On the Vaidya limit of the Tolman model

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We show that the only Tolman models which permit a Vaidya limit are those having a dust distribution that is hollow, such as the self-similar case. Thus the naked shell-focusing singularities found in Tolman models that are dense through the origin have no Vaidya equivalent. This also casts light on the nature of the Vaidya metric. We point out a hidden assumption in Lemos' demonstration that the Vaidya metric is a null limit of the Tolman metric, and in generalizing his result, we find that a different transformation of coordinates is required.

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INTRODUCTION

Recently Lemos [1] showed that the Vaidya metric, describing radially directed incoherent radiation (spherically symmetric null dust), can be obtained from the Tolman metric, which represents a spherically symmetric distribution of pressureless matter (dust), by taking a null limit. This surprising and intriguing insight was inspired by the very strong similarities, quantitative as well as qualitative, between the naked shell-focusing singularities (discovered by Eardley and Smarr [2]) that appear in the self-similar forms of these metrics at the moment the crunch singularity forms, see for example [3–11, and extensive references in footnote 2 of 12]. We discuss the nature of an origin of spherical coordinates in Tolman models, and show that a Vaidya limit cannot be extended to such a point. We show that one of Lemos' assumptions can be relaxed if a different coordinate transformation is used.

The incoming Vaidya metric [13,14; see also 15] is

$$ds^2 = 2dvdR - \left(1 - \frac{2M}{R}\right)dv^2 + R^2 d\Omega^2, \qquad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$ is the metric on a two-sphere, the areal radius is positive, $R > 0$, and $M = M(v) > 0$ is an arbitrary function of the null coordinate v, representing the effective gravitational mass inside v. The only nonzero Einstein tensor component and the Kretschmann scalar $K = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ are

$$G_{vv}^V = \frac{2}{R^2}M^*, \qquad (2)$$

$$K^V = \frac{48M^2}{r^6}, \qquad (3)$$

where the asterisk denotes $\partial/\partial v$, and superscripts V and T are used where necessary to distinguish quantities in the Vaidya and Tolman models.

The Tolman metric [16,17] uses synchronous coordi-

nates that are comoving with the dust particles:

$$ds^2 = -dt^2 + \frac{R'^2}{1+f}dr^2 + R^2 d\Omega^2, \qquad (4)$$

where the prime denotes $\partial/\partial r$, $f = f(r)$ is an arbitrary function of coordinate radius that determines the local spatial geometry, as a function of r, see [18,19]. The areal radius $R = R(t,r)$ is a solution of

$$\dot{R}^2 = \frac{2M}{R} + f, \qquad (5)$$

where the overdot denotes $\partial/\partial t$, and $M = M(r) > 0$ is a second arbitrary function. Comparing this equation with its Newtonian analogue for the kinetic plus potential energy of a radially moving particle of mass m at a distance x from the center of a spherically symmetric dust cloud with density distribution $\rho_N(x)$,

$$\frac{m}{2}\left(\dot{x}^2 - \frac{2M_N(x)}{x}\right) = E, \qquad (6)$$

where

$$M_N(x) = \int_0^x 4\pi x^2 \rho_N(x)dx, \qquad (7)$$

we obtain the interpretation that $M(r)$ is the gravitational mass within comoving radius r, and $f(r)$ is twice the energy per unit mass of the particles at r. (The principal difference between these two equations is the replacement of the radial distance x by the areal radius R.) For $f > 0$ (or rather $Rf/M > 0$) the evolution of the areal radius for a collapsing model is hyperbolic:

$$R = \frac{M}{f}(\cosh\eta - 1), \qquad (8)$$

$$(\sinh\eta - \eta) = \frac{f^{3/2}(a-t)}{M}, \qquad (9)$$

where the third arbitrary function $a = a(r)$ gives the time at which $R = 0$ — the big crunch. (Parabolic and elliptic solutions exist for $f = 0$ and $f < 0$.) Since the pressure is zero, the dust particles (which stay at constant r, θ, ϕ) follow geodesics of the spacetime. It can be shown in

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general [20,18] that, for the collapsing models,

$$R' = \left(\frac{M'}{M} - \frac{f'}{f}\right) R + \left[a' - \left(\frac{M'}{M} - \frac{3f'}{2f}\right)(a-t)\right] \dot{R}.$$

$$(10)$$

The density and the Kretschmann scalar are given by

$$8\pi\rho_T = G_{tt}^T = \frac{2M'}{R^2 R'},$$

$$(11)$$

$$K^T = \frac{48M^2}{R^6} - \frac{32MM'}{R^5 R'} + \frac{12M'^2}{R^4 R'^2}.$$

$$(12)$$

LEMOS' METHOD

We here outline the approach used by Lemos, although we find it convenient to delay taking the null limit until a slightly later stage in the working. He initially makes the assumption of self-similarity in both metrics, for simplicity, and later drops it. That assumption is not made here. The Tolman line element (4) may be transformed from coordinates (t, r) to (t, R) by means of

$$dR = \dot{R}dt + R'dr \rightarrow R'dr = dR - \dot{R}dt,$$

$$(13)$$

which leads to

$$ds^2 = -\left(1 - \frac{2M}{R}\right)\frac{dt^2}{(1+f)} - \frac{2\dot{R}}{(1+f)}dtdR$$
$$+ \frac{dR^2}{(1+f)} + R^2 d\Omega^2,$$

$$(14)$$

where the new g_{tt} has been simplified using (5).

Now the limit of interest is that in which f is allowed to diverge, while M and R are both required to remain finite:

$$f \rightarrow +\infty \quad , \quad 0 \le R, M < \infty$$

$$(15)$$

Equation (8) shows that in this limit $\cosh\eta$ must also diverge, so that $\cosh\eta \rightarrow \sinh\eta \rightarrow e^\eta/2$ and (8) plus (9) simplify to

$$R \rightarrow \sqrt{f}(a-t)$$

$$(16)$$

and, for finite R, $(a-t)$ must be vanishingly small. Similarly (5) for collapsing models becomes

$$\dot{R} \rightarrow -\sqrt{f}$$

$$(17)$$

and the derivative of (16) [or alternatively substituting for $(a-t)$ and $\dot{R}$ from (16) and (17) in (10)] gives

$$R' \rightarrow \frac{Rf'}{2f} + a'\sqrt{f}.$$

$$(18)$$

Lemos then states that the transformation

$$v = \frac{t}{\sqrt{f}} + \frac{R}{f}$$

$$(19)$$

converts (14) into the Vaidya metric (1), in the limit $f \rightarrow \infty$. Since (19) and (16) imply that

$$v \rightarrow \frac{a}{\sqrt{f}},$$

$$(20)$$

the new coordinate becomes a function of r only, in the limit, so we can now write $M \rightarrow M(v)$.

We note, however, that a constant f, inherited from the self-similar case, must still be assumed in order to get this result. If we do not make this assumption, then (19) leads to

$$dv = \frac{dt}{\sqrt{f}} + \frac{dR}{f} - \left(\frac{t}{2f^{3/2}} + \frac{R}{f^2}\right)f'dr$$

$$(21)$$

$$= \frac{dt}{\sqrt{f}} + \frac{dR}{f} - \left(\frac{t}{2f^{3/2}} + \frac{R}{f^2}\right)f'\frac{(dR - \dot{R}dt)}{R'}$$

$$(22)$$

and, after substituting for t, $\dot{R}$, and R' from (16)–(18), to

$$dv \rightarrow \left(\frac{dt}{\sqrt{f}} + \frac{dR}{f}\right)(1 - X),$$

$$(23)$$

where

$$X = \frac{R + a\sqrt{f}}{R + 2f^{3/2}(a'/f')},$$

$$(24)$$

so that (14) in the limit becomes

$$ds^2 \rightarrow -\left(\frac{f}{1+f}\right)\left[2 + \frac{1}{f}\left(1 - \frac{2M}{R}\right)\right]dR^2 + \left(\frac{f}{1+f}\right)\left(\frac{2}{1-X}\right)\left[1 + \frac{1}{f}\left(1 - \frac{2M}{R}\right)\right]dvdR$$
$$- \left(\frac{f}{1+f}\right)\left(\frac{1}{1-X}\right)\left(1 - \frac{2M}{R}\right)dv^2 + R^2 d\Omega^2.$$

$$(25)$$

The limiting behavior of X is not at all clear, as the relationship between $a(r)$ and $f(r)$ is arbitrary in general, and the limiting behavior of a is not specified.

THE PROBLEM OF THE ORIGIN AND THE FORM OF $f(r)$

In all Tolman models describing a collapsing dust cloud which exhibit a naked singularity, this singularity occurs at the moment of collapse $t = a$, at the origin ($r = 0$ being

the natural choice). The origin of spherical coordinates is specified by $R(t, r = 0) = 0, \forall\ t$ and we also have $\dot{R}(t, r = 0) = 0, \forall\ t$, which, by Eq. (5), requires $M(0) = 0$ as well as $f(0) = 0$ at the origin, for example, the homogeneous case [the dust filled Friedmann-Lemaître-Robertson-Walker-model]. Can we extend Lemos' result for the null limit to cases where f does not diverge at the origin? Clearly the functional form of $f(r)$ must allow $f(0) = 0$ — for example $f = pr^2$, $p \rightarrow \infty$.

Consider cases with f finite at $r = 0$, such as the

nonparabolic self-similar models. Assuming $R, M \geq 0$, Eq. (5) shows that $\dot{R}(t, r = 0) \geq f \neq 0$. If $r = 0$ is approached along a constant t surface, with $a(r)$ finite near $r = 0$, (8) and (9) show that either (a) $M \to 0$ so that $\eta \to \infty$ and $R \to \sqrt{f}t$, or (b) M remains finite so that η and R also remain finite. Case (a) represents a hollow dust cloud — it can be matched at $r = 0$ onto a vacuum Tolman (Minkowski) spacetime with $M(r) = 0$ and a true origin at some negative r value where $f = 0$. Case (b) either (i) contains more dust inside $r = 0$, with the true origin again at $f = 0$, i.e., $r = 0$ is not the centre of the cloud, (ii) it contains the Schwarzschild vacuum inside $r = 0$, with no origin, or (iii) it contains a dust filled version of the Schwarzschild-Kruskal-Szekeres topology [19]. In (ii) and (iii), f must pass through zero and reach -1 in order to form the throat, rising to $f \geq 0$ in the second sheet, and $M, R < \infty$ everywhere that $f < 0$. Clearly particle world lines having $f(0) > 0$ are not at the origin, but they do collapse to zero and begin the formation of the singularity.

Furthermore, since shell-focusing singularities do form in Tolman models with normal origins, can the detailed similarity between the naked singularities of the two metrics be extended to such cases, or is constant f required?

Note also that the coordinate r is eliminated by the first transformation (13) and then effectively reintroduced, in the limit $v = a(r)/\sqrt{f(r)}$, via the second one (19). Since the Tolman coordinate r is comoving with the dust particles, and the Vaidya coordinate v is comoving with the shells of radiation, one might expect v to be the direct limit of r. This is consistent with the interpretation of f as an energy parameter which goes to infinity, meaning that the Tolman particle geodesics are asymptotically null. Since a particle staying at the origin of spherical symmetry cannot be moving at light speed, this suggests that a Vaidya limit is not achieveable here.

THE NULL LIMIT FOR GENERAL f AND a

Consider approaching the origin on a constant η surface. Equation (8) shows that Rf/M remains constant, whereas (5) shows that both M/R and f go to zero there. Thus the Vaidya limit could be described by

$$\frac{Rf}{M} \to \infty \ , \quad 0 \leq R, M < \infty \quad (26)$$

which does not necessarily require $f \to \infty$ at $r = 0$. The limiting forms (16)–(18) of R, $\dot{R}$, and R' are unchanged by this adjustment.

Starting again from (4), we transform from (t, r) to (R, r) as our coordinates, thus substituting for t rather than r,

$$dR = \dot{R}dt + R'dr \to dt = (dR - R'dr)/\dot{R} \quad (27)$$

and apply (5) to simplify the resulting g_{rr}:

$$ds^2 = -\frac{1}{\dot{R}^2}dR^2 + 2\frac{R'}{\dot{R}^2}dRdr - \left(1 - \frac{2M}{R}\right)\frac{R'^2}{(1+f)\dot{R}^2}dr^2$$
$$+ R^2 d\Omega^2. \quad (28)$$

From (17) and (18), we have the following limiting forms of the extra factors that do not appear in (1)

$$\frac{1}{\dot{R}^2} \to \frac{1}{f}, \quad \frac{R'}{\dot{R}^2} \to \left\{\frac{a'}{\sqrt{f}} + \frac{Rf'}{2f^2}\right\} \ ,$$

$$\frac{R'^2}{(1+f)\dot{R}^2} \to \left(\frac{f}{1+f}\right)\left\{\frac{a'}{\sqrt{f}} + \frac{Rf'}{2f^2}\right\}^2. \quad (29)$$

The limiting transformation (20) takes care of the first term in the curly brackets $(a'/\sqrt{f})$, but not the second $(Rf'/2f^2)$, and without knowing both $a(r)$ and $f(r)$ [i.e., $f(a)$], it cannot be discounted. The second term is dominant if

$$\left(\frac{Rf'}{2f^2}\right)\bigg/\left(\frac{a'}{\sqrt{f}}\right) = \frac{R}{2f^{3/2}}\frac{df}{da} \to \infty. \quad (30)$$

An example of an $f(a)$ that makes the second term dominant almost everywhere is

$$f = a\ln(p) + \sin(p^n a), \quad n = \text{const}, \quad p \to \infty, \quad (31)$$
$$\frac{df/da}{f^{3/2}} \to \frac{f + ae^{nf/a}\cos(ae^{nf/a})}{af^{3/2}} \quad (32)$$

but this wildly oscillating form is very unrealistic. The conditions for no shell crossings [18] for collapsing hyperbolic Tolman models require $f' > 0$ and $a' > 0$ wherever $M' > 0$, i.e., $df/da > 0$, so adding a linear term to remove negative gradients gives a vanishing second term

$$f = 2p^n a + \sin(p^n a), \quad n = \text{const}, \quad p \to \infty, \quad (33)$$
$$\frac{df/da}{f^{3/2}} \to \frac{2 + \cos(f/2)}{2a\sqrt{f}}. \quad (34)$$

The most rapid uniform divergence of df/da relative to f we have been able to find for $df/da > 0$ still leaves $(df/da)/f^{3/2}$ vanishing. It is expressed in terms of computer notation ˆ for raising to the power:

$$f = ap\hat{\ }(p\hat{\ }[p\hat{\ }\cdots(p\hat{\ }a)]), \quad p \to \infty, \quad (35)$$

$$\frac{df/da}{f^{3/2}} \to \frac{\ln(f)\ln\ln(f)\ln\ln\ln(f)\cdots}{a\sqrt{f}}. \quad (36)$$

However, at a single point (or a finite number of discrete points) the divergence behavior can always be made arbitrarily rapid, e.g.,

$$f = pa + a^{p^n}, \quad n = \text{const}, \quad p \to \infty, \quad (37)$$
$$\frac{df/da}{f^{3/2}} \to \frac{1}{a\sqrt{f}}, \quad 0 < a < 1, \quad (38)$$
$$\to f^{n-3/2}, \quad a = 1, \quad (39)$$
$$\to \frac{\ln(f)}{\ln(a)\sqrt{f}}, \quad a > 1. \quad (40)$$

Consequently, we now introduce the transformation

$$v = \int_0^r \frac{a'}{\sqrt{f}}dr - \frac{R}{2f}, \quad (41)$$

$$dv = \left(\frac{a'}{\sqrt{f}} + \frac{Rf'}{2f^2}\right)dr - \frac{dR}{2f}, \quad (42)$$

which incorporates both terms in the curly brackets of

(29), and which converts (28) to

$$ds^2 \to -\frac{1}{4f(1+f)}\left(1-\frac{2M}{R}\right)dR^2$$
$$+\left[2-\frac{1}{(1+f)}\left(1-\frac{2M}{R}\right)\right]dvdR$$
$$-\left(\frac{f}{1+f}\right)\left(1-\frac{2M}{R}\right)dv^2 + R^2 d\Omega^2. \qquad (43)$$

It is already clear from (29) as well as this equation that $f \to \infty$ is indeed required everywhere to obtain the Vaidya metric as the limit.

$$ds^2 \to 2dvdR - \left(1-\frac{2M}{R}\right)dv^2 + R^2 d\Omega^2. \qquad (44)$$

(The alternative transformation

$$v = \int_0^r \frac{a'}{\sqrt{1+f}}dr - \sqrt{\frac{1+f}{f}}R,$$

$$(45)$$

$$dv = \sqrt{\frac{f}{1+f}}\left(\frac{a'}{\sqrt{f}} + \frac{Rf'}{2f^2}\right)dr - \sqrt{\frac{1+f}{f}}dR$$

does not succeed in removing the factor of $f/(1+f)$, and also leads to the wrong limit.) In the limit (15) then, it is evident from (41) that v becomes a function of r only, so that $M \to M(v)$ holds once again. No assumptions about the functional form of f or the limiting behavior of a were made to obtain the Vaidya metric as the null limit, and we find that the second term of (41) becomes negligible, even if the second term in the parentheses of (42) does not. The new transformation (41) can also be rewritten in the limit as

$$dv \to \frac{dt}{\sqrt{f}} + \frac{dR}{2f} \qquad (46)$$

in order to recover (1) from (14). Equations (46) and (41) are the revised versions of (23) and (20).

The overall transformation from Tolman to asymptotically Vaidya coordinates then is

$$v = \int_0^r \frac{a'(r)}{\sqrt{f(r)}}dr - \frac{R(t,r)}{2f(r)}, \qquad (47)$$

$$R = R(t,r), \qquad (48)$$

where $R(t,r)$ is given by (8) and (9). Using the following limiting values of two of the partial derivatives of the inverse transformation,

$$\left.\frac{\partial r}{\partial v}\right|_R = \frac{f}{R'}, \qquad \left.\frac{\partial t}{\partial v}\right|_R = \sqrt{f}, \qquad (49)$$

the Kretschmann scalar and the density may be converted to their Vaidya forms. Thus

$$M^* = \left.\frac{\partial M}{\partial v}\right|_R = \left.\frac{\partial M}{\partial r}\right|_t \left.\frac{\partial r}{\partial v}\right|_R = M'\frac{f}{R'}, \quad \text{i.e.,} \quad \frac{M'}{R'} = \frac{1}{f}M^*$$

$$(50)$$

ensures that the last two terms on the right of (12) vanish,

giving (3) in the limit. For the "density," (2) is obtained from (11) in the limit by writing

$$G_{vv}^V = \left(\left.\frac{\partial t}{\partial v}\right|_R\right)^2 G_{tt}^T = f\frac{2M'}{R^2 R'} = \frac{2}{R^2}M^*. \qquad (51)$$

The strengths of singularities are variously defined by, e.g., [21,22]

$$\Psi_G = \lim_{\lambda \to 0} \lambda^2 G_{\alpha\beta}k^\alpha k^\beta \quad \text{or} \quad \Psi_R = \lim_{\lambda \to 0} \lambda^2 R_{\alpha\beta}k^\alpha k^\beta$$

$$(52)$$

where k^α is the tangent vector to a null geodesic with parameter λ that hits the singularity at $\lambda = 0$. From the above, and since Ψ is a scalar, it is clear that the strengths of the Vaidya singularity, as measured along radial geodesics are given by the limits of the corresponding Tolman expressions.

CONCLUSIONS

Lemos originally demonstrated that the Vaidya model is a null limit of the Tolman model, by taking the limit $f \to \infty$ and assuming $f =$const in this limit. His transformation was completely valid for models with constant f. However, Tolman shell-focusing singularities also occur in models with matter at the origin. The existence of a normal origin of spherical coordinates at $r = 0$, $(a - t) > 0$ in the Tolman model requires $f(r = 0) = 0$, and we have found this cannot be made consistent with a null limit. The Vaidya limit does indeed require $f \to \infty$, so it cannot be extended to a spherical origin, where $f(0) = 0$, or a Schwarzschild-Kruskal-Szekeres-type topology, which requires $f = -1$ in the throat.

Thus we conclude that every Vaidya model is the limit of a hollow Tolman model, acquiring its arbitrary $M(v)$ from a combination of $M(r)$ and $a(r)$, and must itself be hollow. If $M(r = 0) = 0$, $M(v = 0) = 0$, then $r = 0$, $v = 0$ is a collapsing shell of finite size surrounding Minkowski vacuum, and the limiting Vaidya model can form a shell focusing. If $M(r = 0) > 0$, $M(v = 0) > 0$ then it surrounds Schwarzschild vacuum, and no shell focusing forms. In this latter case, the shells of incoming radiation (having f divergent) cannot pass through the throat (where $f = -1$) and must hit the future singularity first. A dust filled interior is not possible in the limit, since a coordinate line cannot be comoving with both a dust particle and a light ray, but it may be possible to have an intervening vacuum region. Since $t = a$ on the singularity, and R is only finite on a collapsing shell of radiation where $(a - t)$ is infinitesimal, the radiation is all at infinite R for any finite value of $(a - t)$.

If we assume that M^* is finite, then it is apparent from (41) that a finite change in M and v requires an infinite change in a. It is interesting to note that a collapsing, unbound (i.e., hyperbolic) dust cloud of finite total mass may also have $f, a \to \infty$ and M finite in the asymptotic regions. At finite $(a - t)$, R is infinite, from (8) and (9), but as these particles collapse towards the crunch, R becomes finite when $(a - t)$ is infinitesimal, and the Vaidya limit is achieved. In terms of Tolman time, this is in-

finitely long after the initial formation of the singularity, but only a finite retarded time in Vaidya coordinates.

The new coordinate transformation (47) and (48) [or (27) and (42)], makes no assumptions about the three arbitrary Tolman functions f, M, and a, and in particular the relationship between f and a, beyond those normally made for a general Tolman model, and the limit $f \to \infty$. Several important physical quantities, the Einstein tensor, the Kretschmann scalar, the null geodesics, and the strengths of the singularities, all have the correct limit. This generalizes Lemos' unification of the two metrics and their shell focusing singularities.

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