Anisotropic wormhole: Tunneling in time and space

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We discuss the structure of a gravitational Euclidean instanton obtained through coupling of gravity to electromagnetism. This Euclidean solution can be interpreted as a tunneling to a hyperbolic space (baby universe) or alternatively as a static wormhole that joins two asymptotically flat spaces of a Reissner-Nordström type solution.

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I. INTRODUCTION

Wormholes (WH's) are classical Euclidean solutions for the gravitational field coupled to matter or gauge fields that asymptotically connect two four-dimensional manifolds; they are interpreted as tunneling between the two asymptotic configurations. If a WH can be joined at $t=0$ to a hyperbolic universe whose spatial three-dimensional hypersurface is compact, the Euclidean solution can be interpreted as nucleating a baby universe (BU) from an asymptotic region and gives the semiclassical amplitude for creating a BU in that space. The BU then evolves according to its equations of motion.

A large amount of attention has been devoted to explicit WH solutions. In particular, Giddings and Strominger [1] and Myers [2] have discussed WH's generated from coupling the gravitational field to an antisymmetric tensor of rank 3 (the axion), with topology $R\times S^3$; Halliwell and Laflamme [3] have discussed solutions in the presence of a conformal massless field, and Coule and Maeda [4] have examined the case of the axion field coupled to a scalar Klein-Gordon field (in both cases with topology $R\times S^3$); Hawking [5] and Hosoya and Ogura [6] have dealt with gravity coupled to a Yang-Mills field. The magnetic monopole solution in four dimensions has been investigated by Keay and Laflamme [7]; its topology is $R\times S^1\times S^2$.

In this paper we shall investigate a different WH solution of topology $R\times S^1\times S^2$ generated by the electromagnetic (EM) field.

The outline of the paper is the following. In the next section we shall present the Euclidean solutions for gravitational and EM fields in the cases of zero and nonzero cosmological constants; then, in Sec. III, we shall discuss their interpretation as instantons describing a gravitational tunneling.

Finally, in Sec. IV, we shall deal with a different type of continuation to a hyperbolic space leading to an alternative interpretation: a static Reissner-Nordström- (RN-) type solution joined by a WH to a second RN space. According to the usual interpretation, this is evidence of a quantum tunneling: The WH yields the amplitude for a transition between two RN spaces. We shall discuss in detail the transition probability for the particle cross between the two spaces.

This way of looking at the WH as a quantum bridge connecting two classical hyperbolic spaces opens the way to the interesting speculation that singularities in the classical domain of physical, hyperbolic solutions in general relativity can be avoided by Euclidean solutions joining two spaces, as happens in the RN case that we discuss here.

We shall use natural units in Secs. II and III and geometrized units in Sec. IV.

II. EUCLIDEAN SOLUTION

Let us consider the Euclidean action for gravity minimally coupled to the EM field:

$$S_E=\int_\Omega d^4x\sqrt{g}\left[-\frac{M_P^2}{16\pi}(R+2\Lambda)+\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right]+\int_{\partial\Omega}d^3x\sqrt{h}\,\frac{M_P^2}{8\pi}\mathbf{K}\ . \tag{2.1}$$

Here Ω is a compact four-dimensional manifold, M_P is the Planck mass, R is the curvature scalar, Λ is the cosmological constant, $F_{\mu\nu}$ the usual EM field tensor with coupling constant e, $\mathbf{K}$ is the trace of the extrinsic

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curvature of the boundary $\partial\Omega$ of Ω, and h is the determinant of the induced metric over $\partial\Omega$. The EM field prevents spatially homogeneous and isotropic solutions of the field equations; hence, we look for a solution of the form

$$ds^2=dt^2+a^2(t)d\chi^2+b^2(t)d\Omega_2^2 \,, \tag{2.2}$$

where χ is the coordinate of the one-sphere, $0\le\chi<2\pi$, and $d\Omega_2^2$ represents the line element of the two-sphere. The line element (2.2) is known as the Kantowski-Sachs type [8] and describes a $R\times H$ space, where H is a three-dimensional homogeneous and nonisotropic hypersurface with topology $S^1\times S^2$.

Let us first discuss the case $\Lambda=0$. For the EM field, we choose the ansatz

$$A_\mu=A(t)\delta_{\chi\mu} \,. \tag{2.3}$$

The only nonvanishing component of the EM field is, of course,

$$F_{t\chi}=-F_{\chi t}=\dot{A}(t) \,. \tag{2.4}$$

From the equations of motion,

$$\partial_\mu(\sqrt{g}F^{\mu\nu})=0 \,, \tag{2.5}$$

we obtain

$$\dot{A}=K\frac{a}{b^2} \,, \tag{2.6}$$

where K is a constant of motion. Substituting (2.6) into (2.1), one recovers after some algebra (details are given in Appendix A) the scale factors of the one- and two-spheres. The solution is

$$ds^2=dt^2+\bar{q}^{\,2}\frac{t^2}{q^2+t^2}d\chi^2+(q^2+t^2)d\Omega_2^2 \,, \tag{2.7a}$$

$$A(t)=-\frac{\bar{q}qeM_P}{2\sqrt{\pi}}\frac{1}{\sqrt{q^2+t^2}} \,. \tag{2.7b}$$

q is connected to K by

$$q^2=\frac{4\pi}{e^2}\frac{K^2}{M_P^2} \,;$$

$\bar{q}$ is an integration constant with dimension of length whose value remains arbitrary.

Let us now study the asymptotic behaviors of the solution (2.7a). When $t\rightarrow\pm\infty$, the metric coefficients remain well defined, $a^2\rightarrow\bar{q}^{\,2}$, $b^2\rightarrow t^2$, and the line element becomes

$$ds^2=dt^2+\bar{q}^{\,2}d\chi^2+t^2d\Omega_2^2 \,. \tag{2.8}$$

Clearly, the manifold becomes flat with topology $R^3\times S^1$. At $t=0$ the metric is singular. This is only due to the choice of the coordinates that cover only half of the manifold (2.7a). Indeed, in the neighborhood of $t=0$, defining $\bar{q}=q$, (2.7a) becomes

$$ds^2=dt^2+t^2d\chi^2+q^2d\Omega_2^2 \,; \tag{2.9}$$

hence, we see that the singularity $t=0$ can be removed going to Cartesian coordinates in the (t,χ) plane. This particular case has been classified by Gibbons and Hawking [9] as a "bolt" singularity. In the neighborhood of $t=0$, the topology is locally $R^2\times S^2$ with R^2 contracting to zero as $t\rightarrow 0$. New variables can be defined such that the whole Euclidean space is represented by a single chart. Let us define

$$t=q\tan\frac{\xi}{2} \,, \tag{2.10}$$

where ξ is defined in the interval $(-\pi,\pi)$. Introduce now the new coordinates u,v as

$$u=\frac{1-\cos(\xi/2)}{\sin(\xi/2)}e^{1/\cos(\xi/2)}\cos\chi \,, \tag{2.11a}$$

$$v=\frac{1-\cos(\xi/2)}{\sin(\xi/2)}e^{1/\cos(\xi/2)}\sin\chi \,; \tag{2.11b}$$

then, from (2.10) and (2.11), the line element (2.7a) takes on a singularity-free form given by

$$ds^2=q^2\left[1+\frac{q}{\sqrt{t^2+q^2}}\right]^2e^{-2\sqrt{t^2+q^2}/q}(du^2+dv^2)$$
$$+(q^2+t^2)d\Omega_2^2 \,. \tag{2.12}$$

Recalling

$$u^2+v^2=\frac{1-\cos(\xi/2)}{1+\cos(\xi/2)}e^{2/\cos(\xi/2)} \,, \tag{2.13a}$$

$$\frac{v}{u}=\tan\chi \,, \tag{2.13b}$$

we see that the geodesics at constant χ are the straight lines passing through the origin, while the geodesics at fixed t are circles of radius

$$r=\left[\frac{1-\cos(\xi/2)}{1+\cos(\xi/2)}\right]^{1/2}e^{1/\cos(\xi/2)} \,. \tag{2.14}$$

Let us now consider the case $\Lambda\neq 0$. Using the same ansatz (2.3) for the EM field and introducing for convenience a new Euclidean time coordinate $b\equiv b(t)$, a solution is now given by

$$ds^2=\frac{b^2}{\lambda b^4+b^2-q^2}db^2+\bar{q}^{\,2}\frac{\lambda b^4+b^2-q^2}{b^2}d\chi^2+b^2d\Omega_2^2 \,, \tag{2.15a}$$

$$A(b)=-\bar{q}\frac{K}{b} \,, \tag{2.15b}$$

with $\lambda=\Lambda/3$. This solution reduces to (2.7) when $\lambda=0$ and $b^2=q^2+t^2$.

Let us separately discuss $\lambda>0$ and $\lambda<0$. In the first case, it is easy to see that the line element (2.15a) is defined for

$$b^2>q_0^2\equiv\frac{\sqrt{1+4\lambda q^2}-1}{2\lambda} \,. \tag{2.16}$$

With the transformation

$$b^2=q_0^2+\tau^2 \,, \tag{2.17}$$

(2.15a) takes the form

$$ds^2=\frac{\tau^2}{\lambda(q_0^2+\tau^2)^2+q_0^2+\tau^2-q^2}d\tau^2+\bar{q}^2\frac{\lambda(q_0^2+\tau^2)^2+q_0^2+\tau^2-q^2}{q_0^2+\tau^2}d\chi^2+(q_0^2+\tau^2)d\Omega_2^2\ , \tag{2.18}$$

where now $-\infty<\tau<+\infty$.

The asymptotic form of (2.18) for $\tau^2\to\infty$ is

$$ds^2=\frac{1}{\lambda\tau^2}d\tau^2+\lambda\bar{q}^2\tau^2d\chi^2+\tau^2d\Omega_2^2\ . \tag{2.19}$$

Contrary to the $\lambda=0$ case, now this is not a flat Euclidean space. Let us redefine the Euclidean time by

$$|\tau|=\exp(\sqrt{\lambda\tau'^2})\ , \tag{2.20}$$

so that the asymptotic form (2.19) becomes

$$ds^2=d\tau'^2+e^{2\sqrt{\lambda\tau'^2}}(\lambda\bar{q}^2d\chi^2+d\Omega_2^2)\ . \tag{2.21}$$

This line element defines an anisotropic universe whose scale factors expand exponentially; their ratio is fixed by the cosmological constant.

For $\lambda<0$, (2.15a) is defined when

$$q_-^2<b^2<q_+^2\ , \tag{2.22}$$

where

$$q_\pm^2=\frac{1\pm\sqrt{1-4|\lambda|q^2}}{2|\lambda|}\ . \tag{2.23}$$

In this latter case, we can cast the line element (2.15a) into the form (2.2) introducing a new transformation

$$\tau=\frac{1}{2\sqrt{|\lambda|}}\arcsin\left[\frac{2|\lambda|b^2-1}{\sqrt{1-4|\lambda|q^2}}\right]\ . \tag{2.24}$$

We then have

$$ds^2=d\tau^2+\frac{\bar{q}^2}{2}\frac{[1-4|\lambda|q^2]\cos^2[2\sqrt{|\lambda|}\tau]}{1+\sqrt{1-4|\lambda|q^2}\sin[2\sqrt{|\lambda|}\tau]}d\chi^2+\frac{1}{2|\lambda|}\{1+\sqrt{1-4|\lambda|q^2}\sin[2\sqrt{|\lambda|}\tau]\}d\Omega_2^2\ , \tag{2.25a}$$

$$A(\tau)=-\frac{\bar{q}K\sqrt{2|\lambda|}}{[1+\sqrt{1-4|\lambda|q^2}\sin[2\sqrt{|\lambda|}\tau]]^{1/2}}\ . \tag{2.25b}$$

The important feature of (2.25a), which will be relevant in the forecoming discussion, is that of being a periodic solution in the Euclidean time τ.

III. SOLUTION IN HYPERBOLIC SPACETIME: GRAVITATIONAL TUNNELING

The instantons (2.7) and (2.25) can be joined to real, hyperbolic universes; these are the *bounce* solutions of the gravitational tunneling.

To find the hyperbolic manifolds describing the tunneling spacetimes, we have to investigate hyperbolic solutions of the coupled gravity and EM fields with the same symmetry as just discussed in the Euclidean case. Using the hyperbolic version of the action (2.1) when $\Lambda=0$, a solution is given by

$$ds^2=-dt^2+\bar{q}^2\sin^2\left[\frac{t}{q}\right]d\chi^2+q^2d\Omega_2^2\ , \tag{3.1a}$$

$$A(t)=\frac{\bar{q}eM_p}{2\sqrt{\pi}}\cos\left[\frac{t}{q}\right]\ . \tag{3.1b}$$

Here, as before, the topology is $R\times H$, χ is defined in the interval $[0,2\pi[$ and $-\infty<t<+\infty$. The line element (3.1a) describes a nonisotropic universe with a constant two-sphere radius and a periodic one-sphere scale factor taking values in the interval $[0,\bar{q}]$. In the neighborhood of $t=0$, the line element (3.1a) with $\bar{q}=q$ reduces to the form

$$ds^2=-dt^2+t^2d\chi^2+q^2d\Omega_2^2 \tag{3.2}$$

and the EM field (3.1b) is well defined for all t's. However, as for the Euclidean case (2.9), the singularity at $t=0$ can be removed; indeed, the curvature tensor is regular there even if at $t=0$ the physical size in χ is zero. In fact, in the neighborhood of $t=0$ the topology is locally $R^{1,1}\times S^2$ and the three-dimensional spatial hypersurface H of (3.1a) becomes homotopic to S^2 and a point. Thus (3.1a) represents a universe which periodically reproduces itself with period πq.

The Euclidean solution (2.7) can be joined to the hyperbolic solution (3.1) at $t=0$. In fact, solutions (2.7) with $t\in]-\infty,0[$ and (3.1) with $t\in]0,\infty[$ satisfy Darmois conditions for a change of signature at $t=0$ [10]; namely, the first and second fundamental forms of the three-dimensional hypersurfaces H in (2.7a) and (3.1a) coincide smoothly for $t\to0^\pm$. Moreover, also the EM field is continuous with its derivative on the hypersurface $t=0$, where the change of signature occurs. Evidently, the EM field is well behaved on the matching hypersurface $t=0$ because both the Euclidean and hyperbolic manifolds are well defined at $t=0$. The regularity of solution (2.7) and its asymptotic behavior for $t\to\pm\infty$ where the line element reduces to (2.8) and the EM field vanishes allow one to interpret that solution as an instanton which provides a tunneling between a flat vacuum hyperbolic region and the manifold (3.1).

In conclusion, solution (2.7) describes the nucleation of a nonisotropic BU at $t=0$ starting from an original flat spacetime. The hypersurface of the signature change is in our case two dimensional, and this corresponds to the particular situation of a BU nucleated in the phase of maximum shrinkage of the spatial metric.

In the general case $\lambda\neq0$, the Euclidean line element (2.15a) can be interpreted in a similar way. For instance, in the case $\lambda<0$ the instanton (2.25) describes a tunneling between the hyperbolic universes

$$ds^2=-dt^2+\sin^2\left[\omega_-\left[t+\frac{\pi}{4\sqrt{|\lambda|}}\right]\right]d\chi^2+q_-d\Omega_2^2 \tag{3.3a}$$

and

$$ds^2=-dt^2+\sinh^2\left[\omega_+\left[t-\frac{\pi}{4\sqrt{|\lambda|}}\right]\right]d\chi^2+q_+d\Omega_2^2\ , \tag{3.3b}$$

where

$$\omega_\pm^2=\sqrt{1-4|\lambda|q^2}/q_\pm^2\ . \tag{3.4}$$

Here the tunneling between the latter manifolds occurs when the one-sphere radius is zero; they notably describe an oscillating BU in (3.3a) and an ever expanding universe for $t>\pi/4\sqrt{|\lambda|}$ in (3.3b).

Let us now compute the probability amplitude for the formation of a BU in the case $\Lambda=0$, starting from (2.1). On the field equations, $R=0$ and the Euclidean action reduces to

$$S_E=\frac{1}{4e^2}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}\ . \tag{3.5}$$

From (2.7) we obtain

$$\begin{aligned}S_E&=\frac{4\pi^2K^2}{e^2}\int_0^{+\infty}dt\frac{a}{b^2}\\&=\pi M_P^2\bar{q}q=2\pi^{3/2}\bar{q}\frac{K}{eM_P}\ .\end{aligned} \tag{3.6}$$

The probability Γ of formation of a BU in a Planck volume and in a Planck time is given by

$$\Gamma=e^{-S_E}=\exp(-\pi M_P^2\bar{q}q)\ . \tag{3.7}$$

In order to have a probability of the order of unity, the constants appearing in the solution satisfy

$$\bar{q}q\approx\frac{1}{\pi M_P^2}\ . \tag{3.8}$$

Hence the nucleation probability (3.7) is maximum for BU's with dimension of order of the Planck length.

IV. STATIC WORMHOLE INTERPRETATION

Solution (2.7) can also be interpreted as a Euclidean WH joining two isometric, asymptotically flat spacetimes, described by RN type of solutions.

To see this let us make a change of coordinates in (2.2) by substituting $\chi\to iT$, with T having the dimension of a length; for clarity, we shall put $t\equiv r$. Throughout this section we shall use geometrized units (velocity of light and gravitational constant equal to 1). Solving for the hyperbolic version of the action (2.1), we obtain the solution

$$ds^2=-\frac{r^2}{r^2-Q^2}dT^2+dr^2+(r^2-Q^2)d\Omega_2^2\ , \tag{4.1a}$$

$$A(r)=-\frac{K}{\sqrt{r^2-Q^2}}\ , \tag{4.1b}$$

where Q is a constant. Solution (4.1) is defined for $r^2>Q^2$, and at $|r|=|Q|$ there is a curvature singularity.

In the region $|r|<|Q|$, we have no hyperbolic solution; however, solution (2.7), with $t\equiv r$, reduces to (4.1) if we Wick rotate the coordinate χ as $\bar{q}\chi=iT$ and impose that the electric field remains real. Now, since (2.7) with $t\equiv r$ is well behaved for $r\in]-\infty,\infty[$, we can match the two branches of solution (4.1) with the Euclidean solution (2.7) at some $r^2=Q^2+\epsilon^2$ with ϵ arbitrary, via the complexification of χ as stated. With this procedure, setting $\epsilon\to0^\pm$ we obtain a Euclidean WH which joins the two branches of the solution (4.1).

The parameter Q enter (4.1) can be given a particular interpretation. Denoting $R^2\equiv r^2-Q^2$, we obtain

$$ds^2=-\left[1+\frac{Q^2}{R^2}\right]dT^2+\left[1+\frac{Q^2}{R^2}\right]^{-1}dR^2+R^2d\Omega_2^2\ , \tag{4.2}$$

where the radial coordinate R ranges in $]0,\infty[$.

The line element in the form (4.2) can be regarded as of a RN solution with effective gravitational mass equal to $-Q^2/2R$. However, the constant Q is not a real charge since there are no physical charges in the field, as can be deduced from Eq. (2.5), but is a measure of the electric field flux through the WH throat at $R=0$. In fact, since the electric field is radial in R, its integral flux through a sphere containing the origin $R=0$ is equal to

$$\Phi=4\pi Q\ . \tag{4.3}$$

Therefore the constant Q only fixes the amount of flux that we want through any given surface containing the origin, similar to what is done for the axionic field in [1]. Thus the electric field extends smoothly beyond the WH throat to the asymptotic infinities of the isometric spacetimes, generating in both cases an *apparent* charge Q.

Let us now discuss the traversability of the WH. Clearly, in order to cross the WH, a classical particle must be able to reach it. We shall study the equation of motion for a test particle, having an electric charge per unit mass $\tilde{q}$, total specific energy E, and specific angular momentum L with respect to the flat infinity that approaches $R=0$. This is relevant since the particle can cross the WH throat only if it gets to $R=0$ (classically or via quantum tunneling). We assume the motion in the equatorial plane, $\theta=\pi/2$.

The momenta and equations of motion are

$$P_T\equiv-\left[\frac{R^2+Q^2}{R^2}\dot{T}+\frac{\beta}{R}\right]=-E\ , \tag{4.4a}$$

$$P_R\equiv\frac{R^2}{R^2+Q^2}\dot{R}\ , \tag{4.4b}$$

$$P_\phi\equiv R^2\dot{\phi}=L\ , \tag{4.4c}$$

$$\begin{aligned}\dot{R}^2&=\left[E-\frac{\beta}{R}\right]^2-\left[1+\frac{L}{R^2}\right]\left[1+\frac{Q^2}{R^2}\right]\\&\equiv(E-V_+)(E-V_-)\ ,\end{aligned} \tag{4.4d}$$

where $V_\pm$ are the potential barriers given by [11]

$$V(R;\beta,Q,L)_\pm=R^{-2}[\beta R\pm(R^2+Q^2)^{1/2}(R^2+L)^{1/2}]\ , \tag{4.5}$$

with $\beta=\bar{q}Q$. We shall study analytically the graph of the function V_+. The behavior of V_- is easily deduced from the relation

$$V_-(\beta)=-V_+(-\beta)\ . \tag{4.6}$$

The analysis of the potential barriers (4.5) with $L\neq 0$ shows that the barriers are repulsive for all values of the parameters. On the contrary, when $L=0$, namely, when the motion is strictly radial, there is a class of trajectories which can reach the WH throat at $R=0$. We shall discuss extensively this latter case, referring to Appendix B for the general situation.

The potential barrier V_+ for $L=0$ reads, from (4.5),

$$V_+(R;\beta,Q)=R^{-1}[\beta+(R^2+Q^2)^{1/2}]\ . \tag{4.7}$$

When $R\to\infty$, V_+ behaves as

$$V_+\approx 1+\frac{\beta}{R}\ ,$$

while when $R\to 0$ we have

$$V_+\approx\frac{\beta+|Q|}{R}\to\begin{cases}+\infty\ , & \beta>-|Q|\ ,\\ 0\ , & \beta=-|Q|\ ,\\ -\infty\ , & \beta<-|Q|\ .\end{cases} \tag{4.8}$$

The conditions $V_+=1$ and 0 are satisfied, respectively, when $\beta=\beta_1\equiv R-(R^2+Q^2)^{1/2}$ and $\beta=\beta_0\equiv-(R^2+Q^2)^{1/2}$. They are shown in Fig. 1(a), while the graphs of V_+ are shown in Fig. 1(b) for the cases $\beta>-|Q|$, $\beta=-|Q|$, and $\beta<-|Q|$.

We may repeat the analysis for the case of V_- using the symmetry (4.6). The conclusion is that the point $R=0$ can be reached if and only if

$$\beta^2\geq Q^2\ . \tag{4.9}$$

In this case the particles may reach the Euclidean WH and eventually emerge in the other region [12,13]. The transition probability $T_{\rm WH}$ for tunneling by the Euclidean WH is proportional to $\exp(-2S_{\rm cl})$ where $S_{\rm cl}$ is given by (for clarity, we leave here the natural units)

$$S_{\rm cl}=\pi q\bar{q}M_P^2\frac{\sqrt{2}-1}{\sqrt{2}}\ . \tag{4.10}$$

The transition probability characterizes the WH and is independent of the particle's properties since the latter, provided they satisfy (4.9), all reach the WH throat, regardless of their energy.

The radial particles which do not satisfy (4.9) or those which have a nonzero angular momentum may cross the WH, reaching $R=0$ as a result of a quantum tunneling with nonzero quantum probability. Indeed, let us go back to Eq. (4.4d) and use it to establish the equation for the wave function, taking into account that $P_R\to -i d/dR$ is given by Eq. (4.4b):

$$-\frac{d^2}{dR^2}\Psi=\frac{1}{(R^2+Q^2)^2}[R^2(RE-\beta)^2-(R^2+L)(R^2+Q^2)]\Psi\ . \tag{4.11}$$

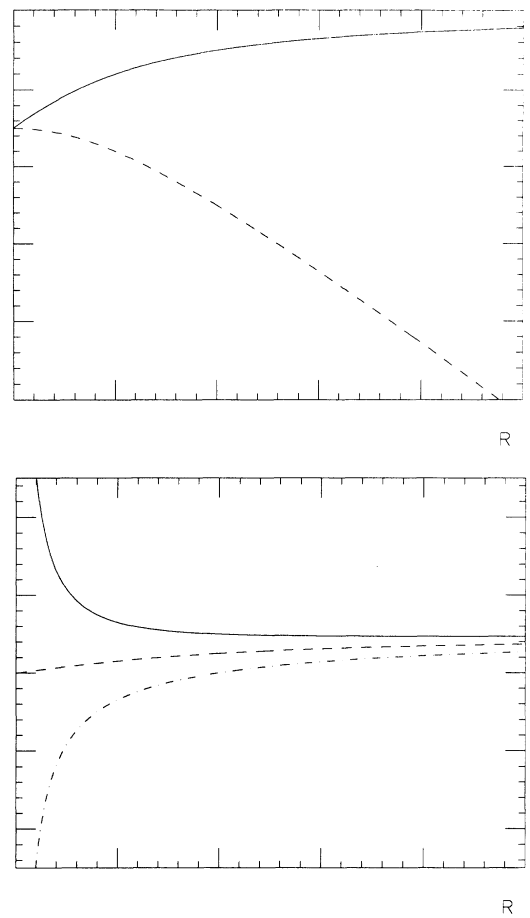

FIG. 1. (a) Plot of the functions β_1 (solid line) and β_0 (dashed line) when $L=0$. (b) Behavior of the effective potential V_+ when $\beta>-|Q|$ (solid line), $\beta=-|Q|$ (dashed line), and $\beta<-|Q|$ (dot-dashed line).

The overall transition probability is given by

$$T=T_0^2T_{\rm WH}\ , \tag{4.12}$$

where $T_{\rm WH}$ is due to the tunneling by the Euclidean WH and T_0 is the usual quantum transition probability of the barrier from $R=R_0$, where R_0 is the classical turning point, to $R=0$.

For the evaluation of T_0 in the WKB approximation, the relevant quantity is

$$\alpha=\int_0^{R_0}dR\frac{1}{R^2+Q^2}[(R^2+L)(R^2+Q^2)-R^2(ER-\beta)^2]^{1/2}\ , \tag{4.13}$$

which is finite and has a particularly simple expression for $L=0$.

V. CONCLUSIONS

In this paper we have discussed Euclidean solutions of the Einstein equations for gravity coupled to the EM field. These solutions describe a tunneling to a BU or a static WH depending on the coordinate chosen to be complexified. In Sec. III we have seen that solution (2.7) describes the nucleation of a BU starting from a flat region. The probability amplitude for this process is given in (3.7); on the contrary, solution (2.25) describes a tunneling between an oscillating universe (3.3a) and a nonoscillating universe (3.3b). In the last section, we have used the Euclidean solution in order to obtain a finite traversability amplitude between two spacetimes of RN type; we may call this a space-tunneling WH. We stress here that the motivations for interpreting solution (4.2) as an overcharged RN spacetime, even if there are no physical charges in the source field, arise from the fact that the constant flux of the electric field through the WH throat can be regarded as originating from an apparent charge. This latter interpretation of a space-tunneling WH is in the direction of the proposal by Wheeler ([14]; see also [15]) of a pair of extreme RN black holes identified at their throats. In the present case, the joining of two RN spacetimes happens through quantum tunneling.

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APPENDIX A

Let us deduce here solution (2.7). The Einstein equations are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi G T_{\mu\nu} . \tag{A1}$$

Substituting Eqs. (2.4) and (2.6) in the expression for $T_{\mu\nu}$, we may find its components in terms of the scale factors

$$\begin{aligned} T_{tt}&=\frac{1}{2e^2}\frac{K^2}{b^4} ,\\ T_{\chi\chi}&=\frac{1}{2e^2}\frac{K^2}{b^4}a^2 ,\\ T_{ij}&=-\frac{1}{2e^2}\frac{K^2}{b^4}g_{ij} . \end{aligned} \tag{A2}$$

The ensuing equations for the two scale factors are then

$$\frac{\dot b^2}{b^2}-\frac{1}{b^2}+2\frac{\dot a\dot b}{ab}=\frac{q^2}{b^4} , \tag{A3a}$$

$$2\frac{\ddot b}{b}+\frac{\dot b^2}{b^2}-\frac{1}{b^2}=\frac{q^2}{b^4} , \tag{A3b}$$

$$\frac{\ddot b}{b}+\frac{\ddot a}{a}+\frac{\dot a\dot b}{ab}=-\frac{q^2}{b^4} . \tag{A3c}$$

In (A3b) only $b(t)$ appears; by the substitution $\dot b^2=f$, (A3b) takes the form

$$f'+\frac{1}{b}f-\frac{1}{b}=\frac{q^2}{b^3} . \tag{A4}$$

Putting then $b=e^h$, we get

$$\frac{df}{dh}+f=1-q^2e^{-2h} , \tag{A5}$$

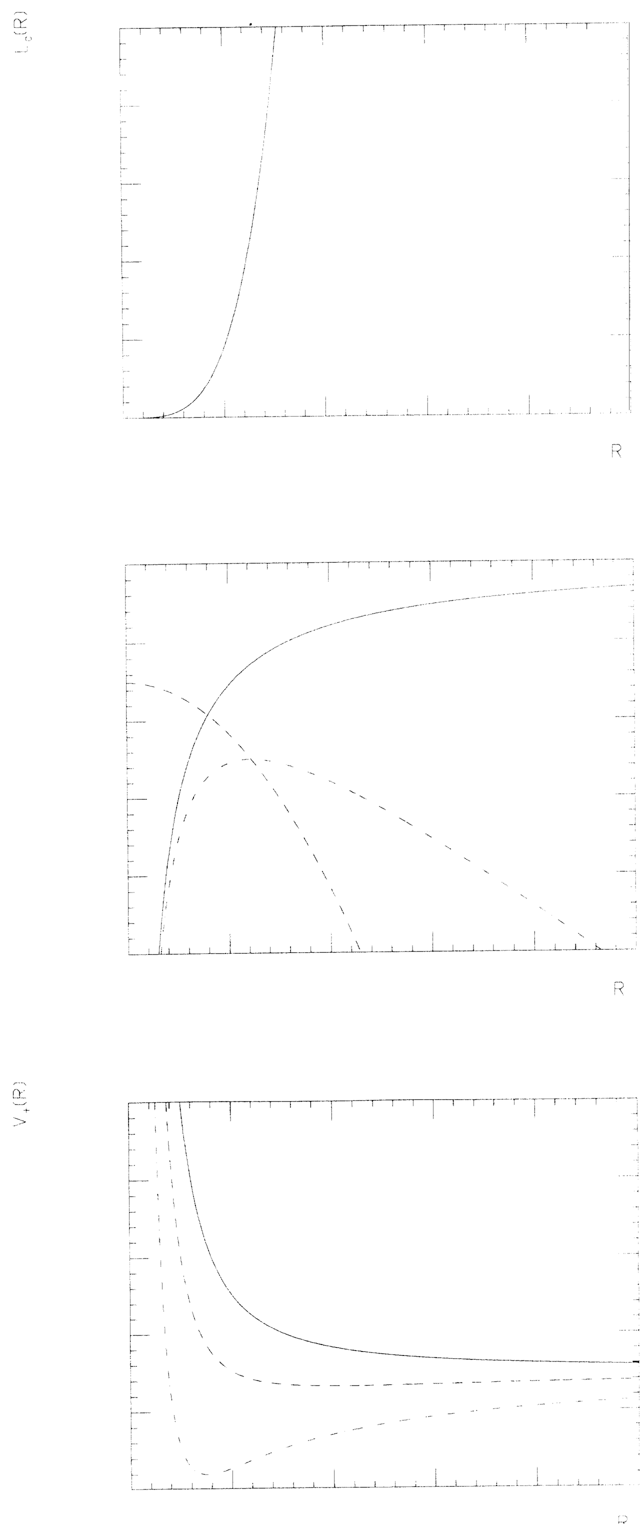

FIG. 2. (a) Plot of the function L_c, which is the locus of the points where the function β_0 has a maximum. (b) Plots of the function β_1 (solid line), which identifies where $V_+=1$, and of the function β_0 (dot-dashed line), which identifies where $V_+=0$. The locus of the maxima of β_0 is along the dashed curve, plot of the function β_{0c}. (c) Behavior of the effective potential V_+ as function of R. The solid line represents the case $\beta>0$, the dashed line for $\beta<0$ and $\beta>\beta_{0c}$, and the dot-dashed line when $\beta<0$ and $\beta<\beta_{0c}$.

whose general solution is

$$f=K_1e^{-h}+1-q^2e^{-2h} \; . \tag{A6}$$

K_1 is an integration constant. In what follows we shall only consider $K_1=0$. In the old variables, then,

$$\dot{b}^2=1-\frac{q^2}{b^2} \; . \tag{A7}$$

So finally

$$b(t)=\sqrt{q^2+t^2} \; . \tag{A8}$$

By substitution of (A8) into (A3a), the equation for the remaining scale factor is

$$\frac{\dot{a}}{a}=\frac{q^2}{t(q^2+t^2)} \; , \tag{A9}$$

whose solution is

$$a(t)=\pm\bar{q}\frac{t}{\sqrt{q^2+t^2}} \; , \tag{A10}$$

where $\bar{q}$ is an integration constant.

The signs $\pm$ refer, respectively, to the submanifolds with $t>0$ or $t<0$, having defined $a(t)$ as non-negative.

APPENDIX B

We give here the details of the equations of motion in the case $L\neq 0$. The asymptotic behavior of V_+ for large R is

$$V\approx\frac{\beta+R}{R}\rightarrow\begin{cases}1_+ \; , & \beta\geq 0 \; , \\ 1_- \; , & \beta<0 \; .\end{cases}$$

For $R\rightarrow 0$ we have

$$V_+\approx\frac{|Q|\sqrt{L}}{R^2}\rightarrow+\infty \; .$$

The condition $V_+=1$ is equivalent to

$$\beta=R^{-1}[R^2-(R^2+Q^2)^{1/2}(R^2+L)^{1/2}]\equiv\beta_1 \; . \tag{B1}$$

Clearly, $\beta_1<0$ always and $\lim_{R\rightarrow\infty}\beta_1=0$, $\lim_{R\rightarrow 0}\beta_1=-\infty$. The function β_1 is plotted in Fig. 2(b). The condition $V_+=0$ is satisfied when

$$\beta=-R^{-1}(R^2+Q^2)^{1/2}(R^2+L)^{1/2}\equiv\beta_0 \; . \tag{B2}$$

Here again $\beta_0<0$ always; the graph of $\beta_0(R)$ is easily deduced from its limits

$$\lim_{R\rightarrow\infty}\beta_0=\lim_{R\rightarrow 0}\beta_0=-\infty$$

and from the locus of its critical points, namely,

$$L=R^4Q^{-2}\equiv L_c \; .$$

The function L_c is plotted in Fig. 2(a) in the $(L-R)$ plane. From (B1) and (B2) we find $\beta_1=R+\beta_0$; hence, $\beta_1\geq\beta_0$, the equality sign holding only in the limit $R\rightarrow 0$. The value of β_0 at its maximum is given by

$$\beta_0(R;Q,L_c)\equiv\beta_{0c}=-\frac{R^2+Q^2}{|Q|} \; ,$$

and its graph is plotted in Fig. 2(b) (dashed line). We are now in the position to draw the potential curves $V_+(R;\beta,Q,L)$ as function of R for any given set of values (β,Q,L). They are shown in Fig. 2(c) for three different values of β, namely, (1) $\beta>0$, (2) $\beta<0$ and $\beta>\beta_{0c}$, and (3) $\beta<0$ and $\beta<-\beta_{0c}$.

The classical motion is only allowed when the total energy E of the charged test particle satisfies the condition $E\geq V$; hence, when the angular momentum L is different from zero, we see by a direct inspection of Fig. 2 that (i) the WH throat $R=0$ cannot be reached classically since the field is repulsive to all particles, either charged or not, and (ii) a sea of negative energy particles is allowed in the vicinity of the throat. This effect is a well-known property of the RN solution and allows for electric field energy extraction via quantum tunneling.

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