General-relativistic celestial mechanics. IV. Theory of satellite motion

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The basic equations needed for developing a complete relativistic theory of artificial Earth satellites are explicitly written down. These equations are given both in a local, geocentric frame and in the global, barycentric one. They are derived within our recently introduced general-relativistic celestial mechanics framework. Our approach is more satisfactory than previous ones, especially with regard to its consistency, completeness, and flexibility. In particular, the problem of representing the relativistic gravitational effects associated with the quadrupole and higher multipole moments of the moving Earth, which caused difficulties in several other approaches, is easily dealt with in our approach thanks to the use of previously developed tools: the definition of relativistic multipole moments and transformation theory between reference frames. With this last paper in a series we hope to indicate the way of using our formalism in specific problems in applied celestial mechanics and astrometry.

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I. INTRODUCTION

High-precision experimental studies of the orbital motion of artificial Earth satellites have become one of the major tools of modern geodesy and geophysics by providing a wealth of information for the determination of the Earth's gravity field including time variabilities, plate motion, or the problem of Earth's rotation. The main type of technique which is currently used is satellite laser ranging (SLR) to dedicated satellites such as the Laser Geodynamics Satellite (LAGEOS). In the near future, several other techniques should also provide high-accuracy gravitational data: tracking by means of the global positioning system (GPS), gravitational measurements from orbiting gradiometers, use of drag-free satellites, etc. Thanks to the use of high pulse rates, sharp temporal, spatial and spectral filtering techniques combined with a highly advanced and reliable control of emitting and receiving telescopes on the basis of precise orbital prediction codes, rms residuals in SLR data now fall into the cm level. This is comparable to the Schwarzschild radius of the Earth,

$$r_{\rm Schw} \equiv 2GM_\oplus/c^2 \sim 0.88 \text{ cm} ,$$

representing the length scale for relativistic effects present in a local reference frame attached to the Earth. Because of this fact, it is now widely recognized that any modern SLR theory should be based upon Einstein's theory of gravity. Relativistic effects in SLR affect both the motion of the satellite ("relativistic forces") and the propagation of laser pulses through a curved spacetime ("gravitational time delay"). Moreover, relativity plays a crucial role for the relation between observed clock readings and theoretically convenient time scales, and, more generally, for the relations between several reference frames having different states of motion. The aim of the present paper is to explicitly write down the basic equations needed for developing a complete consistent relativistic theory of artificial Earth satellites.

From a heuristic point of view, one expects that there should exist, within a relativistic framework, the analogue of a Newtonian geocentric reference frame, following the Earth in its motion around the Sun. In such a frame, the influence of the external bodies (Sun, Moon, planets) should be relatively small and representable as some kind of tidal forces, while the dominant relativistic effects would result from the curved geometry generated by the Earth. The latter curved geometry should in turn be representable in terms of a sequence of multipole moments: mass monopole (total mass of the Earth $M_\oplus$), spin dipole (angular momentum $\mathbf{S}_\oplus$), mass quadrupole, spin quadrupole, etc. Similarly to the well-known anomalous perihelion precession of the inner planets (notably the famous 43 arc sec/century advance of Mercury's perihelion) the leading relativistic effect on the orbit of an Earth satellite is expected to be a secular drift in the argument of the perigee,

$$\delta\omega = 6\pi[GM_\oplus/c^2a(1-e^2)] \text{ rad/orbit} ,$$

associated with the mass monopole (or the corresponding "Schwarzschild field") of the Earth. For LAGEOS the latter perigee advance amounts to about 3 arc sec/yr [1,2] and might become measurable in the near future despite the small eccentricity of the LAGEOS orbit. Of next importance might be the effects associated with the spin dipole $\mathbf{S}_\oplus$ of the rotating Earth ("gravitomagnetic" or "Lense-Thirring" effects [3,4]). For satellite orbits this leads to an additional perigee precession of satellite orbits and a secular drift of the nodes of order

$$\delta\Omega \sim (GS_{\oplus}/c^2R^3)P_{\text{orbit}} \text{ rad/orbit} ,$$

where $S_{\oplus}$ is the angular momentum of the Earth. For LAGEOS this nodal drift is of the order of 3 arc sec/century, roughly comparable with the classical effect from the $l=12$ mass multipole moments. Two assessment studies (one from the Istituto di Fisica dello Spazio Interplanetario at Frascati, Italy and the other from the University of Texas in Austin) and an evaluation by some NASA advisory panel came to the conclusion that $\dot{\Omega}_{\text{LT}}$ might be measurable with 10% accuracy in three years if (following a suggestion of Ciufolini [5] and Bertotti, Ciufolini, and Bender [6]) a further LAGEOS satellite is placed in orbit with about the same orbital parameters as that of LAGEOS but with an inclination $180°-I_{\text{LAG}}$. Relativistic effects from the higher mass moments of the Earth might become significant in the near future especially in relation with drag-free satellites flying on low orbits. Some consequences of post-Newtonian forces related with the mass quadrupole (oblateness) of the Earth are discussed in Soffel *et al.* [7] and Heimberger, Soffel, and Ruder [8]. Finally, a consistent definition of the concept of "tidal forces" within a relativistic setting clearly necessitates the use of at least two different reference systems: a "geocentric" system where such forces appear as the residual effects of some external forces that largely cancel, and a "barycentric" system needed to describe the global motion of all the bodies constituting the solar system, and thereby to determine the time dependence of the local tidal effects.

This preliminary, heuristic discussion of the motion of a satellite in a geocentric frame clearly shows that a prerequisite for a consistent SLR theory is to dispose of a well-defined relativistic theory of spatiotemporal reference frames, able to describe, in a fully explicit manner, the relativistic effects generated by bodies of arbitrary shapes in several different reference frames. Such a theory has been provided by our previous series of publications ([9–11], referred to in the following as papers I–III). The aim of the present paper is to show in detail how to apply the formalism of papers I–III to the problem of the motion of an artificial Earth satellite.

In our opinion, the previous attempts made in the literature to tackle the relativistic satellite problem have all been unsatisfactory. A first type of approach [12,13] tried to work only in a global (barycentric) coordinate system. This introduces many apparent, large contributions in the equations of motion and in the laser propagation effects which cancel when computing local observables (such as the proper time elapsed at the station during the bounce of a laser beam onto a satellite). Worse, as, until paper II, the only known relativistic equations of motion assumed spherically symmetric bodies (Lorentz-Droste, Einstein-Infeld-Hoffmann), this approach missed a numerically important term in the barycentric representation of the gravitational field of the Earth associated with the oblateness of the Earth. This created problems for many years in the University of Texas Orbit Program and Interpolation Algorithm (UTOPIA) in the form of a spurious signal of 100 m amplitude in the along-track component of a satellite orbit with a period of 280 days. The missing term in the barycentric equations of motion has been recently put back by brute force [14] by using some transformation rules for the (coordinate) acceleration vector between the barycentric and some geocentric frame (see also Huang *et al.* [15]). In our approach all the terms associated with arbitrary multipole moments are automatically taken into account (for completeness, we shall exhibit in Appendix A the compatibility of the acceleration transformation rules derived in Huang *et al.* [15] with our general formalism).

A second type of approach [16,17] defined geocentric coordinates as generalized Fermi normal coordinates [18–20] and took into account the nonlinear interaction between the gravitational field of the Earth and that of the Sun. This approach, however, has serious drawbacks which stem from the fact that it is not the first step of a clearly defined algorithm: the background spacetime of this scheme is not well defined; the coupling of the quadrupole moment of the Earth with the external gravitational field is neglected from the start; there is no flexibility in the definition of the rotational state of the geocentric coordinates so that the relation between the barycentric and the geocentric frames necessarily involves a time-dependent rotation. Furthermore, calculations of higher-order tidal terms require higher-order derivatives of the Riemann curvature tensor in the background spacetime [21] and, as we have demonstrated in paper I, the use of curvature components are ill-adapted tools for the description of tidal effects in the *N*-body problem.

A more recent approach, due to Brumberg and Kopejkin [22–24,27,28] and Voinov [25,26] comes closer to achieving a satisfactory theory of relativistic reference frames. However, as already remarked in paper I, we believe that the Brumberg-Kopejkin approach has several drawbacks when compared to the formalism of papers I–III: *ad hoc* assumptions about the structure of various expansions are made, which are only partially justified by some later consistency checks; the scheme is confined to a particular model for the matter (isentropic perfect fluid) and restricts itself to the harmonic gauge; their formalism involves physically and mathematically ill-defined "multipole moments," which lead to a bad definition of the origin of the geocentric frame and to awkward relations between the "multipole moments" appearing in different coordinate systems. Moreover, their cumbersome technique of matching of individual multipoles leads to the strange situation that within their framework it might be more convenient to derive the geocentric satellite equations of motion not as a geodesic equation in the geocentric metric but rather by transforming the barycentric geodesic equation (see, e.g., Refs. [27,28]).

In our new framework of relativistic celestial mechanics (papers I–III) all these problems are circumvented or solved in a satisfactory manner: the whole scheme is developed in a constructive way by proving a number of theorems; the structure of the stress-energy tensor is left completely open; the gravitational field locally generated by each body is skeletonized by well-defined relativistic multipole moments recently introduced by Blanchet and

Damour [29], while the external gravitational field is expanded in terms of a particular new set of relativistic tidal moments. Our treatment of the gauge degree of freedom is new: the spatial coordinate grid is fixed by algebraic conditions leading to special harmonic *spatial* coordinates while a convenient gauge flexibility is left open in the time coordinate. In this way one can easily deal either with harmonic *spacetime* coordinates or "standard" post-Newtonian coordinates in all frames. In both the barycentric and geocentric reference frames we use a particular exponential parametrization of the metric tensor which has the effect of linearizing the field equations, as well as the transformation laws under a change of reference system. This linearity plays, in fact, a crucial role in our formalism; e.g., in each frame there is a canonical and unique way to split the metric into a locally generated part (for example, due to the gravitational action of the Earth itself, when considering the vicinity of the Earth), and an externally generated part, due to the action of the other bodies in the system and of the inertial forces in the accelerated (geocentric) system. Hence, when we talk about the "external" metric we deal with a well-defined concept. In paper I we were able to derive simple transformation rules for the metric potentials (that fully determine the metric) between barycentric and geocentric frames without referring to multipole expansions. These simple and compact transformation laws [Eqs. (4.12), (4.53), and (4.55) of paper I] constitute one of the central pillars of our scheme; they make the matching of individual multipoles used in the Brumberg-Kopejkin scheme completely superfluous.

One further attractive feature of our scheme consists in that the rotational state of the geocentric spatial coordinate grid, as described by some time-dependent orthogonal matrix $R_a^i(T)$ is left open provided that R_a^i is only slowly changing with time

$$\frac{dR_a^i(T)}{dT}=O(2)\ , \tag{1.1}$$

where $O(2)\equiv O(1/c^2)$. As already remarked in paper I, two choices for R_a^i are obviously preferred. The first choice is simply a global fixing, i.e., an alignment of the geocentric spatial coordinate lines with respect to the barycentric ones:

$$R_a^i(T)=\delta_{ia}\ . \tag{1.2}$$

This defines a geocentric kinematically nonrotating frame where, however, relativistic Coriolis forces have to be taken into account. The relativistic Coriolis effects are discussed in Sec. III below and in Appendix C.

The second choice consists of using a particular time dependence of the matrix R_a^i leading to an effacement of relativistic Coriolis effects in the geocentric frame (this is sometimes called a "dynamically nonrotating frame"). As discussed in detail in paper III, this effacement condition in our formalism can simply be expressed as the vanishing of the central, external "gravitomagnetic field," $H_a(T)=0$, or, equivalently, according to theorem 2 of paper III, by the Fermi-transport condition of the vectorial basis $e_\alpha^\mu(T)$ with respect to the external metric $\bar{g}_{\mu\nu}$. For details see Sec. IV B of paper III.

In the formulation of satellite equations of motion we will use the barycentric and geocentric coordinate times t and T, respectively, as fundamental time scales. These two time scales now carry the names temps-coordonnée barycentrique (TCB) $\equiv t$ and temps-coordonnée géocentrique (TCG) $\equiv T$. The normalization of the TCB time scale with respect to the physical second of the International System of Units [which is used to measure the proper times $d\tau=c^{-1}(-g_{\mu\nu}dz^\mu dz^\nu)^{1/2}$] is defined by the requirement that the time-time component of the barycentric metric, g_{00}, tend to -1 at spatial infinity. The normalization of the TCG time scale is defined by what we called the "weak effacement condition" $\overline{W}(T,\mathbf{0})=0$ or, in other words, by the requirement that the time-time component $\overline{G}_{00}$ of the *external* metric in the geocentric frame [uniquely defined in our formalism by Eqs. (5.18) of paper I] equals -1 at the *origin* of the geocentric frame.

For completeness, let us note that the terrestrial time (TT) scale [the old TDT (terrestrial dynamical time)] is defined as $\mathrm{TT}\equiv k_E T$, where the constant k_E is chosen so that TT directly measures the proper time on the *rotating* geoid: $k_E=1-6.97\times10^{-10}$. The relation to the international atomic time (TAI) is then simply given by $\mathrm{TT}=\mathrm{TAI}+32.184$ s. Note also that the Jet Propulsion Laboratory (JPL) ephemerides of the solar system such as DE200 use a time scale [called barycentric dynamical time (TDB)], which is ideally supposed to be just a rescaled version of TCB. However, there are by now well-known problems in the definition of TDB in relation with TT. Moreover, the scale factor between TDB and TCB generates useless inconveniences by affecting the measurement of the masses of the bodies of the solar system [12,13,30–32].

Finally, let us mention that the present paper is primarily directed toward treating the motion of *artificial* Earth satellites, causing negligible gravitational forces on the other bodies. The motion of the Moon poses a different problem. Within our formalism, the simplest way to study the motion of the Moon is to treat it on a par with all the other bodies of the solar system and to integrate its global-frame (barycentric) translational (paper II) and rotational (paper III) equations of motion. It would, however, be interesting to develop also a local-frame approach to the motion of the Moon (note that our formalism is flexible enough to use either a local frame with origin at the geocenter or one with origin at the Blanchet-Damour center of mass of the Earth-Moon subsystem).

This paper is organized as follows. We start with a discussion of satellite equations of motion in the local geocentric frame. In Sec. II we recall the form of the metric in the geocentric system. The external part of the metric, which describes the tidal forces and is given by our potentials $\overline{W}_\alpha$, is characterized in two different ways: (i) by means of a closed form expression and (ii) by means of an expansion in terms of our gravitoelectric and gravitomagnetic tidal moments G_L and H_L. In Sec. III the satellite equations of motion in the geocentric frame are discussed. The relativistic forces acting on a satellite are split into a local, an external, and a mixed part: the local part describes the action of the Earth's gravity field, the external part is the action of other bodies as well as inertial forces, and the mixed part is bilinear in the local and the external gravitational fields. The local part of the acceleration contains the Schwarzschild and Lense-Thirring accelerations, as well as their higher-multipole analogues ("relativistic forces" associated with higher Earth's mass multipoles). The external force is determined by $\overline{W}_\alpha$ and $(\overline{E}_a,\overline{B}_a)$ and characterized in the ways indicated above. Especially for high-flying Earth-orbiting satellites it might be advantageous to dispose of equations of motion in the global barycentric frame. In Sec. IV we discuss the explicit form of the metric and satellite acceleration in the global barycentric coordinate system. Section V contains some brief concluding remarks. Finally, in Appendix A we discuss the transformation properties of "relative position vectors" of satellites and corresponding expressions for the relative satellite accelerations. These transformation rules relate the satellite equations of motion from Secs. III and V. In Appendix B some technical details concerning the form of the barycentric metric are given. Some guidance on how to treat the geodetic precession in the framework of satellite dynamics is given in Appendix C.

II. THE METRIC IN THE GEOCENTRIC FRAME

In the Introduction we recalled the heuristic expectation that there should exist some Einsteinian analogue of the Newtonian, geocentric (nonrotating) frames following the Earth in its motion around the Sun. The formalism of papers I–III has led to a precise definition of such relativistic geocentric frames, as being what we called the "local" coordinate systems $X^\alpha=(cT,X^a)$ associated with the moving Earth [we follow the notation of our previous papers to which we refer the reader in case a notation used below is not explicitly redefined]. In the present, more specialized paper, it is convenient to use a time-honored terminology and to use the words "geocentric" to refer to the local frame X^α, and "barycentric" to refer to the "global" coordinate system x^μ used to describe the overall motion of the solar system. Let us recall that, in the local geocentric system with coordinates $X^\alpha=(cT,X^a)$, the metric is written in our usual exponential form

$$G_{00}=-\exp\left[-\frac{2}{c^2}W\right], \tag{2.1a}$$

$$G_{0a}=-\frac{4}{c^3}W_a , \tag{2.1b}$$

$$G_{ab}=\delta_{ab}\exp\left[+\frac{2}{c^2}W\right]+O(4), \tag{2.1c}$$

and the metric potentials $W_\alpha\equiv(W,W_a)$ are split into *local* and *external* parts (see Sec. IV of paper I):

$$W_\alpha=W_\alpha^+ +\overline{W}_\alpha . \tag{2.2}$$

Here, the locally generated part W_α^+ results from the gravitational action of the Earth and the externally generated part $\overline{W}_\alpha$ from all the other massive bodies in the solar system as well as from inertial effects. According to Eq. (6.9) of paper I, the local parts, in the skeletonized harmonic gauge, can be written as

$$W^+(T,\mathbf{X})=G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L[R^{-1}M_L^A(T\pm R/c)]+O(4), \tag{2.3a}$$

$$W_a^+(T,\mathbf{X})=-G\sum_{l\geq 1}\frac{(-)^l}{l!}\left[\partial_{L-1}\left[R^{-1}\frac{d}{dT}M_{aL-1}^A\right]+\frac{l}{l+1}\epsilon_{abc}\partial_{bL-1}(R^{-1}S_{cL-1}^A)\right]+O(2), \tag{2.3b}$$

where

$$f(T\pm R/c)\equiv[f(T+R/c)+f(T-R/c)]/2$$

with $R\equiv(\delta_{ab}X^aX^b)^{1/2}$ and where (as in our previous papers) L is a shorthand notation for a multispatial index $a_1a_2\cdots a_l$. In Eqs. (2.3) M_L^A and S_L^A are the relativistic [Blanchet-Damour (BD)] mass and spin moments of the Earth [see Eqs. (6.11) of paper I], A being a label which refers to the Earth. The external potentials $\overline{W}_\alpha$ according to Eq. (5.5) of paper II can be written as

$$\overline{W}_\alpha(X^\beta)=\sum_{B\neq A}W_\alpha^{B/A}(X^\beta)+\overline{W}_\alpha''(X^\beta)+O(4,2), \tag{2.4}$$

where the "B over A" term $W_\alpha^{B/A}$ denotes the contribution from body B and $\overline{W}_\alpha''$ results from inertial effects. In the approximation where the external bodies are described as mass monopoles, we find, from Eqs. (A7) and (A8) of paper II,

$$W^{B/A}(t,x^i)=\frac{GM_B}{r_B}\left[1+\frac{2}{c^2}(\mathbf{v}_B-\mathbf{v}_A)^2-\frac{1}{c^2}\overline{w}_B(\mathbf{z}_B)-\frac{1}{2c^2}(\mathbf{n}_B\cdot\mathbf{v}_B)^2-\frac{1}{2c^2}\mathbf{a}_B\cdot\mathbf{r}_B\right]+O(4), \tag{2.5a}$$

$$W_a^{B/A}(t,x^i)=R_{ia}^A(v_B^i-v_A^i)\frac{GM_B}{r_B}+O(2). \tag{2.5b}$$

If needed (e.g., for a more accurate treatment of the gravitational effect of the Moon) the contributions of the higher-multipole moments of the external bodies can be straightforwardly obtained from the results of paper II (see also Appendix B below). Note that we have expressed $W_\alpha^{B/A}$ not in terms of the local coordinates X^β, but instead in terms of the corresponding global ones $[x^\mu=(ct,x^i)]$. In Eqs. (2.5) $r_B\equiv(\delta_{ij}r_B^ir_B^j)^{1/2}$, $n_B^i\equiv r_B^i/r_B$ with $r_B^i(t,x^j)\equiv x^i-z_B^i(t)$, $v_B^i=dz_B^i/dt$, and $a_B^i=dv_B^i/dt$, where $z_B^i(t)$ denotes the barycentric spatial coordinates of the central point (origin) of body B as a function of the barycentric time coordinate (usually chosen to be the center of mass of B). According to Eqs. (4.15) and (5.10) of paper II the inertial part $\overline{W}''_\alpha$ has the form

$$\overline{W}''(T,X^a)=G''+G''_aX^a+\frac{1}{2}G''_{ab}\hat{X}^{ab}+\frac{1}{10c^2}\mathbf{X}^2X^a\left[\frac{d^2}{dT^2}G''_a\right]+\frac{1}{c^2}\frac{\partial}{\partial T}\Lambda''+O(4)\ , \tag{2.6a}$$

$$\overline{W}''_a(T,X^a)=-\frac{3}{10}\hat{X}^{ab}\left[\frac{d}{dT}G''_b\right]+\frac{1}{8}\epsilon_{abc}X^bH''_c-\frac{1}{4}\frac{\partial}{\partial X^a}\Lambda''+O(2)\ , \tag{2.6b}$$

with [see also Eqs. (6.30) of paper I]

$$G''(T)=c^2\ln\left[\frac{dT}{d\tau_f}\right]+O(4)\ , \tag{2.7a}$$

$$G''_a(T)=-A_a^A+O(4)\ , \tag{2.7b}$$

$$G''_{ab}(T)=\frac{3}{c^2}A^A_{\langle a}A^A_{b\rangle}+O(4)\ , \tag{2.7c}$$

$$H''_a(T)=\epsilon_{abc}\left[V_b^AA_c^A+c^2\frac{dR^A_{ib}}{dT}R^A_{ic}\right]+O(2)\ . \tag{2.7d}$$

In these equations τ_f is the global Minkowski proper time along the geocenter:

$$d\tau_f=c^{-1}(-f_{\mu\nu}dz^\mu_Adz^\nu_A)^{1/2}\ , \tag{2.8}$$

and

$$A_a^A\equiv f_{\mu\nu}e_a^{A\mu}\frac{d^2z^\nu_A}{d\tau_f^2} \tag{2.9}$$

denotes a certain local-frame projection of the global Minkowski acceleration of the central point of body A. It is explicitly given by

$$A_a^A=R^A_{ia}\left[\frac{d^2z^i_A}{dt^2}-\frac{1}{c^2}\overline{w}_A(\mathbf{z}_A)a^i_A+\frac{1}{2c^2}(\mathbf{v}_A\cdot\mathbf{a}_A)v^i_A+\frac{1}{c^2}\mathbf{v}^2_Aa^i_A\right]+O(4)\ . \tag{2.10}$$

In Eqs. (2.6) the function $\Lambda''(T,\mathbf{X})$ must vanish at the origin [because of the definition of $G''(T)\equiv W''(T,\mathbf{0})+O(4)$]. Therefore, we can write $\Lambda''(T,\mathbf{X})=C_a(T)X^a+D(T,\mathbf{X})$ with $D(T,\mathbf{X})=O(\mathbf{X}^2)$ as $X^a\to 0$. The value of $C_a(T)$ [and thereby the value of $\overline{W}''_a(T,\mathbf{0})$] can be changed at will by appropriately choosing the free datum 3 of Sec. V A of paper I, i.e., $\epsilon_a(T)$. Similarly, the value of $G''(T)$ can be changed at will by an appropriate rescaling of the local (or geocentric) time coordinate $T\equiv$TCG along the world line of the geocenter. We assume that these choices are made so as to satisfy the four weak effacement conditions of the *full* (tidal + inertial) *external* gravitational potentials:

$$\overline{W}_\alpha(T,\mathbf{0})\equiv 0\ . \tag{2.11}$$

The value of the higher terms $D(T,\mathbf{X})=O(\mathbf{X}^2)$ in Λ'' can also be arbitrarily changed by an appropriate choice of the free datum 5 Sec. V A of paper I, i.e., $\xi(T,\mathbf{X})\equiv c^3\xi^0$. However, this freedom corresponds to a transformation of the local time coordinate $\delta T=\xi/c^4$ which is of no physical importance at the first post-Newtonian level.

Our formalism gives us the flexibility of giving two types of expansions for the external potentials $\overline{W}_\alpha$, or equivalently for the *external gravitoelectric* and *gravitomagnetic* fields $\overline{E}_a$ and $\overline{B}_a$, which play an important role in the equations of motion of a satellite [see Eqs. (3.4) and (3.16) below]. These quantities are defined by

$$\overline{E}_a=\partial_a\overline{W}+\frac{4}{c^2}\partial_T\overline{W}_a\ , \tag{2.12a}$$

$$\overline{B}_a=-4\epsilon_{abc}\partial_b\overline{W}_c\ . \tag{2.12b}$$

$\overline{E}_a$ and $\overline{B}_a$ can be given as closed-form expressions in terms of the masses and global coordinates of the external bodies, or as tidal series in powers of the local coordinates X^a. The latter tidal expansion will be recalled in the next section. Let us show here how to compute a closed-form representation of $\overline{E}_a$ and $\overline{B}_a$ (when approximating the external bodies as mass monopoles). Inserting the linear split (2.4) into the definitions (2.12), we get a corresponding linear decomposition of $\overline{E}_a$ and $\overline{B}_a$:

$$\overline{E}_a=\sum_{B\neq A}E_a^{B/A}+\overline{E}''_a\ , \tag{2.13a}$$

$$\overline{B}_a=\sum_{B\neq A}B_a^{B/A}+\overline{B}''_a\ , \tag{2.13b}$$

with

$$E_a^{B/A}=D_aW^{B/A}+\frac{4}{c^2}D_TW_a^{B/A}\ , \tag{2.14a}$$

$$B_a^{B/A}=-4\epsilon_{abc}D_bW_c^{B/A}\ . \tag{2.14b}$$

From Eq. (4.16) of paper II we get [see also Eq. (6.31) of paper I]

$$\overline{E}''_a=G''_a+G''_{ab}X^b-\frac{1}{c^2}\mathbf{X}^a\left[\frac{d^2}{dT^2}G''\right]+\frac{1}{6c^2}\mathbf{X}^2\left[\frac{d^2}{dT^2}G''_a\right]-\frac{1}{c^2}\hat{X}^{ab}\left[\frac{d^2}{dT^2}G''_b\right]+\frac{1}{2c^2}\epsilon_{abc}X^b\left[\frac{d}{dT}H''_c\right]+O(4)\ , \tag{2.15a}$$

$$\overline{B}''_a=H''_a-2\epsilon_{abc}X^b\left[\frac{d}{dT}G''_c\right]+O(2)\ . \tag{2.15b}$$

As in paper II, we will express the differentiations with respect to the local-A-frame coordinates, $D^A_\alpha\equiv\partial/\partial X^\alpha_A$, in terms of global-frame differentiations, $\partial_\mu\equiv\partial/\partial x^\mu$. We get

$$D_a^A = \frac{\partial x^\mu}{\partial X_A^\alpha}\frac{\partial}{\partial x^\mu} = \frac{\partial f^\mu(X^\alpha)}{\partial X^\alpha}\partial_\mu , \tag{2.16}$$

where

$$x^\mu = f^\mu(X^\alpha) = z^\mu(T) + e_a^\mu(T)Y^a(T,\mathbf{X}) + O(3,4) , \tag{2.17}$$

with

$$Y^a(T,\mathbf{X}) = X^a + \frac{1}{c^2}\left[\frac{1}{2}A_a^A\mathbf{X}^2 - X^a(\mathbf{A}^A\cdot\mathbf{X})\right] . \tag{2.18}$$

This leads us to [see also Eq. (4.16) of paper I]

$$D_T = \frac{\partial}{\partial T} = \partial_t + v_A^i\partial_i + O(2) , \tag{2.19a}$$

$$D_a = \frac{\partial}{\partial X^a} = e_a^{Ai}\left[\partial_i + \frac{1}{c^2}v_A^i\partial_t\right] + \frac{1}{c^2}e_b^{Ai}[A_b^AX^a - X^bA_a^A - \delta_{ab}(\mathbf{A}^A\cdot\mathbf{X})]\partial_i + O(4) . \tag{2.19b}$$

Inserting the expressions for $W_a^{B/A}$ into (2.14) and performing the derivatives as indicated we finally get the B/A parts of the external gravitoelectric and gravitomagnetic fields:

$$E_a^{B/A} = -R_a^{Ai}\frac{GM_Bn_B^i}{r_B^2}\left[1 + \frac{2}{c^2}(\mathbf{v}_B - \mathbf{v}_A)^2 - \frac{1}{c^2}\overline{w}^A(\mathbf{z}_A) - \frac{1}{c^2}\overline{w}^B(\mathbf{z}_B) - \frac{3}{2c^2}(\mathbf{n}_B\cdot\mathbf{v}_B)^2 - \frac{1}{2c^2}\mathbf{a}_B\cdot\mathbf{r}_B\right]$$
$$- R_a^{Ai}\frac{GM_B}{c^2r_B^2}[(\mathbf{n}_B\cdot\mathbf{v}_B)(5v_B^i - 3v_A^i) - \tfrac{7}{2}r_Ba_B^i + 4r_Ba_A^i] - \frac{1}{c^2}R_a^{Aj}v_A^i\left[4v_B^j - \frac{7}{2}v_A^j\right]\frac{GM_Bn_B^i}{r_B^2}$$
$$- \frac{1}{c^2}R^{Aib}[A_b^AX^a - X^bA_a^A - \delta_{ab}(\mathbf{A}^A\cdot\mathbf{X})]\frac{GM_Bn_B^i}{r_B^2} + O(4) , \tag{2.20a}$$

$$B_a^{B/A} = 4\epsilon_{abc}R_b^{Ai}R_c^{Aj}\left[(v_B^j - v_A^j)\frac{GM_Bn_B^i}{r_B^2}\right] + O(2) . \tag{2.20b}$$

To conclude this section, let us recall that our formalism leaves open some flexibility in the definition of a local, geocentric frame. First, the choice of the origin of the local frame is free. However, we repeatedly made it clear that there is a preferred choice, namely, that for which the post-Newtonian mass dipole $M_a^A(T)$ vanishes (this defines a relativistically mass-centered local frame). Second, the rotational state of the spatial local coordinate grid is free, as long as it evolves with post-Newtonian slowness, Eq. (1.1). We shall return below to the effects associated with this rotational flexibility.

III. SATELLITE EQUATIONS OF MOTION IN THE LOCAL GEOCENTRIC FRAME

The relativistic equations of motion of a satellite in a geocentric frame can be expanded in powers of three independent parameters: (i) the parameter $\epsilon \sim eR_\oplus/R$ (where $R_\oplus$ is the radius of the Earth and e is some measure of the lack of sphericity of the Earth's mass distribution) which corresponds to a *multipole expansion*; (ii) the parameter $\eta \sim R/D$ (where D is a characteristic distance between the Earth and the external bodies) which corresponds to a *tidal expansion*; and (iii) the relativistic parameter $\zeta \sim v^2/c^2 \sim GM/c^2R$, which corresponds to a *post-Newtonian expansion.* In the present section we first write the full multipole and tidal expansion of the satellite equations of motion at post-Newtonian accuracy [i.e., neglecting only terms of order $\zeta^2 = O(c^{-4})$]. Then we write in more detail some approximations of the equations of motion, obtained by retaining only the leading *combined* corrections of order $\zeta\epsilon^n$ or $\zeta\eta^m$.

Neglecting the satellite's own gravity field [and nongravitational forces (atmospheric drag, radiation pressure, etc.), which must be added separately] the satellite's center of mass $(Z_S^\alpha) \equiv (cT, Z_S^a)$ will follow a geodesic. In our metric (2.1) a geodesic can be derived from the Lagrangian

$$\mathcal{L} = -c[-G_{\alpha\beta}(Z_S^\gamma)dZ_S^\alpha/dT\,dZ_S^\beta/dT]^{1/2} + c^2 ,$$

which reads

$$\mathcal{L} = \frac{1}{2}\mathbf{V}^2 + W - \frac{1}{2c^2}W^2 + \frac{3}{2c^2}W\mathbf{V}^2 + \frac{1}{8c^2}\mathbf{V}^4 - \frac{4}{c^2}W_aV^a + O(4) , \tag{3.1}$$

where $\mathbf{V}$ is the geocentric coordinate velocity of the satellite, $V^a \equiv dZ_S^a/dT$ (as it leads to no ambiguities in the geocentric frame we simplify the notation by suppressing the label S on the velocity of the satellite). The geocentric coordinate acceleration of the satellite is then given by

$$\frac{d^2Z_S^a(T)}{dT^2} = [W_{,a} + c^{-2}(-4WW_{,a} - 4W_{,b}V^bV^a + W_{,a}\mathbf{V}^2 - 3W_{,T}V^a + 4W_{a,T} + 8W_{[a,b]}V^b)]_{X^a = Z_S^a(T)} + O(4) , \tag{3.2}$$

where $W_{[a,b]} \equiv \frac{1}{2}(W_{a,b} - W_{b,a})$. In terms of the full (local + external) gravitoelectric and gravitomagnetic fields E_a and B_a, defined by

$$E_a = \partial_a W + \frac{4}{c^2}\partial_T W_a \ , \tag{3.3a}$$

$$B_a = -4\epsilon_{abc}\partial_b W_c \ , \tag{3.3b}$$

this acceleration can also be written in the form

$$\frac{d^2 Z_S^a}{dT^2} = \left[\left[1 + \frac{\mathbf{V}^2}{c^2} - \frac{4W}{c^2} \right] \left[E_a + \frac{1}{c^2}\epsilon_{abc} V^b B_c \right] - \frac{1}{c^2}(4V^b \partial_b W + 3\partial_T W) V^a \right]_{X_a = Z_S^a(T)} + O(4) \ . \tag{3.4}$$

Note the appearance in Eq. (3.4) of the "Lorentz force" $\mathbf{E} + \mathbf{V} \times \mathbf{B}/c^2$, which also played an important role in our study in papers I–III of the consequences of the relativistic "Euler equations" for the continuous matter distribution constituting the Earth and the other bodies in the solar system.

The linear split of the metric potentials in (2.2) induces a corresponding split of the acceleration terms on the right-hand side of (3.2) according to

$$\frac{d^2 Z_S^a}{dT^2} = F_{\rm loc}^a + F_{\rm ext}^a + F_{\rm mix}^a \ , \tag{3.5}$$

where $F_{\rm loc}^a$ results entirely from the gravitational action of the Earth, $F_{\rm ext}^a$ results from the other massive bodies in the solar system together with inertial effects and where

$$F_{\rm mix}^a = -\frac{4}{c^2}(\overline{W} W_{,a}^+ + W^+ \overline{W}_{,a}) = -\frac{4}{c^2}(\overline{W} E_a^+ + W^+ \overline{E}_a) \tag{3.6}$$

is *bilinear* in the local and external gravitational fields [for simplicity we henceforth drop all the $O(4)$ error terms in the equations of motion], and $F_{\rm loc}^a$ is simply obtained by inserting W_α^+ from Eqs. (2.3) instead of W_α on the right-hand side of (3.2) ($M^{(n)} \equiv d^n M/dT^n$, etc.). We write

$$F_{\rm loc}^a = F_{\rm loc}^{a[0]} + F_{\rm loc}^{a[2]} \ , \tag{3.7}$$

where

$$F_{\rm loc}^{a[0]} \equiv G \sum_{l \ge 0} \frac{(-)^l}{l!} M_L^A \Phi_{aL}^A \ , \tag{3.8}$$

and

$$F_{\rm loc}^{a[2]} = G \sum_{l \ge 0} \frac{(-)^l}{l!} \frac{1}{c^2} \left[\frac{1}{2} M_L^{A(2)} \partial_{aL} R - 4 M_L^A \Phi_{aL}^A \left[G \sum_{k \ge 0} \frac{(-)^k}{k!} M_K^A \Phi_K^A \right] - 4 M_L^A \Phi_{bL}^a V^b V^a + M_L^A \Phi_{aL}^A \mathbf{V}^2 - 3 M_L^{A(1)} \Phi_L^A V^a \right.$$
$$- 4 M_{aL-1}^{A(2)} \Phi_{L-1}^A + \frac{4l}{l+1}\epsilon_{abc} S_{bL-1}^{A(2)} \Phi_{cL-1}^A - 8 M_{(L-1)[a}^{A(1)} \Phi_{b]L-1}^A V^b$$
$$\left. - \frac{8l}{l+1}\epsilon_{dc[a} \Phi_{b]dL-1}^A S_{cL-1}^A V^b \right] \ . \tag{3.9}$$

Here we have introduced the useful notation

$$\Phi_L^A \equiv \Phi_{a_1 \cdots a_l}^A \equiv \frac{\partial^l}{\partial X^{a_1} \cdots \partial X^{a_l}} \left[\frac{1}{R_A} \right] = (-)^l (2l-1)!! \frac{\hat{N}_A^{a_1 \cdots a_l}}{R_A^{l+1}} \ , \tag{3.10}$$

with $N_A^a \equiv X_A^a / R_A$ and the caret denoting as usual a symmetric trace-free projection. $F_{\rm loc}^{a[0]}$ might be called the "quasi-Newtonian" local acceleration of the satellite being just the gradient of a "quasi-Newtonian" local scalar potential

$$U_{\rm loc}^{[0]}(T, \mathbf{X}) \equiv G \sum_{l \ge 0} \frac{(-)^l}{l!} M_L^A(T) \partial_L \left[\frac{1}{R_A} \right] \ , \tag{3.11}$$

while $F_{\rm loc}^{a[2]}$ will be referred to as the "relativistic" part of the local acceleration.

We want to stress that although Eq. (3.8) looks perfectly "Newtonian" the mass moments of the Earth, M_L^A (or the equivalent spherical multipole coefficients C_{lm} and S_{lm}) are the full post-Newtonian (BD) moments defined in Eq. (6.11) of paper I. As only these moments (together with their spin analogues S_L^A) enter the satellite equations of motion, it is clear that they are directly measurable from high-precision satellite orbital data. In particular, there would be no *observable* meaning to split M_L^A into Newtonian and post-Newtonian contributions [in spite of the fact that the *theoretical* definition of M_L^A, Eq. (6.11) of paper I, does contain explicit post-Newtonian contributions].

To discuss in more detail the post-Newtonian acceleration of an Earth-orbiting satellite we make several approximations in the total acceleration (3.5): we neglect all higher spin moments S_{aL} for $l \ge 1$; we neglect all time derivatives of M_L and S_a; among the explicit c^{-2} terms we only keep the mass monopole and quadrupole terms; the c^{-2} terms quadratic in M_{ab}^A or bilinear in M_{ab}^A and S_c are neglected. At this point it is also convenient to enforce that the origin of the local system be taken so as to make the BD mass dipole vanish:

$$M_a^A(T) = 0$$

("relativistic mass-centered local frame"). The "relativistic" part of the local acceleration might be split according to

$$F^{a[2]}_{\rm loc}=F^a_{\rm Schw}+F^a_{RQ}+F^a_S+\cdots \ . \tag{3.12}$$

Here, $F^a_{\rm Schw}$ denotes the Schwarzschild acceleration, resulting from the mass monopole of the Earth:

$$F^a_{\rm Schw}=\frac{GM_A}{c^2R^3}\left[\left[4\frac{GM_A}{R}-\mathbf{V}^2\right]X^a+4(\mathbf{X}\cdot\mathbf{V})V^a\right] \ . \tag{3.13}$$

F^a_{RQ} denotes those relativistic accelerations which are linear in the mass-quadrupole moments:

$$F^a_{RQ}=\frac{GM^A_{bc}}{c^2}\left[-2GM_A(\Phi^A_{bc}\Phi^A_a+\Phi^A\Phi^A_{abc})\right.$$
$$\left.-2V^aV^d\Phi^A_{bcd}+\tfrac{1}{2}\mathbf{V}^2\Phi^A_{abc}\right] \ . \tag{3.14}$$

Finally, F^a_S is the spin-orbit ("Lense-Thirring") acceleration, resulting from the spin vector (= angular momentum) of the Earth:

$$F^A_S=\frac{4G}{c^2}V^bS^A_d\epsilon_{cd[a}\Phi^A_{b]c} \ . \tag{3.15}$$

[The (satellite spin) × orbit and spin-spin contributions to the equations of motion have been discussed in paper II, and are negligible in practical applications.]

Next we come to the external (or "tidal") accelerations which can be written as

$$F^a_{\rm ext}=\left[1+\frac{\mathbf{V}^2}{c^2}-\frac{4\overline{W}}{c^2}\right]\left[\overline{E}_a+\frac{1}{c^2}\epsilon_{abc}V^b\overline{B}_c\right]$$
$$-(4V^b\partial_b\overline{W}+3\partial_T\overline{W})V^a \ . \tag{3.16}$$

For some high-flying satellites it might be advantageous to evaluate the right-hand side of (3.16) by using the closed-form expressions for $\overline{W}$ and $(\overline{E}_a,\overline{B}_a)$ given in the last section. In most cases, however, an expansion in terms of the relativistic tidal moments G_L and H_L introduced in paper I will be more appropriate. Using the expansions [see Eq. (6.23) of paper I]

$$\overline{E}_a=\sum_{l\ge 0}\frac{1}{l!}\left[\hat{X}^LG_{aL}+\frac{1}{2(2l+3)c^2}\mathbf{X}^2\hat{X}^LG^{(2)}_{aL}-\frac{7l-4}{(2l+1)c^2}\hat{X}^{aL-1}G^{(2)}_{L-1}+\frac{l}{(l+1)c^2}\epsilon_{abc}\hat{X}^{bL-1}H^{(1)}_{cL-1}\right]+O(4) \ , \tag{3.17a}$$

$$\overline{B}_c=\sum_{l\ge 0}\frac{1}{l!}\left[\hat{X}^LH_{cL}-\frac{4l}{l+1}\epsilon_{cde}\hat{X}^{dL-1}G^{(1)}_{eL-1}\right]+O(2) \ , \tag{3.17b}$$

and

$$\overline{W}=\sum_{l\ge 0}\frac{1}{l!}\hat{X}^LG_L+O(2) \ , \tag{3.18}$$

we can decompose the external acceleration in "quasi-Newtonian" and "relativistic" parts:

$$F^a_{\rm ext}=F^{a[0]}_{\rm ext}+F^{a[2]}_{\rm ext} \tag{3.19}$$

with

$$F^{a[0]}_{\rm ext}=\sum_{l\ge 0}\frac{1}{l!}\hat{X}^LG_{aL} \ , \tag{3.20}$$

and

$$F^{a[2]}_{\rm ext}=\sum_{l\ge 0}\frac{1}{l!}\frac{1}{c^2}\left[\frac{1}{2(2l+3)}\mathbf{X}^2\hat{X}^LG_{aL}-\frac{7l-4}{2l+1}\hat{X}^{aL-1}G^{(2)}_{L-1}+\frac{l}{l+1}\epsilon_{abc}\hat{X}^{bL-1}H^{(1)}_{cL-1}-4\hat{X}^LG_{aL}\left[\sum_{k\ge 0}\frac{1}{k!}G_K\hat{X}^K\right]\right.$$
$$\left.-4\hat{X}^LG_{bL}V^bV^a+\hat{X}^LG_{aL}\mathbf{V}^2-3\hat{X}^LG^{(1)}_LV^a+\epsilon_{abc}V^b\left[\hat{X}^LH_{cL}-\frac{4l}{l+1}\epsilon_{cde}\hat{X}^{dL-1}G^{(1)}_{eL-1}\right]\right] \ . \tag{3.21}$$

Note that although expression (3.20) for $F^{a[0]}_{\rm ext}$ looks Newtonian, the quantities G_{aL} are the full post-Newtonian electriclike tidal moments, which contain c^{-2} terms when written in terms of the multipole moments of the other bodies. Equation (3.20) contains for $l=0$ a spatially uniform contribution given by the tidal-dipole moment G_a [see, e.g., Eq. (6.6b) of paper II],

$$G_a=-\sum_{l\ge 2}\frac{1}{l!}\left[\frac{M^A_L}{M_A}\right]G_{aL}+O(2)$$
$$\simeq-\frac{1}{2}\left[\frac{M^A_{bc}}{M_A}\right]G_{abc}+\cdots \ , \tag{3.22}$$

which is present even in the Newtonian limit as $M^A_{bc}\neq 0$

(only if the geocenter were to follow a geodesic would G_a vanish).

For practical applications it might be sufficient to retain the full post-Newtonian accuracy only in the terms which are linear in the local space coordinate X^a. This implies that in $F^{a[0]}_{\rm ext}$ the full post-Newtonian tidal-quadrupole matrix G_{ab} should be used. According to Eqs. (4.27) and (4.29) of paper III, i.e., in the approximation where the external bodies can be described as mass monopoles, G_{ab} is given by

$$G_{ab}=\sum_{B\neq A} R^i_a R^j_b \left[\frac{3GM_B}{r^3_{AB}}\right]\Bigg[n^{\langle ij\rangle}_{AB}+\frac{1}{c^2}\{n^{\langle ij\rangle}_{AB}[2\mathbf{v}^2_{AB}-2\overline{w}_A(\mathbf{z}_A)-\overline{w}_B(\mathbf{z}_B)-\tfrac{3}{2}(\mathbf{n}_{AB}\cdot\mathbf{v}_B)^2-\tfrac{1}{2}\mathbf{a}_B\cdot\mathbf{r}_{AB}] + a^{\langle i}_{AB}r^{j\rangle}_{AB}+v^{\langle i}_{AB}v^{j\rangle}_{AB}-2(\mathbf{n}_{AB}\cdot\mathbf{v}_{AB})n^{\langle i}_{AB}v^{j\rangle}_{AB} -(\mathbf{n}_{AB}\cdot\mathbf{v}_A)n^{\langle i}_{AB}(v^{j\rangle}_A-2v^{j\rangle}_B)\}\Bigg]+O(4)\ , \tag{3.23}$$

where the angular brackets denote the symmetric trace-free part of a tensor, e.g.,

$$a^{\langle i}b^{j\rangle}\equiv\tfrac{1}{2}(a^ib^j+b^ia^j)-\tfrac{1}{3}a^sb^s\delta^{ij}\ .$$

[If needed the contribution of the higher-multipole moments M^B_L can be straightforwardly computed from the result of paper II.] For the higher tidal moments, G_L with $l>2$ the Newtonian limit

$$G_L=d^A_L\overline{w}_A(\mathbf{z}_A)+O(2)\ , \tag{3.24}$$

with

$$d^A_L\equiv R_{a_1}^{Ai_1}\cdots R_{a_l}^{Ai_l}\partial_{i_1\cdots i_l}\ ,$$

and

$$\overline{w}_A(t,x^i)=\sum_{B\neq A}GM_B/r_B+O(2)\ ,$$

might be sufficient. Hence, we have, for the "quasi-Newtonian" tidal acceleration,

$$F^{a[0]}_{\rm ext}\simeq G_a+G_{ab}X^b+\sum_{l\geq 2}\frac{1}{l!}\hat{X}^L d^A_{aL}\overline{w}(\mathbf{z}_A)\ . \tag{3.25}$$

As for the "relativistic" part of the tidal acceleration, assuming the weak effacement condition $G(T)=0$, it reads

$$F^{a[2]}_{\rm ext}\simeq\frac{1}{c^2}(\epsilon_{abc}V^bH_c-\tfrac{1}{2}\epsilon_{abc}H^{(1)}_bX^c+G_{ab}X^b\mathbf{V}^2 -4V^aG_{bc}V^bX^c+\epsilon_{abc}V^bX^dH_{cd}) \tag{3.26}$$

in the approximation where only those c^{-2} terms which are at most linear in X^a are retained, and where $c^{-2}G_a$ terms are also neglected. The first expression on the right-hand side of (3.26) is nothing but a relativistic Coriolis force in the local geocentric frame:

$$F^{a[2]}_{\rm Cor}=\frac{1}{c^2}\epsilon_{abc}V^bH_c=2(\mathbf{\Omega}_{\rm Cor}\times\mathbf{V})^a\ , \tag{3.27}$$

where

$$\mathbf{\Omega}_{\rm Cor}\equiv-\frac{1}{2c^2}\mathbf{H}\ .$$

We discussed in detail in paper III how $\mathbf{\Omega}_{\rm Cor}$ arises as a universal Larmor-type term in the precession of gyroscopes. This post-Newtonian Coriolis force vanishes for $H_c=0$, a condition which defines a dynamically nonrotating geocentric frame, i.e., a frame in which both Coriolis effects and the universal gravitational Larmor effects are absent. For such a Coriolis-effaced geocentric system the local spatial coordinate grid must precess with respect to the barycentric one, i.e., the rotation matrix R^i_a will be a function of time obtained by integrating Eqs. (4.19) of paper III (an approximate treatment of this time-dependent orthogonal matrix is given in Appendix C).

Inserting the Newtonian expressions of the electriclike tidal-quadrupole matrix, G_{ab} [Eq. (3.23)] and of the magneticlike tidal quadrupole

$$H_{ab}=\sum_{B\neq A}R_a^{Ai}R_b^{Aj}\left[\frac{3GM_B}{r^3_{AB}}\right][-2\epsilon_{ikl}(v^B_l-v_l^A)\hat{n}^{jk}_{AB} -2\epsilon_{jkl}(v^B_l-v_l^A)\hat{n}^{ik}_{AB}]$$

we find the following explicit expression for $F^{a[2]}_{\rm ext}$:

$$F^{a[2]}_{\rm ext}\simeq 2(\mathbf{\Omega}_{\rm Cor}\times\mathbf{V})^a+[(d\mathbf{\Omega}_{\rm Cor}/dT)\times\mathbf{X}]^a +\sum_{B\neq A}\left[\frac{3GM_B}{r^3_{AB}}\right]\frac{n^{\langle ij\rangle}_{AB}}{c^2}[R_a^{Ai}R_b^{Aj}\mathbf{V}^2X^b-4R_b^{Ai}R_c^{Aj}V^aV^bX^c -2V^b(v^k_B-v^k_A)X^d\epsilon_{abc}(\epsilon_{def}R_c^{Ai}+\epsilon_{cef}R_d^{Ai})R_e^{Aj}R_f^{Ak}]\ . \tag{3.28}$$

In Eq. (3.28), V^a denotes, as above, the velocity of the satellite in a geocentric frame, while $v^k_B-v^k_A$ denotes the relative barycentric velocity of body B with respect to the Earth.

Finally, we come to the mixed acceleration terms from Eq. (3.6). Using relation (4.15a) of paper II and

$$\overline{W}_{,a}=\sum_{l\geq 0}\frac{1}{l!}G_{La}\hat{X}^L+O(2) \tag{3.29}$$

we can write $F^a_{\rm mix}$ as

$$F^a_{\rm mix} = -\frac{4G}{c^2}\sum_{k\geq 0}\sum_{l\geq 0}\frac{(-)^k}{k!}\frac{1}{l!}\hat{X}^L M_K^A(G_L\Phi^A_{aK}+G_{La}\Phi^A_K)$$
$$\simeq -4\left[\frac{GM_A}{c^2R}\right](G_{ab}X^b-\tfrac{1}{2}G_{bc}N^{bc}X^a) , \quad (3.30)$$

where $N^{bc}\equiv N^bN^c$ with $N^b\equiv X^b/R$. Inserting the Newtonian expression for G_{ab} we can also write more explicitly

$$F^a_{\rm mix}\simeq 2\left[\frac{GM_A}{c^2R}\right]\sum_{B\neq A}\frac{GM_B}{r^3_{AB}}[X^a-6n^a_{AB}(\mathbf{n}_{AB}\cdot\mathbf{X}) +3(\mathbf{n}_{AB}\cdot\mathbf{N})^2X^a] . \quad (3.31)$$

To end this section, note that the explicit results given here for the coordinate equations of motion in a geocentric frame have to be completed in practical applications by taking into account the relativistic gravitational time delay effects in the time of flight of a laser pulse bounced back on the satellite (as well, evidently, as the time dilation factors between coordinate time and proper time for the clocks used in the timing). At the post-Newtonian accuracy the time delay and time dilation effects in a geocentric frame are entirely described by the scalar potential W, taken at the Newtonian approximation, i.e., by the sum of the local potential $U^{[0]}_{\rm loc}$ of Eq. (3.11) and $\bar{w}_A$ of Eq. (3.24). However, for the time delay we expect in most cases the effects of higher-multipole moments and of tidal moments to be negligible so that it will be enough to take into account only the usual (Shapiro) logarithmic time delay associated with the mass of the Earth (which corresponds to an equivalent range effect of order $GM_\oplus/c^2\sim 0.88$ cm). On the other hand, a higher accuracy might be needed in the computation of time dilation effects necessary to retain the effects of higher multipole moments and tidal potentials.

IV. SATELLITE EQUATIONS OF MOTION IN THE GLOBAL BARYCENTRIC FRAME

For completeness, and also to allow one to compare and contrast our approach with previous ones, we discuss also the form taken by the satellite equations of motion written by using only global, barycentric coordinates $(x^\mu)=(ct,z^i)$. This form might be useful for high-flying satellites.

A. The global (barycentric) metric

As in the geocentric system the metric in the global barycentric frame with coordinates $x^\mu=(ct,x^i)$ is written in exponential form:

$$g_{00}=-\exp\left[-\frac{2}{c^2}w\right] , \quad (4.1a)$$

$$g_{0i}=-\frac{4}{c^3}w_i , \quad (4.1b)$$

$$g_{ij}=\delta_{ij}\exp\left[+\frac{2}{c^2}w\right]+O(4) . \quad (4.1c)$$

For the problem of Earth-orbiting satellites we write

$$w_\mu = w^A_\mu + \bar{w}^A_\mu , \quad (4.2)$$

where w^A_μ results from the gravitational action of the Earth ("body A") and $\bar{w}^A_\mu$ results from all other massive bodies in the solar system. To derive the post-Newtonian acceleration of a satellite we will consider all bodies $B\neq A$ in the mass-monopole approximation. Therefore, from Eqs. (7.14) and (7.19) of paper I (or from Appendix B), we get

$$\bar{w}^A = \sum_{B\neq A}\frac{GM_B}{r_B}+\frac{3}{2}\sum_{B\neq A}\frac{GM_B}{r_B}\frac{\mathbf{v}_B^2}{c^2} - \sum_{B\neq A}\sum_{C\neq B}\frac{G^2M_BM_C}{c^2r_Br_{BC}} +\frac{1}{2c^2}\sum_{B\neq A}GM_B\frac{\partial^2}{\partial t^2}r_B(t)+O(4) , \quad (4.3a)$$

$$\bar{w}^A_i = \sum_{B\neq A}\frac{GM_B}{r_B}v^i_B+O(2) , \quad (4.3b)$$

with $[n^i_B(t)\equiv(x^i-z^i_B)/|\mathbf{x}-\mathbf{z}_B|]$

$$\frac{\partial^2}{\partial t^2}r_B(t)=\frac{\mathbf{v}_B^2}{r_B}-\frac{(\mathbf{n}_B\cdot\mathbf{v}_B)^2}{r_B}-\mathbf{n}_B\cdot\mathbf{a}_B . \quad (4.4)$$

For the Earth we shall retain its multipole structure. However, among the relativistic terms only those arising from the mass, the mass quadrupole, and the spin dipole will be kept. Since we neglect the multipole moments for all bodies other than the Earth we will drop the index A for the Earth on all multipole terms other than the mass. In the following we will also assume the local geocentric A system to be relativistically mass centered, i.e.,

$$M^A_a(T_A)=0 . \quad (4.5)$$

Accordingly, the barycentric metric potentials for the Earth will be approximated by

$$w^A_\mu = w^A_\mu|_{\rm LD}+w^A_\mu|_S+w^A_\mu|_Q+w^A_\mu|_{\rm higher} . \quad (4.6)$$

Here, $w^A_\mu|_{\rm LD}$ denotes the mass-monopole Lorentz-Droste (Einstein-Infeld-Hoffmann) part given by (see Appendix B for derivations and more details)

$$w^A|_{\rm LD}=\frac{GM_A}{r_A}\left[1+\frac{3}{2c^2}\mathbf{v}_A^2-\frac{\bar{w}_A(\mathbf{z}_A)}{c^2}\right] +\frac{1}{2c^2}GM_A\partial_{tt}r_A(t) , \quad (4.7a)$$

$$w^A_i|_{\rm LD}=\frac{GM_A}{r_A}v^i_A . \quad (4.7b)$$

The spin and quadrupole parts $w^A_\mu|_S$ and $w^A_\mu|_Q$ read (see Appendix B)

$$w^A|_S=-\frac{2G}{c^2}\epsilon_{jkl}v^k_AS_l\varphi^A_j , \quad (4.8a)$$

$$w^A_i|_S=\frac{G}{2}\epsilon_{ijk}S_k\varphi^A_j , \quad (4.8b)$$

and

$$w^A|_Q=\frac{1}{2}Gm_{jk}\left[1+\frac{3}{2c^2}\mathbf{v}_A^2-\frac{3\overline{w}_A(\mathbf{z}_A)}{c^2}\right]\varphi^A_{jk}+\frac{2G}{c^2}m_{jk}a^k_A\varphi^A_j-\frac{1}{2c^2}Gv^l_Av^A_{\langle k}m_{j\rangle l}\varphi^A_{jk}+\frac{G}{4c^2}m_{jk}\partial_{ttjk}r_A(t)\ , \tag{4.9a}$$

$$w^A_i|_Q=\tfrac{1}{2}Gm_{jk}v^i_A\varphi^A_{jk}\ , \tag{4.9b}$$

while the gravitational potentials associated with the higher-multipole moments of the Earth are approximated as

$$w^A|_{\text{higher}}\simeq\sum_{l\geq 3}\frac{(-)^l}{l!}Gm_{j_1\cdots j_l}\varphi^A_{j_1\cdots j_l}\ , \tag{4.9c}$$

$$w^A_i|_{\text{higher}}\simeq 0\ . \tag{4.9d}$$

Here, $s_i=R_a^{Ai}S_a, m_{jk}\equiv R_a^{Aj}R_b^{Ak}M_{ab}$, etc. and

$$\varphi^A_L\equiv\varphi^A_{i_1\cdots i_l}\equiv\frac{\partial^l}{\partial x^{i_1}\cdots\partial x^{i_l}}\left[\frac{1}{r_A}\right]=(-)^l(2l-1)!!\frac{\hat{n}_A^{i_1\cdots i_l}}{r_A^{l+1}}\ . \tag{4.10}$$

B. The barycentric satellite acceleration

Since the metric (4.1) in the global barycentric system is one of the same form as the one in the geocentric system [Eqs. (2.1)], the satellite equations of motion in the barycentric frame are obtained from Eq. (3.2) simply by replacing geocentric quantities $(X^\alpha, W_\alpha, Z^a_S(T))$ by corresponding barycentric ones $(x^\mu, w_\mu, z^i_S(t))$:

$$\frac{d^2z^i_S(t)}{dt^2}=[w_{,i}+c^{-2}(-4ww_{,i}-4w_{,j}v^j_Sv^i_S+w_{,i}\mathbf{v}_S^2-3w_{,t}v^i_S+4w_{i,t}+8w_{[i,j]}v^j_S)]_{x^i=z^i_S(t)}\ , \tag{4.11}$$

where $\mathbf{z}_S(t)$ is the barycentric coordinate position of the satellite, $v^i_S\equiv dz^i_S(t)/dt$ is its barycentric velocity, and $w_{[i,j]}\equiv\frac{1}{2}(w_{i,j}-w_{j,i})$. We write

$$\frac{d^2z^i_S}{dt^2}=a^i_{\text{LD}}(\mathbf{z}_S,\mathbf{v}_S)+a^i_S+a^i_Q+a^i_{\text{higher}}\ . \tag{4.12}$$

Here, $a^i_{\text{LD}}(\mathbf{z}_S,\mathbf{v}_S)$ is the Lorentz-Droste (Einstein-Infeld-Hoffmann) acceleration of the satellite for a system of mass monopoles [see Eq. (7.20) of paper I with A = satellite]. For the convenience of further discussions below, we write $\mathbf{a}_{\text{LD}}$ in the general form

$$\mathbf{a}_{\text{LD}}(\mathbf{z}_K,\mathbf{v}_K)=-\sum_{B\neq K}\frac{GM_B}{r^2_{KB}}\mathbf{n}_{KB}\left[1+\frac{1}{c^2}\left\{\mathbf{v}_K^2+2\mathbf{v}_B^2-4\mathbf{v}_K\cdot\mathbf{v}_B-\frac{3}{2}(\mathbf{n}_{KB}\cdot\mathbf{v}_B)^2\right\}-4\sum_{C\neq K}\frac{GM_C}{c^2r_{KC}}-\sum_{C\neq B}\frac{GM_C}{c^2r_{BC}}\left\{1+\frac{1}{2}\frac{r_{KB}}{r_{CB}}\mathbf{n}_{KB}\cdot\mathbf{n}_{CB}\right\}\right]-\frac{7}{2}\sum_{B\neq K}\sum_{C\neq B}\mathbf{n}_{BC}\frac{G^2M_BM_C}{c^2r_{KB}r^2_{BC}}+\sum_{B\neq K}(\mathbf{v}_K-\mathbf{v}_B)\frac{GM_B}{c^2r^2_{KB}}(4\mathbf{n}_{KB}\cdot\mathbf{v}_K-3\mathbf{n}_{KB}\cdot\mathbf{v}_B)\ , \tag{4.13}$$

where

$$r_{KB}\equiv|\mathbf{z}_K(t)-\mathbf{z}_B(t)|\ ,$$

$$\mathbf{n}_{KB}\equiv[\mathbf{z}_K(t)-\mathbf{z}_B(t)]/r_{KB}\ .$$

The first term on the right-hand side of Eq. (4.12) is simply obtained by taking $K=S$ (=satellite) in Eq. (4.13) and by neglecting M_S in the sums where it could appear (on the other hand, the sums over B and C *must* evidently include the label A = Earth).

$\mathbf{a}_S$ and $\mathbf{a}_Q$ in Eq. (4.12) are the barycentric accelerations caused by the Earth's spin and mass-quadrupole moments, respectively. These accelerations originate in the $w^A_\mu|_S$ and $w^A_\mu|_Q$ terms of Eqs. (4.8) and (4.9). All the acceleration effects resulting from inserting $w^A_\mu|_{\text{LD}}$, Eq. (4.7), into the satellite equations of motion (4.11) are contained in a^i_{LD}. The spin acceleration a^i_S results from the $w^A_\mu|_S$ terms in $w^A_{,i}$, $4c^{-2}w^A_{i,t}$, and $8c^{-2}w^A_{[i,j]}v^j_S$. Neglecting time derivatives of the spin vector, these contributions together yield ($\mathbf{v}_{SA}\equiv\mathbf{v}_S-\mathbf{v}_A$)

$$a^i_S=\frac{4G}{c^2}v^j_{SA}s_l\epsilon_{kl[i}\varphi^A_{j]k}\ , \tag{4.14}$$

in agreement with Eq. (6.32) of paper II. Note, that a^i_S depends only upon the *relative* velocity of the satellite with respect to the geocenter v_{SA}. This results from the fact that s_l are the components (projected onto the global system) of the spin vector of the Earth as measured in the local mass-centered geocentric (A) system. Additional spin-orbit terms as they appear in other frameworks (see, e.g., Eq. (5.1.12) of Brumberg [27]) do *not* occur in our framework.

Let us now consider the quadrupole acceleration terms in the global barycentric system. We first split a^i_Q in the following way:

$$a^i_Q=a^{i[0]}_Q+a^{i[2]}_Q\ , \tag{4.15}$$

where $a^{i[0]}_Q$ is the quasi-Newtonian acceleration of the satellite due to the quadrupole mass moments (C_{2m} and S_{2m} in the usual spherical representation) of the Earth:

$$a^{i[0]}_Q=\tfrac{1}{2}Gm_{jk}\varphi^A_{ijk}\ . \tag{4.16}$$

Let us stress again that although the expression (4.16) for $a^{i[0]}_Q$ looks perfectly "Newtonian" the mass-quadrupole moments m_{jk} are the (projected) full post-Newtonian quadrupole moments of the Earth. It should be stressed also that the decomposition into "quasi-Newtonian" plus "relativistic" parts is frame dependent. The barycentric quasi-Newtonian accelerations differ by $O(c^{-2})$ from their geocentric counterparts (when such natural counterparts exist). The "relativistic" part, $a_Q^{i[2]}$ will be split into three parts according to

$$a_Q^{i[2]}=a_{RQ,\mathrm{mix}}^i+a_{RQ,1}^i+a_{RQ,2}^i \ . \tag{4.17}$$

Here, $a_{RQ,\mathrm{mix}}^i$ denotes that part of $-4c^{-2}ww_{,i}$ that contains products of the Earth's quadrupole moment and external masses; terms of order $c^{-2}m_{ij}m_{jk}$ will be neglected. $a_{RQ,\mathrm{mix}}^i$ is given by

$$a_{RQ,\mathrm{mix}}^i=-\frac{2Gm_{jk}}{c^2}(\overline{w}_{A,i}\varphi_{jk}^A+\overline{w}_A\varphi_{ijk}^A) \tag{4.18}$$

with

$$\overline{w}_A=\sum_{B\neq A}\frac{GM_B}{r_B}+O(2) \ .$$

The two remaining quadrupole-acceleration terms read

$$a_{RQ,1}^i=\frac{Gm_{jk}}{c^2}[-2GM_A(\varphi_{jk}^A\varphi_i^A+\varphi^A\varphi_{ijk}^A) \\ -2v_{SA}^i v_{SA}^l\varphi_{jkl}^A+\tfrac{1}{2}\mathbf{v}_{SA}^2\varphi_{ijk}^A] \tag{4.19}$$

and

$$a_{RQ,2}^i=\frac{Gm_{jk}}{c^2}\{2\varphi_{ij}^A a_A^k+2\varphi_{jk}^A a_A^i+\tfrac{1}{4}\partial_{ttijk}r_A(t)-\varphi_{ijk}^A[\mathbf{v}_A\cdot\mathbf{v}_S-\tfrac{1}{4}\mathbf{v}_A^2+\tfrac{3}{2}\overline{w}_A(\mathbf{z}_A)]-\tfrac{1}{2}\varphi_{jkl}^A v_S^i v_A^l\} \\ -\frac{G}{2c^2}v_A^l v_{\langle k}^A m_{j\rangle l}\varphi_{ikj}^A \ . \tag{4.20}$$

Here, $a_{RQ,1}^i$ is the barycentric counterpart of the geocentric relativistic quadrupole acceleration Eq. (3.14). Let us remark that $\partial_{ttijk}r_A(t)$ (where the first two indices denote global-time derivatives) is given by [see Eq. (A18) of paper II]

$$\partial_{ttijk}r_A(t)=v_A^l v_A^m\partial_{ijklm}r_A(t)-a_A^l\partial_{ijkl}r_A(t) \ , \tag{4.21a}$$

with

$$\partial_{i_1\cdots i_m}(r_A)=\frac{(-)^{m-1}(2m-3)!!}{r_A^{m-1}}\left[\hat{n}_A^{i_1\cdots i_m}-\frac{m(m-1)}{(2m-1)(2m-3)}\delta^{(i_1 i_2}\hat{n}_A^{i_3\cdots i_m)}\right] \ . \tag{4.21b}$$

Finally the acceleration arising from the higher-multipole moments of the Earth is treated at the Newtonian approximation:

$$a_{\mathrm{higher}}^i\simeq\sum_{l\geq 3}\frac{(-)^l}{l!}Gm_{j_1\cdots j_l}\varphi_{ij_1\cdots j_l} \ .$$

C. The relative barycentric satellite acceleration

For practical applications (especially if numerical integrations are required) one usually will employ an expression for the barycentric satellite acceleration relative to the barycentric position of the Earth taken at the same barycentric coordinate time t. Let us denote

$$r_{SA}^i(t)\equiv z_S^i(t)-z_A^i(t) \ . \tag{4.22}$$

When computing the relative acceleration

$$\frac{d^2r_{SA}^i}{dt^2}=\frac{d^2z_S^i(t)}{dt^2}-\frac{d^2z_A^i(t)}{dt^2} \tag{4.23}$$

we need to insert the equations of motion of the Earth. Contrary to the satellite, the extended structure of the Earth causes its center of mass *not* to follow a geodesic of the Earth-external metric $\overline{g}_{\mu\nu}^A(x^\lambda)$. The full structure (at the first post-Newtonian approximation) of the translational equations of motion of an extended body has been discussed in detail in Sec. VI of paper II. The deviation from a geodesic motion

$$[-G_a^A\equiv\overline{g}_{\mu\nu}^A(z_A)e_a^{A\mu}\overline{\nabla}_{\overline{u}_A}\overline{u}_A^\nu\neq 0]$$

is measured by the "force term" $\hat{\Phi}_a^A$ defined by Eqs. (6.12) and (6.14) of paper II:

$$-M^A G_a^A=\hat{\Phi}_a^A+O(4) \ . \tag{4.24}$$

If we retain in $\hat{\Phi}_a$ only the Newtonian contributions, and the lowest-order relativistic one, we can write

$$\hat{\Phi}_a^A\simeq\sum_{l\geq 2}\frac{1}{l!}M_L^A G_{aL}^A+\frac{1}{2c^2}S_b^A H_{ab}^A \ . \tag{4.25}$$

In view of the smallness of $\hat{\Phi}_a^A$ we can safely neglect terms of order $\hat{\Phi}_a\times(v^2/c^2)$ and write the barycentric coordinate acceleration of the Earth (in the approximation where the other bodies are treated as mass monopoles) as

$$\frac{d^2z_A^i}{dt^2}=a_{\mathrm{LD}}^i(z_A,v_A)+a_\Phi^i \ , \tag{4.26a}$$

with

$$a_\Phi^i \simeq R_a^{Ai} \frac{\hat{\Phi}_a^A}{M^A} . \tag{4.26b}$$

Therefore, the relative acceleration of a satellite can be written as (note the minus sign in the last term)

$$\frac{d^2 r_{SA}^i}{dt^2} = \Delta a_{\rm LD}^i + a_S^i + a_Q^i + a_{\rm higher}^i - a_\Phi^i , \tag{4.27}$$

where

$$\Delta a_{\rm LD}^i \equiv a_{\rm LD}^i(\mathbf{z}_S, \mathbf{v}_S) - a_{\rm LD}^i(\mathbf{z}_A, \mathbf{v}_A) . \tag{4.28}$$

We have given above the explicit expressions of the spin, quadrupole, and higher-multipole accelerations. From Eq. (4.25) we see that the lowest-order approximation of the extra term $-a_\Phi^i$ reads

$$-a_\Phi^i \simeq -\frac{1}{2} R_a^{Ai} \frac{M_{bc}^A}{M^A} G_{abc}^A \simeq -\frac{1}{2} \frac{m_{jk}^A}{M^A} \partial_{ijk} \overline{w}^A . \tag{4.29}$$

In order to explicate in detail the structure of the term $\Delta a_{\rm LD}^i$ it is convenient to introduce some notation. Let, for any pair of bodies (K,B),

$$\varepsilon_{KB} \equiv \mathbf{v}_K^2 + 2\mathbf{v}_B^2 - 4\mathbf{v}_K \cdot \mathbf{v}_B - \tfrac{3}{2}(\mathbf{n}_{KB} \cdot \mathbf{v}_B)^2 - 4\overline{w}_K - \overline{w}_B - \frac{1}{2} r_{KB} (\mathbf{n}_{KB} \cdot \mathbf{a}_B) - \frac{7}{2} \frac{GM_K}{r_{KB}} , \tag{4.30}$$

where, as above, $\overline{w}_B = \sum_{C \neq B} GM_C / r_{BC}$ is the gravitational potential generated by all the other bodies at the center of mass of B and where $\mathbf{a}_B$ is the acceleration of body B, given with sufficient accuracy by $\mathbf{a}_B \simeq -\sum GM_C \mathbf{n}_{BC} / r_{BC}^2$, and

$$\mathbf{b}_{KB} \equiv \frac{GM_B}{r_{KB}^2} \left[-\left(1 + \frac{1}{c^2} \varepsilon_{KB} \right) \mathbf{n}_{KB} + \frac{1}{c^2} (4\mathbf{n}_{KB} \cdot \mathbf{v}_K - 3\mathbf{n}_{KB} \cdot \mathbf{v}_B)(\mathbf{v}_K - \mathbf{v}_B) \right] . \tag{4.31}$$

With this notation the general Lorentz-Droste acceleration (4.13) reads

$$\mathbf{a}_{\rm LD}(K) = \sum_{B \neq K} \mathbf{b}_{KB} - \frac{7}{2} \sum_{\substack{B \neq K \\ C \neq K \\ B \neq C}}{}' \frac{G^2 M_B M_C}{c^2 r_{KB} r_{BC}^2} \mathbf{n}_{BC} . \tag{4.32}$$

In other words, $\mathbf{b}_{KB}$ represents essentially the (barycentric frame) relativistic generalization of the accelerative force on body K caused by body B *(two-body* interaction), while we have separated out the explicit *three-body* effects in the last term (accelerative force on K caused jointly by two other bodies, B and C). Note that the "two-body" interaction $\mathbf{b}_{KB}$ (which can be usefully thought of as a diagram of the form K-B) contains, however, some implicit three-body effects (in $\overline{w}_K$, $\overline{w}_B$, and $\mathbf{a}_B$), and that the sum over the three-body effects in Eq. (4.32) runs only on diagrams K-B-C containing three different points [the term K-B-K present in Eq. (4.13) has been incorporated in the effective K-B interaction as the last term in Eq. (4.30)].

With this notation in hand it is easy to find that $\Delta \mathbf{a}_{\rm LD}$ can be written as (where A labels the Earth)

$$\Delta \mathbf{a}_{\rm LD} = \mathbf{b}_{SA} - \frac{7}{2} \sum_B{}' \frac{G^2 M_A M_B}{c^2 r_{AB}^2} \left(\frac{1}{r_{SA}} - \frac{1}{r_{SB}} \right) \mathbf{n}_{AB} + \sum_B{}' (\mathbf{b}_{SB} - \mathbf{b}_{AB}) - \frac{7}{2} \sum_{B,C}{}' \frac{G^2 M_B M_C}{c^2 r_{BC}^2} \left(\frac{1}{r_{SB}} - \frac{1}{r_{AB}} \right) \mathbf{n}_{BC} , \tag{4.33}$$

where the primes over the summation symbols indicate that B and C are different from S and A, and different from each other. The first term on the right-hand side of Eq. (4.33) represents the barycentric-frame post-Newtonian acceleration primarily caused by the mass of the Earth, including $O(c^{-2})$ corrections due to velocity $(\mathbf{v}_S, \mathbf{v}_A)$, potential $(\overline{w}_S, \overline{w}_A)$, and acceleration $(\mathbf{r}_{SA} \cdot \mathbf{a}_A)$ effects. [When using this term to solve for the relative barycentric position $\mathbf{r}_{SA}(t)$ one can advantageously replace $\mathbf{v}_S$ by $\mathbf{v}_{SA} + \mathbf{v}_A$ to reexpress the velocity-dependent corrections in terms of $\mathbf{v}_{SA}$ and $\mathbf{v}_A$.] The last two terms in Eq. (4.33) have the form of differential ("tidal") effects: $F(S)$-$F(A)$.

Finally, let us recall that when working in barycentric coordinates one must be very careful in transforming coordinate-dependent quantities into observables. In particular, there are many cancellations between large "coordinate effects" in the orbital motion and in the calculation of the laser time of flight [2].

V. CONCLUDING REMARKS

This paper has shown explicitly how to apply the new formulation of the relativistic theory of reference systems proposed in papers I–III to the specific problem of the motion of artificial satellites. Among the main advantages of our approach with respect to previous attempts one can note (i) its completeness (no *ad hoc* unjustiifed approximations are made, and all multipole and tidal contributions within post-Newtonian accuracy are included), (ii) its consistency (all the concepts and quantities entering our scheme are clearly defined and their transformation properties under a change of coordinate system are known), and (iii) and its linearity (although details can get messy when keeping all the terms, the remarkable linear properties of the variables used in our scheme allow one to keep always a clear conceptual overall grasp of what the formalism does for you). We think that these features of the formalism proposed in papers I–III makes it an ideal tool for dealing with many practical aspects of relativistic celestial mechanics and astrometry in the solar system (such as very long base line interferometry [33], global positioning system, lunar laser ranging, high-precision ephemeris programs). We also feel that the conceptual simplicity of our formalism should be very useful in clarifying the many conceptual subtleties that arise when trying to shift from a Newtonian view to a consistent Einsteinian description of the experiments and observations made in our local (noncosmological) part of the Universe.

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APPENDIX A: TRANSFORMATION RULES FOR "POSITION VECTOR" AND SATELLITE ACCELERATION

This appendix serves several purposes. It provides transformation rules between the global barycentric and the local geocentric expressions for the satellite acceleration and, hence, a useful piece of information if one wants to explicitly relate our barycentric with our geocentric satellite equations of motion by direct transformations. Such relations between the various forms of the satellite equations of motion by means of direct transformations might be useful as algebraic checks. It is clear from papers I–III (especially from the various transformation rules for the metric potentials, etc.) that the overall consistency of our formalism has already been explicitly proven and needs not be checked in all applications. However, checks of the complicated algebra that arises in applications are always useful (note that this situation is different in the Brumberg-Kopejkin formalism, whose overall consistency is less clear and where it might be advantageous to derive the geocentric satellite equations of motion from the global ones by direct transformations). Another purpose of this appendix is to make contact between the general methods of our formalism and some more *ad hoc* procedures used recently [14,15] to deal with the relativistic effects of the Earth quadrupole moment when seen in the barycentric frame. Let us start by recalling that the general relation between global barycentric coordinates (ct,x^i) and local geocentric ones (cT,X^a) is written in the form [Eq. (1.6) of paper I]

$$x^\mu(X^\alpha)=z^\mu(T)+e^\mu_a(T)X^a+\xi^\mu(T,X^a)\ , \qquad \text{(A1)}$$

where ξ^μ contains these terms which are at least quadratic in the local space coordinate X^a. In paper I it was shown that $\xi^0=O(3)$ and

$$\xi^i=\frac{1}{c^2}e^i_a(T)\left[\frac{1}{2}A_a\mathbf{X}^2-X^a(\mathbf{A}\cdot\mathbf{X})\right]+O(4)\ . \qquad \text{(A2)}$$

Here,

$$A_a=e^i_a\frac{d^2z^i}{dT^2}+O(2) \qquad \text{(A3)}$$

is essentially the barycentric acceleration of the geocenter projected onto the geocentric coordinate lines. Let us now consider the three events in Fig. 1 denoted by e_S, e_t, and e_T. e_S refers to the satellite at barycentric coordinate time t and barycentric coordinate x^i. In the local geocentric frame e_S has coordinates (cT,X^a), related to (ct,x^i) by Eq. (A1). e_t (e_T) denotes the intersection of the $t=$const ($T=$const) hypersurface through e_S with the central world line of the Earth, labeled A [usually chosen as the post-Newtonian (BD) geocenter], given by $X^a=0$. These two events have coordinates:

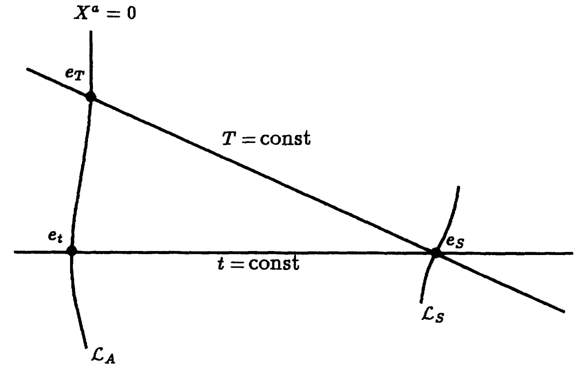

FIG. 1. Three events of importance for the description of relativistic satellite motion where $v^i_{SA}\equiv v^i_S-v^i_A$ is the barycentric satellite velocity relative to the Earth.

$$e_t:\ (t,z^i_A(t))\ \ (T_{\rm sim},\mathbf{0})\ , \qquad \text{(A4a)}$$

$$e_T:\ (t_{\rm sim},z^i_A(t_{\rm sim}))\ \ (T,\mathbf{0})\ . \qquad \text{(A4b)}$$

Using relation (A1) one finds that

$$T_{\rm sim}=T+\frac{1}{c}\frac{e^0_a}{e^0_0}X^a+O(4)\ , \qquad \text{(A5a)}$$

$$t_{\rm sim}=t-\frac{1}{c}e^0_aX^a+O(4)\ . \qquad \text{(A5b)}$$

Furthermore, the differential relation between t and T along the world line of the satellite $\mathcal{L}_S$ is given by

$$\left[\frac{dt}{dT}\right]_{\mathcal{L}_S}=e^0_0(T)+\frac{1}{c}\left[\frac{d}{dT}e^0_a\right]X^a_S+\frac{1}{c}e^0_aV^a_S+O(4)$$
$$=1+\frac{1}{c^2}\left[\frac{1}{2}\mathbf{v}^2_A+\bar{w}_A(\mathbf{z}_A)+R^i_a(a^i_AX^a_S+v^i_AV^a_S)\right]$$
$$+O(4)\ , \qquad \text{(A6)}$$

where $V^a_S=dX^a_S/dT$ is the satellite velocity in the geocentric frame. From Eqs. (A4) one finds for the transformation rule of the relative position vector (relative coordinates in our charts),

$$r^i_{SA}(t)=\left[e^i_a-\frac{e^i_0e^0_a}{e^0_0}\right]X^a_S+\xi^i(T,X^a_S)$$
$$=\left[1-\frac{\bar{w}_A(\mathbf{z}_A)}{c^2}\right]R^j_a(T)\left[\delta^{ij}-\frac{1}{2c^2}v^i_Av^j_A\right]X^a_S$$
$$+\frac{1}{c^2}R^i_a[\tfrac{1}{2}A^a_A\mathbf{X}^2_S-X^a_S(\mathbf{A}_A\cdot\mathbf{X}_S)]\ , \qquad \text{(A7)}$$

where

$$r^i_{SA}(t)\equiv(x^i_S-z^i_A)(t)\ . \qquad \text{(A8)}$$

For completeness, let us note in passing that if the relative barycentric position vector r^i_{SA} is referred to time $t_{\rm sim}$, as was done in the paper by Ries, Huang, and Watkins [14], we get

$$r_{SA}^i(t_{\rm sim})=r_{SA}^i(t)-\frac{1}{c^2}v_{SA}^i R_a^j v_A^j X^a+O(4)\ . \tag{A9}$$

The transformation of the satellite acceleration can be achieved by means of

$$\frac{d^2r_{SA}^i(t)}{dt^2}=\frac{d^2r_{SA}^i}{dT^2}\left[\frac{dt}{dT}\right]_{\mathcal{L}_S}^{-2}-\frac{(dr_{SA}^i/dT)(d^2t/dT^2)_{\mathcal{L}_S}}{(dt/dT)_{\mathcal{L}_S}^3}$$
$$=\frac{d^2r_{SA}^i}{dT^2}\left[\frac{dt}{dT}\right]_{\mathcal{L}_S}^{-2}-v_{SA}^i\left[\frac{d^2t}{dT^2}\right]+O(4)\ , \tag{A10}$$

with

$$\left[\frac{d^2t}{dT^2}\right]_{\mathcal{L}_S}=\frac{1}{c^2}[\mathbf{v}_A\cdot\mathbf{a}_A+\dot{\bar{w}}_A(\mathbf{z}_A)$$
$$+R_a^i(\dot{a}_A^iX_S^a+2a_A^iV_S^a+v_A^iA_S^a)]+O(4)\ . \tag{A11}$$

The value of $d^2r_{SA}^i/dT^2$ to be inserted in Eq. (A10) is obtained by differentiating (A7). In terms of $A_S^a(T)\equiv d^2X_S^a/dT^2$ one finds in obvious notation (e.g., $\mathbf{v}_A\cdot\mathbf{V}_S\equiv R_a^iv_A^iV_S^a$, etc.)

$$\frac{d^2r_{SA}^i}{dT^2}=R_a^iA_s^a+2\left[\frac{dR_a^i}{dT}\right]V_S^a+\left[\frac{d^2R_a^i}{dT^2}\right]X_S^a$$
$$-\frac{1}{c^2}[\bar{w}_A(\mathbf{z}_A)A_S^i+\tfrac{1}{2}(\dot{\mathbf{a}}_A\cdot\mathbf{X}_S+2\mathbf{a}_A\cdot\mathbf{V}_S+\mathbf{v}_A\cdot\mathbf{A}_S)v_A^i+(\mathbf{a}_A\cdot\mathbf{X}_S+\mathbf{v}_A\cdot\mathbf{V}_S)a_A^i+\tfrac{1}{2}(\mathbf{v}_A\cdot\mathbf{X}_S)\dot{a}_A^i$$
$$+2\dot{\bar{w}}_A(\mathbf{z}_A)V_S^i+\ddot{\bar{w}}_A(\mathbf{z}_A)X_S^i+(\ddot{\mathbf{a}}_A\cdot\mathbf{X}_S+2\dot{\mathbf{a}}_A\cdot\mathbf{V}_S+\mathbf{a}_A\cdot\mathbf{A}_S)X_S^i+2(\dot{\mathbf{a}}_A\cdot\mathbf{X}_S+\mathbf{a}_A\cdot\mathbf{V}_S)V_S^i$$
$$+(\mathbf{a}_A\cdot\mathbf{X}_S)A_S^i-(\mathbf{A}_S\cdot\mathbf{X}_S+\mathbf{V}_S^2)a_A^i-2(\mathbf{V}_S\cdot\mathbf{X}_S)\dot{a}_A^i-\tfrac{1}{2}\ddot{a}_A^i\mathbf{X}_S^2]\ . \tag{A12}$$

This result represents a generalization of that found by Huang *et al.* [15].

APPENDIX B: DERIVATION OF THE BARYCENTRIC METRIC

The metric potentials $w_\mu\equiv(w,w_i)$ yielding the barycentric metric (4.1) can be written as a sum

$$w_\mu=\sum_B w_\mu^B\ , \tag{B1}$$

where w_μ^B is the contribution from body B of our N-body system of massive bodies in the solar system, i.e., $B=$Earth (body A), Moon, Sun, other planets, etc. w_μ^B was already evaluated in paper II. From Eq. (5.33) of paper II taking as "local X_A^α system" the global x^μ system (by putting $v_A^i=0, R_{ia}^A=\delta_{ia}$ in paper II) we get

$$w^B(x^\mu)=G\sum_{k\geq 0}\frac{(-)^k}{k!}\partial_{\nu_1\cdots\nu_k}\left[\frac{\tilde{m}_{\nu_1\cdots\nu_k}^B(S_B)}{\rho_B}\right]_{\pm}$$
$$+O(4)\ , \tag{B2a}$$

$$w_i^B(x^\mu)=G\sum_{k\geq 0}\frac{(-)^k}{k!}\partial_{\nu_1\cdots\nu_k}\left[\frac{\tilde{p}_{i\nu_1\cdots\nu_k}^B(S_B)}{\rho_B}\right]_{\pm}$$
$$+O(2)\ , \tag{B2b}$$

where

$$\tilde{m}_{\nu_1\cdots\nu_k}^B(S_B)$$
$$\equiv e_{b_1\cdots b_k}^{B\nu_1\cdots\nu_k}\left[\mathcal{M}_{b_1\cdots b_k}^B-\frac{k}{c^2}A_c^B\mathcal{M}_{cb_1\cdots b_k}^B\right]+O(4)\ , \tag{B3a}$$

$$\tilde{p}_{i\nu_1\cdots\nu_k}^B(S_B)\equiv e_{b_1\cdots b_k}^{B\nu_1\cdots\nu_k}\mathcal{P}_{ib_1\cdots b_k}^B+O(2)\ , \tag{B3b}$$

and

$$\mathcal{M}_L^B\equiv\left[1+\frac{2}{c^2}\mathbf{v}_B^2\right]M_L^B+\frac{4}{(l+1)c^2}v_B^iR_{ib}^BM_{bL}^{(1)B}$$
$$-\frac{4l}{(l+1)c^2}v_B^iR_{ib}^BS_{b\langle L\rangle}^{B*}\ , \tag{B4a}$$

$$\mathcal{P}_{iL}^B\equiv v_B^iM_L^B+\frac{1}{l+1}R_{ia}^BM_{aL}^{(1)B}-\frac{l}{l+1}R_{ia}^BS_{a\langle L\rangle}^{B*}\ . \tag{B4b}$$

Here,

$$S_{b\langle L\rangle}^{B*}\equiv\epsilon_{cb\langle b_l}S_{L-1\rangle c}^B \tag{B5}$$

and all remaining notations are explained in paper II. M_L^B and S_L^B are the relativistic, Blanchet-Damour mass- and spin-multipole moments of body B, defined in the corresponding rest frame of body B as functions of T_B, the local coordinate time in the B frame ($M_{aL}^{(1)B}\equiv dM_{aL}^B/dT_B$, etc.). As was shown in paper II, the result (B2) for $w_\mu^B(x)$ can be written in quasi-Newtonian form. Using Eqs. (A7) and (A8) of paper II we get

$$w^B(t,x^i)=G\sum_{k\geq 0}\frac{(-)^k}{k!}\partial_{j_1\cdots j_k}\left[\frac{\overline{m}_{j_1\cdots j_k}^B(t)}{r_B}+\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}[\overline{m}_{j_1\cdots j_k}^B(t)r_B(t)]\right]+O(4)\ , \tag{B6a}$$

$$w_i^B(t,x^i)=G\sum_{k\geq 0}\frac{(-)^k}{k!}\partial_{j_1\cdots j_k}\left[\frac{\overline{p}^B_{ij_1\cdots j_k}}{r_B}\right]+O(2)\ , \tag{B6b}$$

with

$$\overline{m}^B_{j_1\cdots j_k}(t)=\left[(e_0^{B0})^{-k-1}\left[1+\frac{k}{2c^2}(\mathbf{v}_B)^2\right]R^{Bj_1}_{b_1}\cdots R^{Bj_k}_{b_k}\left[\mathcal{M}^B_{b_1\cdots b_k}-\frac{k+1}{c^2}A_c^B\mathcal{M}^B_{cb_1\cdots b_k}-\frac{k}{2c^2}V_c^BV^B_{\langle b_k}\mathcal{M}^B_{b_1\cdots b_{k-1}\rangle c}-\frac{1}{c^2}V_c^B\frac{d}{dT_B}\mathcal{M}^B_{cb_1\cdots b_k}\right]\right]_{z_B^0=ct}+O(4)\ , \tag{B7a}$$

$$\overline{p}^B_{ij_1\cdots j_k}(t)=\{R^{Bj_1}_{b_1}\cdots R^{Bj_k}_{b_k}\mathcal{P}^B_{ib_1\cdots b_k}\}_{z_B^0=ct}+O(2)\ , \tag{B7b}$$

and [assuming the weak effacement condition in Eq. (5.12) of Paper I]

$$e_0^{B0}=1+\frac{1}{c^2}\left[\frac{1}{2}\mathbf{v}_B^2+\overline{w}(\mathbf{z}_B)\right]+O(4)\ . \tag{B8}$$

For the problem of Earth-orbiting satellites we write

$$w_\mu=w_\mu^A+\overline{w}_\mu^A\ , \tag{B9}$$

where w_μ^A results from the gravitational action of the Earth ("body A") and $\overline{w}_\mu^A$ from all other massive bodies in the solar system. To derive the post-Newtonian acceleration of a satellite we will consider all bodies $B\neq A$ in the mass-monopole approximation. Therefore, from Eq. (7.14) of paper I we get

$$\overline{w}^A(x)=\sum_{B\neq A}\left[\frac{GM_B(1+2\mathbf{v}_B^2/c^2)}{\rho_B}\right]_\pm+O(4)\ , \tag{B10a}$$

$$\overline{w}_i^A(x)=\sum_{B\neq A}\left[\frac{GM_Bv_B^i}{\rho_B}\right]_\pm+O(2)\ . \tag{B10b}$$

Inserting the expression from Eq. (7.19) of paper I this leads to Eqs. (4.3).

Using Eqs. (B7) the components of $\overline{m}^A_{j_1\cdots j_k}$ and $\overline{p}^A_{ij_1\cdots j_k}$ that we will consider for the problem of satellite motion read

$$\overline{m}^A=\left[1+\frac{3}{2c^2}\mathbf{v}_A^2-\frac{\overline{w}_A(\mathbf{z}_A)}{c^2}\right]M_A+O(4)\ , \tag{B11a}$$

$$\overline{m}_j^A=\frac{1}{c^2}R_a^{Aj}(V_b^AM_{ab}^{(1)}+2V_b^A\epsilon_{abc}S_c-2A_b^AM_{ab})+O(4)\ , \tag{B11b}$$

$$\overline{m}^A_{jk}=R_a^{Aj}R_b^{Ak}\left[\left[1+\frac{3}{2c^2}\mathbf{v}_A^2-\frac{3\overline{w}_A(\mathbf{z}_A)}{c^2}\right]M_{ab}-\frac{1}{c^2}V_c^AV^A_{\langle b}M_{a\rangle c}\right]+O(4)\ , \tag{B11c}$$

and

$$\overline{p}_i^A=M_Av_A^i+O(2)\ , \tag{B12a}$$

$$\overline{p}_{ij}^A=\tfrac{1}{2}R_a^{Aj}R_b^{Ai}(M_{ab}^{(1)}+\epsilon_{abc}S_c)+O(2)\ , \tag{B12b}$$

$$\overline{p}_{ijk}^A=R_a^{Aj}R_b^{Ak}M_{ab}v_A^i+O(2)\ . \tag{B12c}$$

Here, $V_c^A\equiv R_{ic}^Av_A^i$, etc. For use in the text it is convenient to separate the various contributions to $\overline{m}\ldots$ and $\overline{p}\ldots$ according to the basic multipole moments that they arise from. Also we neglect all relativistic terms containing time derivatives of the Earth's multipole moments in the local geocentric system. We write

$$\overline{m}^A_{\ldots}=\overline{m}^A_{\ldots}|_{\rm LD}+\overline{m}^A_{\ldots}|_S+\overline{m}^A_{\ldots}|_Q+\overline{m}^A_{\ldots}|_{\rm higher}\ , \tag{B13a}$$

$$\overline{p}^A_{\ldots}=\overline{p}^A_{\ldots}|_{\rm LD}+\overline{p}^A_{\ldots}|_S+\overline{p}^A_{\ldots}|_Q+\overline{p}^A_{\ldots}|_{\rm higher}\ , \tag{B13b}$$

where the index LD refers to the Lorentz-Droste (Einstein-Infeld-Hoffmann) mass-monopole part, S to the spin parts, Q to the mass-quadrupole parts, and "higher" to the higher-order multipole parts. Neglecting $M_L^{(1)}$ one has, in obvious notation for the nonvanishing terms,

$$\overline{m}^A|_{\rm LD}=\left[1+\frac{3}{2c^2}\mathbf{v}_A^2-\frac{\overline{w}_A(\mathbf{z}_A)}{c^2}\right]M_A\ , \tag{B14a}$$

$$\overline{p}_i^A|_{\rm LD}=M_Av_A^i\ , \tag{B14b}$$

$$\overline{m}_j^A|_S=\frac{2}{c^2}\epsilon_{jkl}v_A^ks_l\ , \tag{B15a}$$

$$\overline{p}_{ij}^A|_S=-\tfrac{1}{2}\epsilon_{ijk}s_k\ , \tag{B15b}$$

$$\overline{m}_j^A|_Q=-\frac{2}{c^2}m_{jk}a_A^k\ , \tag{B16a}$$

$$\overline{m}_{jk}^A|_Q=\left[1+\frac{3}{2c^2}\mathbf{v}_A^2-\frac{3\overline{w}_A(\mathbf{z}_A)}{c^2}\right]m_{jk}-\frac{1}{c^2}v_A^lv^A_{\langle k}m_{j\rangle l}\ , \tag{B16b}$$

$$\overline{p}_{ijk}^A|_Q=m_{jk}v_A^i\ , \tag{B16c}$$

and

$$\overline{m}^A_{j_1\cdots j_l}|_{\text{higher}}=m_{j_1\cdots j_l}+O(2)$$

[neglecting the effects associated with the $O(2)$ correction to $m_{j_1\cdots j_l}$ and to $\overline{p}^A_{j_1\cdots j_l}|_{\text{higher}}$]. Here, $m_{jk}\equiv R_a^{Aj}R_b^{Ak}M_{ab}$, etc. (as explained in the text, we drop the label A on all the Earth multipole moments). Equations (4.7)–(4.9) are then obtained by inserting these expression for $\overline{m}^A_{\cdots}$ and $\overline{p}^A_{\cdots}$ into Eqs. (B6).

APPENDIX C: SOME DETAILS ON THE RELATIVISTIC CORIOLIS PRECESSION

As discussed in detail in this and our previous papers there are two preferred choices for the rotational state of the geocentric spatial coordinate grid leading to either a globally fixed (kinematically nonrotating), or a Coriolis effacing (dynamically nonrotating) geocentric reference frame. The globally fixed geocentric frame is defined by

$$R^A_{ia}(T)\equiv\delta_{ia}\ .$$

Then the Coriolis effects in Eqs. (3.27) and (3.28) have to be taken into account in the equation of satellite motion. The Coriolis effacing (dynamically nonrotating) local frame is characterized by the vanishing of relativistic Coriolis forces, Eq. (3.27). In this case the geocentric spatial coordinate grid must precess with respect to the global barycentric one. The exact expression of the needed time-dependent orthogonal matrix $R_a^{Ai}(T)$ is obtained by integrating the differential equation (4.19b) of paper III, which says, in fact, that the vectorial rotational velocity of the Coriolis-effacing frames is identical with Ω_{Cor} which enters the satellite equations of motion of the globally fixed local frames. Let us give here only an approximate treatment based on considering that the main contribution to the right-hand side of Eq. (4.19b) of paper III is the one due to the gravitoelectric field of the Sun $\overline{\mathbf{e}}=\nabla(GM_\odot/r)$, where $r=|\mathbf{z}_\oplus-\mathbf{z}_\odot|$ is the Earth-Sun distance, with the approximation that the barycentric acceleration of the Earth $\mathbf{a}_\oplus\simeq\overline{\mathbf{e}}$. This leads to

$$\mathbf{\Omega}_{\text{Cor}}\simeq\frac{3}{2c^2}\mathbf{v}_\oplus\times\overline{\mathbf{e}}\simeq\frac{3GM_\odot}{2c^2r^3}\mathbf{r}\times\mathbf{v}_\oplus\ , \tag{C1}$$

where $\mathbf{v}_\oplus$ can be equivalently considered, within the present approximate treatment, to be the barycentric or the heliocentric velocity of the Earth. In the further approximation where the direction of the ecliptic plane is fixed ($\mathbf{r}\times\mathbf{v}_\oplus\simeq$const), and taken as the x-y plane of a barycentric coordinate system we can write the rotational matrix of Coriolis-effacing frames as

$$R^A_{ia}(T)=\begin{pmatrix}\cos\Phi & \sin\Phi & 0\\ -\sin\Phi & \cos\Phi & 0\\ 0 & 0 & 1\end{pmatrix} \tag{C2}$$

with

$$\Phi=\int\Omega_{\text{Cor}}dt\ . \tag{C3}$$

For practical calculations we can express Ω_{Cor} by the usual Keplerian elements of the Earth's orbit

$$\mathbf{\Omega}_{\text{Cor}}\simeq\frac{3}{2c^2}\frac{(GM_\odot)^{3/2}}{[a(1-e^2)]^{5/2}}(1+e\cos f)^3\mathbf{k}\ , \tag{C4}$$

where a, e, and f are the semimajor axis, eccentricity, and true anomaly of the Earth's orbit around the Sun and $\mathbf{k}$ is a unit vector (in the usual Euclidean sense; $k^jk^j=1$) pointing along the angular momentum of the Earth motion, i.e., "perpendicular to the ecliptic." From (C3) and (C4) one finds

$$\Phi=\frac{3}{2}\frac{GM_\odot}{a(1-e^2)}(f+e\sin f)\ . \tag{C5}$$

The secular part of Φ, Eq. (C5), amounts to about 2 arc sec/century ("de Sitter precession" or "geodetic precession"). Its presence in the orbital motion of the Moon has been directly confirmed by lunar laser ranging data [34,35].

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