial_\chi\gamma_{\rho\mu}-\partial_\rho\gamma_{\mu\chi})\right] \end{aligned} \tag{3.1}$$

[where $\gamma^{\chi\rho}\equiv(\gamma)^{-1}_{\chi\rho}$] and

$$G_{\mu\nu}(\gamma)\equiv R_{\mu\nu}(\gamma)-\tfrac{1}{2}\gamma_{\mu\nu}(\gamma^{\alpha\beta}R_{\alpha\beta}) \tag{3.2}$$

and the covariant Ricci and Einstein tensors correspondingly.

Let q and g be smooth, nondegenerate, symmetric metrics on M too. We consider γ and g to be vacuum metrics, but q to be nonvacuum:

$$G_{\mu\nu}(\gamma)=G_{\mu\nu}(g)=0\ ,\quad G_{\mu\nu}(q)\neq 0\ . \tag{3.3}$$

Further we shall call γ "the premetric" (it had existed before the waves appeared), g the physical metric, and q the average metric, or the background. The differences between the covariant components of the metrics are denoted

$$h_{\mu\nu}\equiv g_{\mu\nu}-q_{\mu\nu}\ ,\quad \eta_{\mu\nu}\equiv q_{\mu\nu}-\gamma_{\mu\nu}\ . \tag{3.4}$$

By definition, the appropriate values $h^{\mu\nu}$ and $\eta^{\mu\nu}$ will belong to the Riemann space determined by the metric ten-

[1]The formula for the stress-energy tensor and its detailed derivation were presented in our previous paper [19].

sor q; by definition,

$$h^{\mu\nu} \equiv h_{\alpha\beta} q^{\alpha\mu} q^{\beta\nu} = g^{\mu\nu} - q^{\mu\nu} , \tag{3.5a}$$

$$\eta^{\mu\nu} \equiv \eta_{\alpha\beta} q^{\alpha\mu} q^{\beta\nu} = q^{\mu\nu} - \gamma^{\mu\nu} ; \tag{3.5b}$$

i.e., h and η are covariant and contravariant tensors and they are connected with one another via the nonvacuum metric q. Generally, whenever we use the contravariant notation, it will mean that connection between the contravariant indices and the appropriate covariant ones is defined via the metric q only. Treating $h_{\mu\nu}$ as a perturbation of the metric $q_{\mu\nu}$ we expand the Ricci tensor in a power series:

$$R_{\mu\nu}(g) = R_{\mu\nu}(q) + R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(3)}_{\mu\nu}(q,h) + O(h^4) , \tag{3.6}$$

where ([13], formula (35.58))

$$R^{(1)}_{\mu\nu}(q,h) = \tfrac{1}{2}(-h^{\alpha}_{\alpha;\mu\nu} - h_{\mu\nu;\alpha}{}^{;\alpha} + h_{\alpha\mu;\nu}{}^{;\alpha} + h_{\alpha\nu;\mu}{}^{;\alpha}) , \tag{3.7}$$

$$R^{(2)}_{\mu\nu}(q,h) = \tfrac{1}{2}[\tfrac{1}{2} h_{\alpha\beta;\mu} h^{\alpha\beta}{}_{;\nu} + h^{\alpha\beta}(h_{\alpha\beta;\mu\nu} + h_{\mu\nu;\alpha\beta} - h_{\alpha\mu;\nu\beta} - h_{\alpha\nu;\mu\beta}) + h_{\nu}{}^{\alpha;\beta}(h_{\alpha\mu;\beta} - h_{\beta\mu;\alpha}) - (h^{\alpha\beta}{}_{;\beta} - \tfrac{1}{2} h^{\tau}{}_{\tau}{}^{;\alpha})(h_{\alpha\mu;\nu} + h_{\alpha\nu;\mu} - h_{\mu\nu;\alpha})] . \tag{3.8}$$

All the covariant derivatives are taken with respect to the average metric q. In the same way,

$$R_{\mu\nu}(\gamma) = R_{\mu\nu}(q) + R^{(1)}_{\mu\nu}(q,(-\eta)) + R^{(2)}_{\mu\nu}(q,\eta) + O(\eta^3) . \tag{3.9}$$

Since γ and g are vacuum metrics, it follows from (3.6) and (3.9) that

$$\begin{aligned} 0 &= R_{\mu\nu}(g) - R_{\mu\nu}(\gamma) \\ &= R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(1)}_{\mu\nu}(q,\eta) \\ &\quad + R^{(3)}_{\mu\nu}(q,h) + O(h^4) + O(\eta^2) . \end{aligned} \tag{3.10}$$

At this point we accept the following assumptions.

Assumption 1. The waves are weak in the sense that the terms denoted as $O(h^4)$ and $O(\eta^2)$ are small against the preceding terms.[2]

Assumption 2. The components of η are of the order $\simeq h^2$.

Assumption 3. The tensor field $h_{\mu\nu}$ includes modes with wavelengths not exceeding some maximal scale L determined by the observer.

The latter means that we follow the natural procedure of metric separation into background and waves, introduced in [14]. For example, in the measurements of anisotropy of the relic microwave electromagnetic radiation, the Universe as a whole acts as a natural interferometer of size up to about c/H. So the COBE-type experiments are to become the largest possible gravitation-wave detectors, provided we learn to separate the contributions from the scalar and the tensor cosmological perturbations (as established by the authors of [15], such separation may be in principle carried out).

Returning back to (3.10), we can, in the proposed approach, derive the wave equation in vacuum, with two next-after-leading order terms:

$$R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(1)}_{\mu\nu}(q,\eta) + R^{(3)}_{\mu\nu}(q,h) = 0 , \tag{3.11}$$

(where $q_{\mu\nu} \equiv \gamma_{\mu\nu} + \eta_{\mu\nu}$). Note that the term $R^{(1)}_{\mu\nu}(q,\eta)$ has the same meaning as $R^{(0)}_{\mu\nu}$ in the paper [12] by Isaacson, and it is of the same order as $R^{(2)}_{\mu\nu}(q,h)$.

V. WAVE EQUATION FOR THE GRAVITY WAVES IN AN IDEAL MEDIA

To generalize the formalism to the case of spaces with ideal-fluid-like matter and a cosmological term, we shall begin with recalling several simple facts. With the signature $(-+++)$, the cosmological constant Λ is introduced as

$$G_{\mu\nu}(g) + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} ; \tag{4.1}$$

in Friedmann-Robertson-Walker (FRW) models it effectively behaves as an ideal fluid of pressure $p_{\text{eff}} = -(8\pi G c^{-4})\Lambda$ and density $\rho_{\text{eff}} = (8\pi G c^{-4})\Lambda$. In any way, Λ always may be effectively included into $T_{\mu\nu}$.

Now, let the metrics $\gamma_{\mu\nu}, q_{\mu\nu}$, and $g_{\mu\nu}$ have the same meaning as in the vacuum case: the premetric γ had existed before the wave train came (or was produced); g is the physical metric which is decoupled into the smooth background q and the excitations h. These excitations cause some average shift η of the geometry: $h_{\mu\nu} \equiv g_{\mu\nu} - q_{\mu\nu}, \eta_{\mu\nu} \equiv q_{\mu\nu} - \gamma_{\mu\nu}, \eta \simeq h^2$. Obviously,

$$G_{\mu\nu}(\gamma) - kT_{\mu\nu}(\gamma,\varphi) = 0 , \tag{4.2}$$

$$G_{\mu\nu}(g) - kT_{\mu\nu}(g,\varphi) = 0 , \tag{4.3}$$

where $k \equiv 8\pi G c^{-4}$ and φ stands for the fields if matter. All these tensors can be expanded around q, like (3.6) and (3.9). For example,

[2] This is the precise form of the supposal that $|h| \ll |\gamma|$.

$$0=G_{\mu\lambda}(g)-kT_{\mu\lambda}(g,\varphi)=G_{\mu\lambda}(q)+G^{(1)}_{\mu\lambda}(q,h)+G^{(2)}_{\mu\lambda}(q,h)+O(h^3)$$
$$-k[T_{\mu\lambda}(q,\varphi)+T^{(1)}_{\mu\lambda}(q,h,\varphi)+T^{(2)}_{\mu\lambda}(q,h,\varphi)+O(h^3)] \ , \tag{4.4}$$

where

$$G^{(1)}_{\mu\nu}(q,h)=R^{(1)}_{\mu\nu}(q,h)-\tfrac{1}{2}h_{\mu\nu}[q^{\alpha\beta}R_{\alpha\beta}(q)]-\tfrac{1}{2}q_{\mu\nu}[q^{\alpha\beta}R^{(1)}_{\alpha\beta}(q,h)]+\tfrac{1}{2}q_{\mu\nu}[q^{\alpha\beta}R^{(1)}_{\alpha\beta}(q,h)] \ , \tag{4.5}$$

$$G^{(2)}_{\mu\nu}(q,h)=R^{(2)}_{\mu\nu}(q,h)-\tfrac{1}{2}h_{\mu\nu}[q^{\alpha\beta}R^{(1)}_{\alpha\beta}(q,h)]+\tfrac{1}{2}h_{\mu\nu}[h^{\alpha\beta}R_{\alpha\beta}(q)]-\tfrac{1}{2}q_{\mu\nu}[q^{\alpha\beta}R^{(2)}_{\alpha\beta}(q,h)]-\tfrac{1}{2}q_{\mu\nu}[h^{\alpha}_{\chi}h^{\chi\beta}R_{\alpha\beta}(q)] \tag{4.6}$$

and so on; $R^{(1)}_{\alpha\beta}$ and $R^{(2)}_{\alpha\beta}$ are determined by (3.7) and (3.8).

Taking the difference between this expansion and the similar one for γ, neglecting the higher-than-third order terms, one gets a formula similar to (3.11):

$$0=\{G^{(1)}_{\mu\nu}(q,h)-kT^{(1)}_{\mu\nu}(q,h,\varphi)\}+\{G^{(1)}_{\mu\nu}(q,\eta)-kT^{(1)}_{\mu\nu}(q,\eta,\varphi)\}$$
$$+\{G^{(2)}_{\mu\nu}(q,h)-kT^{(2)}_{\mu\nu}(q,h,\varphi)\}+\{G^{(3)}_{\mu\nu}(q,h)-kT^{(3)}_{\mu\nu}(q,h,\varphi)\} \ . \tag{4.7}$$

Generally speaking, even in the linear approximation to the wave equation,

$$G^{(1)}_{\mu\nu}(q,h)-kT^{(1)}_{\mu\nu}(q,h,\varphi)=0 \ , \tag{4.8}$$

there are terms directly dependent on the parameters of the matter. As we restrict ourselves to the purely gravity waves (not coupled matter-gravity oscillations), it appears that in the case of an ideal fluid the terms that depend on the matter parameters can be excluded from the linear approximation. To perform this, we shall recall that in this model the tensor $T_{\mu\nu}$ will read

$$T_{\mu\nu}(q)=(p+\rho)u_{\mu}u_{\nu}+(p-\Lambda/k)q_{\mu\nu} \ . \tag{4.9}$$

If the excitations are purely gravity, then $\delta\rho=\delta\rho=0$, $\delta u_{\mu}=0$, and (in the gauge $h_{\mu}{}^{\nu}u_{\nu}=0$) the first variation of $T_{\mu\nu}$ looks like

$$T^{(1)}_{\mu\nu}(q,h)=(p-\Lambda/k)h_{\mu\nu} \ . \tag{4.10a}$$

Thus, from (4.9)–(4.10a), one gets several very simple and useful relations:

$$T^{(1)}_{\mu\nu}(q,h)=T_{\mu\lambda}(q)h_{\nu}{}^{\lambda} \ , \tag{4.11a}$$

$$T^{(2)}_{\mu\nu}(q,h)=T^{(3)}_{\mu\nu}(q,h)=\cdots=0 \ . \tag{4.12}$$

In case the background shift is taken into account and the total difference between the physical metric g and the premetric γ is $(h+\eta)$, then (4.10)–(4.11) look like

$$T^{(1)}_{\mu\nu}(q,h+\eta)=(p-\Lambda/k)(h_{\mu\nu}+\eta_{\mu\nu}) \ , \tag{4.10b}$$

$$T^{(1)}_{\mu\nu}(q,h+\eta)=T_{\mu\lambda}(q)(h_{\nu}{}^{\lambda}+\eta_{\nu}{}^{\lambda}) \ , \tag{4.11b}$$

and the gauge condition $(h_{\mu}{}^{\nu}+\eta_{\mu}{}^{\nu})u_{\nu}=0$ should be imposed.

Further we shall need the relations (4.10b) and (4.11b). Now we would remind how it comes that in the linear approximation the wave equation does not contain, in a direct form, any parameters of matter.

To exclude the matter-parameters-dependent terms from the wave equation, in the linear approximation it is quite sufficient to transfer to the nontensor quantities $\delta^{(1)}G_{\mu}{}^{\nu}$ and $\delta^{(1)}T_{\mu}{}^{\nu}$:

$$\delta^{(1)}G_{\mu}{}^{\nu}\equiv G_{\mu\chi}(g)g^{\chi\nu}-G_{\mu\chi}(q)q^{\chi\nu}$$
$$=G^{(1)}_{\mu\chi}q^{\chi\nu}+G_{\mu\chi}\delta q^{\chi\nu}=G^{(1)}_{\mu\chi}q^{\chi\nu}+G_{\mu\chi}(-h^{\chi\nu}) \ , \tag{4.13}$$

$$\delta^{(1)}T_{\mu}{}^{\nu}=T^{(1)}_{\mu\chi}q^{\chi\nu}+T_{\mu\chi}(-h^{\chi\nu}) \ , \tag{4.14}$$

and, since in a comoving reference frame, in a traceless gauge, $\delta^{(1)}T_{\mu}{}^{\nu}$ vanishes,[3] one gets the linear approximation in the form

$$\delta^{(1)}G_{\mu}{}^{\nu}=0 \ , \tag{4.15}$$

that is equal (see Appendix A) to

$$-h_{;\mu\nu}-h_{\mu\nu;\alpha}{}^{;\alpha}+2R_{\chi\mu\nu\rho}h^{\chi\rho}+q_{\mu\nu}h_{;\alpha}{}^{;\alpha}+R_{\chi\nu}h^{\chi}{}_{\mu}$$
$$-R_{\chi\mu}h^{\chi}{}_{\nu}+q_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}+2h_{\alpha\{\mu}{}^{;\alpha}{}_{;\nu\}}-q_{\mu\nu}h_{\alpha\beta}{}^{;\alpha\beta}=0 \ . \tag{4.16}$$

However, one should keep in mind that addition and subtraction of mixed tensors, whose superscripts have been moved up by different metrics, is not a mathematically correct procedure and it can lead to mistakes in the high-order terms. (This is shown in Appendix A). So, to write down the wave equation with the higher terms correctly and to exhibit it in the form where the linear part would look conventionally [as (4.16)], we shall contract the tensor equation (4.4) with the tensor $(-h_{\nu}{}^{\lambda}-\eta_{\nu}{}^{\lambda})$ and add the obtained product to (4.7). Taking (4.12) into account, we shall receive

$$0=G^{(1)}_{\mu\nu}(q,h)-h_{\nu}{}^{\lambda}G_{\mu\lambda}(q)+k\{(h_{\nu}{}^{\lambda}+\eta_{\nu}{}^{\lambda})T_{\mu\lambda}(q)-T^{(1)}_{\mu\lambda}(q,h+\eta)\}$$
$$+G^{(1)}_{\mu\nu}(q,\eta)-\eta_{\nu}{}^{\lambda}G_{\mu\lambda}(q)-h_{\nu}{}^{\lambda}G^{(1)}_{\mu\lambda}(q,h)+k(h_{\nu}{}^{\lambda}+\eta_{\nu}{}^{\lambda})T^{(1)}_{\mu\lambda}(q,h)$$
$$-\eta_{\nu}{}^{\lambda}G^{(1)}_{\mu\lambda}(q,h)+G^{(2)}_{\mu\nu}(q,h)+G^{(3)}_{\mu\nu}(q,h)+O(h^4) \ . \tag{4.17}$$

[3]The equalities (4.13) and (4.14) are nontensor, so the sign "−" before $h^{\chi\nu}$ does not contradict the convention (3.5a): in (4.13) and (4.14) we have taken $g^{\chi\nu}\equiv(g_{\chi\nu})^{-1}$ but not $g^{\chi\nu}\equiv g_{\alpha\beta}q^{\alpha\chi}q^{\beta\nu}$.

According to (4.11b), the sum in parentheses vanishes. Then we write

$$k(h_\nu{}^\lambda+\eta_\nu{}^\lambda)T^{(1)}_{\mu\lambda}(q,h)=k(h_\nu{}^\lambda+\eta_\nu{}^\lambda)T^{(1)}_{\mu\lambda}(q,h+\eta)-kh_\nu{}^\lambda T^{(1)}_{\mu\lambda}(q,\eta)+O(h^4)$$

making use of (4.11b),

$$=k(h_\nu{}^\lambda+\eta_\nu{}^\lambda)(h_\lambda{}^\chi+\eta_\lambda{}^\chi)T_{\mu\chi}(q)-kh_\nu{}^\lambda T^{(1)}_{\mu\lambda}(q,\eta)+O(h^4)$$

with the aid of (4.4),

$$\begin{aligned}&=(h_\nu{}^\lambda+\eta_\nu{}^\lambda)(h_\lambda{}^\chi+\eta_\lambda{}^\chi)G_{\mu\chi}(q)+h_\nu{}^\lambda h_\lambda{}^\chi G^{(1)}_{\mu\chi}(q,h)\\&\quad-kh_\nu{}^\lambda h_\lambda{}^\chi T^{(1)}_{\mu\chi}(q,h)-kh_\nu{}^\lambda T^{(1)}_{\mu\lambda}(q,\eta)+O(h^4)\\&=h_\nu{}^\lambda h_\lambda{}^\chi G_{\mu\chi}(q)+(h_\nu{}^\lambda\eta_\lambda{}^\chi+h_\lambda{}^\chi\eta_\nu{}^\lambda)G_{\mu\chi}(q)+h_\nu{}^\lambda h_\lambda{}^\chi G^{(1)}_{\mu\chi}(q,h)\\&\quad-kh_\nu{}^\lambda h_\lambda{}^\chi T^{(1)}_{\mu\chi}(q,h)-kh_\nu{}^\lambda T^{(1)}_{\mu\lambda}(q,\eta)+O(h^4)\end{aligned}$$

making use of (4.11b) once more,

$$\begin{aligned}&=h_\nu{}^\lambda h_\lambda{}^\chi G_{\mu\chi}(q)+(h_\nu{}^\lambda\eta_\lambda{}^\chi+h_\lambda{}^\chi\eta_\nu{}^\lambda)G_{\mu\chi}(q)+h_\nu{}^\lambda h_\lambda{}^\chi G^{(1)}_{\mu\chi}(q,h)-h_\nu{}^\lambda h_\lambda{}^\chi h_\chi{}^\xi G_{\mu\xi}(q,h)\\&\quad-kh_\nu{}^\lambda T^{(1)}_{\mu\lambda}(q,\eta)+O(h^4)\ .\end{aligned}\tag{4.18}$$

Substituting this into (4.17), one will obtain, up to the cubic in $h_{\mu\nu}$ terms inclusively, the wave equation of the gravitational waves, including the nonlinear corrections

$$\begin{aligned}&G^{(1)}_{\mu\nu}(q,h)-h_\nu{}^\lambda G_{\mu\lambda}(q)+\{G^{(1)}_{\mu\nu}(q,\eta)+G^{(2)}_{\mu\nu}(q,h)-\eta_\nu{}^\lambda G_{\mu\lambda}(q)-h_\nu{}^\lambda G^{(1)}_{\mu\lambda}(q,h)+h_\nu{}^\lambda h_\lambda{}^\chi G_{\mu\chi}(q)\}\\&\quad+\{(h_\nu{}^\lambda\eta_\lambda{}^\chi+h_\lambda{}^\chi\eta_\nu{}^\lambda)G_{\mu\chi}(q)+h_\nu{}^\lambda h_\lambda{}^\chi G^{(1)}_{\mu\chi}(q,h)-h_\nu{}^\lambda h_\lambda{}^\chi h_\chi{}^\xi G_{\mu\xi}(q,h)\\&\quad-\eta_\nu{}^\lambda G^{(1)}_{\mu\lambda}(q,h)+G^{(3)}_{\mu\nu}(q,h)-kh_\nu{}^\lambda T^{(1)}_{\mu\lambda}(q,\eta)\}=0\ ,\end{aligned}\tag{4.19}$$

where the first two terms are linear, the next five are quadratic, and the others cubic. This is the equation for the purely gravity waves taken in the gauge

$$u_\lambda(h_\nu{}^\lambda+\eta_\nu{}^\lambda)=0\ ,\tag{4.20}$$

in an ideal medium, with or without the cosmological constant. The equation is written down so that the terms containing the parameters of matter are excluded not only from the linear part but from the quadratic one as well. [It was just for this purpose that we have thread our way through the tedious calculations (4.18).] In the cubic approximation these terms cannot be excluded. Equation (4.19) will become solvable after the background shift $\eta_{\mu\nu}$ is expressed via $h_{\alpha\beta}$. At least in one case one can do it easily. It is the case of relic gravity waves in FRW-spaces. Since the premetric γ was of FRW type then the average metric $q\equiv\gamma+\eta$ belongs to this type as well. Generally, this assumption is wrong. However, in the case of a uniformly distributed primeval radiation one can take that the average shift η depends on the cosmological time only, and the assumption becomes valid. As has been noticed in [14], the background shift $\eta_{\mu\nu}$ effectively reduces the Universe expansion and plays the role of a negative feedback because the creation of gravity waves is effectively equal to production of some sort of matter, and it decelerates the expansion. Recall this more precisely; $\eta_{\mu\nu}$ appears, in this situation, to be a negative variation $\delta a(\eta)$ of the Friedmann radius $a(\eta)$. As is known, the pressure, energy density and Friedmann radius are connected like this [see the formula (112.6) in [16]]: $\delta\rho=-3(\rho+p)\delta a/a$. In our case, it means that

$$\delta a(\eta)/a(\eta)=-\frac{\rho_g}{3\rho_m}\ ,\tag{4.21}$$

where ρ_g and ρ_m are the energy densities of the relic gravitons noise and of the matter correspondingly. The ratio ρ_g/ρ_m for the spatially flat FRW model has been estimated by Allen in his paper [17] and appeared to be about 10^{-18}. Thus, the background shift will look like

$$\begin{aligned}\eta_{ij}&=-\frac{2}{3}a^2\frac{\rho_g}{\rho_m}\delta_{ij}\ ,\\\eta_{00}&=+\frac{2}{3}a^2\frac{\rho_g}{\rho_m}\ ,\end{aligned}\tag{4.22}$$

while all other components of η vanish.

VI. THE QUADRATIC APPROXIMATION

The most attractive feature of the second-order approximation is that it does not directly depend on the parameters of matter. Keeping in the wave equation (4.17) only the linear and the quadratic terms, we obtain

$$G^{(1)}_{\mu\nu}(q,h)-h_\nu{}^\lambda G_{\mu\lambda}(q)+\{G^{(1)}_{\mu\nu}(q,\eta)+G^{(2)}_{\mu\nu}(q,h)-\eta_\nu{}^\lambda G_{\mu\lambda}(q)-h_\nu{}^\lambda G^{(1)}_{\mu\lambda}(q,h)+h_\nu{}^\lambda h_\lambda{}^\chi G_{\mu\chi}(q)\}=0\ ,\tag{5.1}$$

or, the same (up to the higher-than-quadratic order terms),

$$G^{(1)}_{\mu\nu}(q,(h+\eta))-(h_\nu{}^\lambda+\eta_\nu{}^\lambda)G_{\mu\lambda}(q)+G^{(2)}_{\mu\nu}(q,(h+\eta))-(h_\nu{}^\chi+\eta_\nu{}^\chi)\{G^{(1)}_{\mu\chi}(q,(h+\eta))-(h_\chi{}^\lambda+\eta_\chi{}^\lambda)G_{\mu\lambda}(q)\}=0\ .\tag{5.2}$$

Thus, the equation can be expressed via the tensor $\tilde{h}_{\alpha\beta} \equiv h_{\alpha\beta} + \eta_{\alpha\beta} = g_{\alpha\beta} - \gamma_{\alpha\beta}$. This is the full variation of the metric. In the linear approximation, they never make any distinction between $h_{\alpha\beta}$ and $\tilde{h}_{\alpha\beta}$, but in fact they are different quantities. In direct interferometrical measurements, they observe, rigorously speaking, $h_{\alpha\beta}$ rather than $\tilde{h}_{\alpha\beta}$, because the interferometer finds itself not in the initial metric γ but in the average metric q. Anyway, should one express Eq. (5.2) in terms of $\tilde{h}_{\alpha\beta}$, it will look especially simply

$$G^{(1)}_{\mu\nu}(q,\tilde{h}) - \tilde{h}_\nu{}^\lambda G_{\mu\lambda}(q) + G^{(2)}_{\mu\nu}(q,\tilde{h}) - \tilde{h}_\nu{}^\chi \{ G^{(1)}_{\mu\chi}(q,\tilde{h}) - \tilde{h}_\chi{}^\lambda G_{\mu\lambda}(q) \} = 0 \ . \quad (5.3)$$

Now we shall impose the transverse traceless (TT) gauge on the tensor $\tilde{h}_{\alpha\beta}$ (not on $h_{\alpha\beta}$):

$$\tilde{h} \equiv \tilde{h}_\mu{}^\mu = 0 \ , \quad (5.4a)$$

$$\tilde{h}_{\alpha 0} = 0 \ , \quad (5.4b)$$

$$\tilde{h}_{\alpha\beta}{}^{;\beta} = 0 \ . \quad (5.4c)$$

These conditions can be satisfied simultaneously for any solution and on an arbitrary background on an arbitrary initial timelike hypersurface. However, they cannot in general be satisfied simultaneously off it [18]; i.e., it is impossible to localize the degrees of freedom explicitly (though at any time the count of number of "physical" components of the field $\tilde{h}_\mu{}^\nu$ gives the number 2). Still the most primitive cosmological models (and among them the FRW spaces) belong to the class of spaces in which such localization is possible [18].

In FRW universes, when the gauge (5.4) is imposed upon $\tilde{h}_\mu{}^\nu$, one has

$$\tilde{h}_\nu{}^\lambda G_{\mu\lambda}(q) - G^{(1)}_{\mu\nu}(q,\tilde{h}) = \tfrac{1}{2}\tilde{h}_{\mu\nu;\alpha}{}^{;\alpha} - R_{\alpha\mu\nu\delta}\tilde{h}^{\alpha\delta} = \tfrac{1}{2}\tilde{h}_{\mu\nu;\alpha}{}^{;\alpha} + \tilde{h}_{\mu\nu} a^2(\varepsilon + H_1^2) \ . \quad (5.5)$$

Here we used (1) and also the fact that in FRW spaces $R_{\alpha\mu\nu\delta}\tilde{h}^{\alpha\delta} = \frac{1}{2}(R_\mu{}^\mu + R_\nu{}^\nu - \frac{1}{3}R)\tilde{h}_{\mu\nu}$, no summing over μ and ν. This can be easily derived from the vanishing of the Weyl tensor in FRW universes. In these spaces, in the gauge (5.4),

$$-G^{(2)}_{\mu\nu}(q,\tilde{h}) = \tfrac{1}{2}\{ \tfrac{1}{2}\tilde{h}^{\rho\tau}{}_{;\mu}\tilde{h}_{\rho\tau;\nu} + \tfrac{1}{2}\tilde{h}^\tau{}_\nu \tilde{h}_{\mu\tau;\rho}{}^{;\rho} - \tilde{h}^\rho{}_\nu \tilde{h}_{\{\mu}{}^\chi R_{\rho\}\chi} \} + \tfrac{1}{4}q_{\mu\nu}\tilde{h}^{\alpha\tau}R_{\chi\tau}\tilde{h}_\alpha{}^\chi + \tfrac{1}{2}\{ (\tilde{h}^\alpha{}_\nu \tilde{h}_{\mu\chi;\alpha})^{;\chi} - (h^\alpha{}_\chi \tilde{h}_{\mu\nu;\alpha})^{;\chi} + 2(\tilde{h}^\alpha{}_\chi \tilde{h}_{\alpha\{\mu;\nu\}})^{;\chi} - (\tilde{h}^\alpha{}_\nu \tilde{h}_{\mu\alpha;\chi})^{;\chi} - (\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta;\mu})_{;\nu} \} + \tfrac{1}{4}q_{\mu\nu}\{ \tfrac{3}{2}(\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta;\chi})^{;\chi} - (\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\chi;\beta})^{;\chi} \} \ . \quad (5.6)$$

(See appendix B for details.)

VII. SUMMARY OF THE MAIN RESULTS

(1) In vacuum, the gravitational waves themselves create the average geometry background $q_{\mu\nu}$ which differs from the "premetric" $\gamma_{\mu\nu}$ that had existed before the train of waves has come or has been produced; a similar effect takes place in the presence of matter, when the back reaction of the gravity waves produces a shift of the background [formula (2.6)]. The existence of the background shift $\eta_{\mu\nu}$ is an essentially nonlinear phenomenon. For the case of stochastic relic gravity-wave noise, $\eta_{\mu\nu}$ is expressed directly via $h_{\mu\nu}$ [formula (4.22)].

(2) Because of the nonlinear nature of the Einstein equation, the high-order approximations naturally relieve the linearized approach. Among other things, the nonlinear corrections may be of importance for studies of the possible energy cascade in the power spectrums of the primordial gravitons, primordial adiabatical perturbations, and perhaps for modeling of nonlinear interactions between these types of excitations, that might take place during the radiation-dominated stage.

The wave equation for the purely gravitational waves, with quadratic and cubic (in $h_{\alpha\beta}$) terms, looks like (3.11) in vacuum and like (4.19) in an ideal media, with a cosmological constant or without it.

(3) The main result of the paper is Eq. (5.3): it appears that in the quadratic approximation the wave equation acquires such a simple form that it contains directly neither the parameters of matter nor the nonlinear shift η. Since the most part of the Universe age has been the stage of dust domination, where all the scalar-type oscillations were frozen out and only pure gravity waves kept oscillating, it seems that (5.3) may help us to answer the question whether the age of Universe has been long enough to enable the nonlinearity to manifest itself as an energy cascade.

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APPENDIX A

In this appendix we omit the tilde in order to simplify the notation; thus, write $h_{\mu\nu}$ instead of $\tilde{h}_{\mu\nu}$. Since in vacuum or in an ideal-fluid-filled space (perhaps with a cosmological term) $\delta^{(1)}T^\nu_\mu = 0$ in the synchronous gauge (5.4b), so the linear approximation to wave equation is taken, in this gauge, as $\delta^{(1)}G^\lambda_\mu \equiv G^{(1)}_{\mu\chi}q^{\chi\lambda} - G_{\mu\chi}h^{\chi\lambda} = 0$. Multiplying by the background metric $q_{\lambda\nu}$, one will get

$$
\begin{aligned}
0=G^{(1)}_{\mu\nu}-G_{\chi\mu}h_\nu{}^\chi &= R^{(1)}_{\mu\nu}-\tfrac{1}{2}h_{\mu\nu}R-\tfrac{1}{2}q_{\mu\nu}q^{\alpha\beta}R^{(1)}_{\alpha\beta}+\tfrac{1}{2}q_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}-h^\chi{}_\nu R_{\mu\chi}+\tfrac{1}{2}Rh_{\mu\nu}\\
&= R^{(1)}_{\mu\nu}-\tfrac{1}{2}q_{\mu\nu}q^{\alpha\beta}R^{(1)}_{\alpha\beta}+\tfrac{1}{2}q_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}-h^\chi{}_\nu R_{\mu\chi}\\
&= -\tfrac{1}{2}h_{;\mu\nu}-\tfrac{1}{2}h_{\mu\nu;\alpha}{}^{;\alpha}+R_{\chi\mu\nu\rho}h^{\chi\rho}+\tfrac{1}{2}q_{\mu\nu}h_{;\alpha}{}^{;\alpha}\\
&\quad+\tfrac{1}{2}q_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}+h_{\alpha\{\mu}{}^{;\alpha}{}_{;\nu\}}-\tfrac{1}{2}q_{\mu\nu}h_{\alpha\beta}{}^{;\alpha\beta}+\tfrac{1}{2}(R_{\chi\nu}h^\chi{}_\mu-R_{\chi\mu}h^\chi{}_\nu)\ .
\end{aligned}
$$

Thus in the most general form, the linearized equation reads (in the synchronous gauge)

$$
-h_{;\mu\nu}-h_{\mu\nu;\alpha}{}^{;\alpha}+2R_{\chi\mu\nu\rho}h^{\chi\rho}+q_{\mu\nu}h_{;\alpha}{}^{;\alpha}+R_{\chi\nu}h^\chi{}_\mu-R_{\chi\mu}h^\chi{}_\nu+q_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}+2h_{\alpha\{\mu}{}^{;\alpha}{}_{;\nu\}}-q_{\mu\nu}h_{\alpha\beta}{}^{;\alpha\beta}=0\ . \tag{A1}
$$

In the traceless gauge, the first and fourth terms vanish. In a FRW geometry, provided the gauge is synchronous, the seventh term vanishes as well. The transversality condition will make the last two terms zero. Finally, if the spatial curvature is zero, (A1) reduces as

$$
\ddot{h}_m{}^n+2H_1\dot{h}_m{}^n-q^{bb}h_m{}^n{}_{,b,b}=0 \tag{A2}
$$

(Lifshitz equation). The overdots stand for conformal-time derivatives. Should one try to expand this method to the second- or third-order approximations, he will face some contradictions that follow from the incorrect nature of approximation of a nontensor quantity by tensors. For example, in vacuum,

$$
\begin{aligned}
0=\delta R_\mu{}^\xi\equiv(g_{\xi\chi})^{-1}R_{\mu\chi}(g)-(\gamma_{\xi\chi})^{-1}R_{\mu\chi}(\gamma)&=q^{\xi\chi}R^{(1)}_{\mu\chi}(q,h)+\{q^{\xi\chi}R^{(2)}_{\mu\chi}(q,h)-h^{\xi\chi}R^{(1)}_{\mu\chi}(q,h)+q^{\xi\chi}R^{(1)}_{\mu\chi}(q,\eta)\}\\
&\quad+\{q^{\xi\chi}r^{(3)}_{\mu\chi}(q,h)-h^{\xi\chi}R^{(2)}_{\mu\chi}(q,h)+h_\tau{}^\xi h^{\tau\chi}R^{(1)}_{\mu\chi}(q,h)-h^{\xi\chi}R_{\mu\chi}(q)\}+O(h^4)\ .
\end{aligned} \tag{A3}
$$

This result is manifestly erroneous. [Compare it with the proper, tensor, formula (3.11).]

APPENDIX B

As in Appendix A, we omit the tilde and write $h_{\mu\nu}$ instead of $\tilde{h}_{\mu\nu}$. It follows from (3.7) and (3.8) that

$$
\begin{aligned}
-G^{(2)}_{\mu\nu}(q,h)&\equiv -R^{(2)}_{\mu\nu}(q,h)+\tfrac{1}{2}q_{\mu\nu}q^{\alpha\beta}R^{(2)}_{\alpha\beta}(q,h)+\tfrac{1}{2}h_{\mu\nu}q^{\alpha\beta}R^{(1)}_{\alpha\beta}(q,h)\\
&\quad+\tfrac{1}{2}q_{\mu\nu}(h^{\alpha\xi}h_\xi{}^\beta R_{\alpha\beta})-\tfrac{1}{2}q_{\mu\nu}h^{\alpha\beta}R^{(1)}_{\alpha\beta}(q,h)-\tfrac{1}{2}h_{\mu\nu}(h^{\alpha\beta}R_{\alpha\beta})
\end{aligned} \tag{B1a}
$$

$$
\begin{aligned}
&=\tfrac{1}{2}\{\tfrac{1}{2}h^{\rho\tau}{}_{;\mu}h_{\rho\tau;\nu}+\tfrac{1}{2}h^\tau{}_\nu h_{\mu\tau;\rho}{}^{;\rho}-\tfrac{1}{2}h^\tau{}_\nu h_\mu{}^\rho{}_{;\rho;\tau}+\tfrac{1}{2}h_{\tau\rho}{}^{;\rho}{}_{;\mu}h^\tau{}_\nu\\
&\quad-h^\rho{}_\mu h_\nu{}^\chi R_{\chi\rho}-h^{;\tau}(h_{\tau\{\mu;\nu\}}-\tfrac{1}{2}h_{\mu\nu;\tau})-\tfrac{1}{2}h^\tau{}_\nu h_{;\mu\tau}+\tfrac{1}{2}h_{\mu\nu}h_{;\rho}{}^{;\rho}-\tfrac{1}{2}h_{\mu\nu}h_{\tau\rho}{}^{;\tau\rho}\}\\
&\quad+\tfrac{1}{4}q_{\mu\nu}(-\tfrac{1}{2}hh_{\alpha\beta}{}^{;\alpha\beta}+h^{\alpha\tau}R_{\chi\tau}h_\alpha{}^\chi+\tfrac{1}{2}hh^{\alpha\beta}R_{\alpha\beta}+\tfrac{1}{2}hh_{;\tau}{}^{;\tau}\\
&\qquad+\tfrac{1}{2}h^{\alpha\beta}h_{;\alpha\beta}+h^{;\tau}(h_\tau{}^\rho{}_{;\rho}-\tfrac{1}{2}h_{;\tau}))+\tfrac{1}{2}h_{\mu\nu}(q^{\alpha\beta}R^{(1)}_{\alpha\beta}-\tfrac{1}{2}h^{\alpha\beta}R_{\alpha\beta})\\
&\qquad+\tfrac{1}{2}\{(h^\alpha{}_\nu h_{\mu\chi;\alpha})^{;\chi}-(h^\alpha{}_\chi h_{\mu\nu;\alpha})^{;\chi}+2(h^\alpha{}_\chi h_{\alpha\{\mu;\nu\}})^{;\chi}-(h^{\alpha\beta}h_{\alpha\beta;\mu})_{;\nu}-(h^\alpha{}_\nu h_{\mu\alpha;\chi})^{;\chi}\}\\
&\qquad\quad+\tfrac{1}{4}q_{\mu\nu}\{\tfrac{3}{2}(h^{\alpha\beta}h_{\alpha\beta;\chi})^{;\chi}+(h^{\alpha\chi}h_{;\alpha})_{;\chi}-2(h^{\alpha\chi}h_{\alpha\gamma}{}^{;\gamma})_{;\chi}-(h^{\alpha\beta}h_{\alpha\chi;\beta})^{;\chi}\}\ .
\end{aligned} \tag{B1b}
$$

Here the curly brackets in subscripts stand for the symmetrization procedure: for example $h_{\tau\{\mu}{}^{;\tau}{}_{;\nu\}}\equiv\frac{1}{2}(h_{\tau\mu}{}^{;\tau}{}_{;\nu}+h_{\tau\nu}{}^{;\tau}{}_{;\mu})$. In the gauge (5.4),

$$
\begin{aligned}
-G^{(2)}_{\mu\nu}(q,h)&=\tfrac{1}{2}\{\tfrac{1}{2}h^{\rho\tau}{}_{;\mu}h_{\rho\tau;\nu}+\tfrac{1}{2}h^\tau{}_\nu h_{\mu\tau;\rho}{}^{;\rho}-h^\rho{}_\mu h_\nu{}^\chi R_{\chi\rho}\}+\tfrac{1}{4}q_{\mu\nu}h^{\alpha\tau}R_{\chi\tau}h_\alpha{}^\chi-\tfrac{1}{4}h_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}\\
&\quad+\tfrac{1}{2}\{(h^\alpha{}_\nu h_{\mu\chi;\alpha})^{;\chi}-(h^\alpha{}_\chi h_{\mu\nu;\alpha})^{;\chi}+2(h^\alpha{}_\chi h_{\alpha\{\mu;\nu\}})^{;\chi}-(h^{\alpha\beta}h_{\alpha\beta;\mu})_{;\nu}-(h^\alpha{}_\nu h_{\mu\alpha;\chi})^{;\chi}\}\\
&\quad+\tfrac{1}{4}q_{\mu\nu}\{\tfrac{3}{2}(h^{\alpha\beta}h_{\alpha\beta;\chi})^{;\chi}-(h^{\alpha\beta}h_{\alpha\chi;\beta})^{;\chi}\}\ .
\end{aligned} \tag{B2}
$$

The formula is valid in any space where (5.4b) and (5.4c) are compatible. If one restricts the geometry to FRW spaces, the product $h^{\alpha\beta}R_{\alpha\beta}$ will vanish, and one will arrive to the formula (5.6).

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