Nonperturbative QCD and the meson (q-$\bar{q}$) spectrum

G. Preparata and M. Scorletti
Dipartimento di Fisica, Università di Milano, Milano, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, via G. Celoria 16, 20133 Milano, Italy
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We analyze the problem of the meson (q-$\bar{q}$) spectrum in the formulation of nonperturbative QCD that has been called anisotropic chromodynamics, the theory of finite energy quantum fluctuations (the hadrons) around the (assumed) QCD vacuum, the chromomagnetic liquid. We find very satisfactory results for both the spectrum and wave functions, in terms of a *minimal set of inputs*: quark masses (both light and heavy) and the gauge coupling constant.

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I. INTRODUCTION

The last 20 years have seen a definite shift of interest from purely hadronic physics, the physics of long distances, to high momentum transfer (deep inelastic) phenomena, the physics of short distances. The reason, as we know well, for this momentous change in point of view has been well described by Weinberg [1]: In QCD short distances are quite simple, being governed by asymptotic freedom (AF) and perturbative QCD, just like the regime of lamellar flow of the hydrodynamics of the Navier-Stokes equations; long distances, on the other hand, are very complicated, being dominated by the severe quantum fluctuations of the color field, whose analogy again with the turbulent regime of the Navier-Stokes equations appears completely pertinent. And as a solution of the problem of turbulent flow within classical hydrodynamics is slowly coming through massive computer calculations, it is to be expected that the nonperturbative QCD regime, with its strong hadronic fluctuations, can only be conquered by similarly massive computer simulations: The great interest surrounding lattice gauge theories (LGT's) bears witness to the general agreement with Weinberg's point of view.

The work to be presented in the following sections is an attempt that springs from a completely different point of view, which, though recognizing the plausibility of the above attitude, focuses on the peculiarity of QCD and on the important steps, experimental and theoretical, that were taken in the 1960s and led to the construction of the standard model (SM). In particular, the lesson from the quark model, even in its most *naive* form (which was crucial in conjuring up QCD), is that hadrons are not so complicated after all, and in their apparently baroque architecture they reveal to the perceptive eye many elements of order and simplicity that continue to operate at the very high energies where, according to Weinberg, the regime has suddenly changed from turbulent to lamellar. Indeed, these elements of order, all keyed to the fundamental notions of local color fields (the gluons) and colored matter (the quarks), do not appear to change their relevance from long to short distances, the quark footprint being as clear in, say, the baryon wave function as in the structure of $R_{e^+e^-}$ or in the Bjorken scaling of deep inelastic scattering.

It must be stressed that the point of view of this paper has found solid theoretical motivation in a research program developed in the 1980s in which the problem of the QCD ground state was analyzed in a nonperturbative fashion [2]. As discussed in several published papers and reviews, a remarkable phenomenon of chromomagnetic instability of the perturbative vacuum, first discovered by Nielsen and Olesen [3], makes the perturbative QCD ground state unsuitable for describing the true dynamics of QCD even at very short distances in spite of the AF indication that the coupling constant, by increasing the energy, runs (logarithmically) to zero. It is upon this crucial fact and on a reasonable conjecture on the structure of the true QCD ground state that it has been possible to build a complete dynamical strategy [4] to solve QCD in a nonperturbative fashion that is summarized in the next section. The nonperturbative approach to QCD, which has been called anisotropic chromodynamics (ACD), after several sparse attempts, is now being investigated systematically and to a certain depth. This paper deals with the basic problem of computing the meson (q-$\bar{q}$) spectrum from "basic principles," i.e., from the assumption that the chromomagnetic liquid (CML) (which will be described in the next section) is a good approximation to the true QCD ground state. As will be shown, the results are very promising, and the overall picture that emerges is quite satisfactory.

We are well aware of the antagonism, both conscious and unconscious, that this work may excite on the reader, and we understand it. But we ask of him an act of strenuous intellectual sacrifice: to follow through what we do and to dig into it as deeply as he can. He may find it worthwhile.

II. QCD, ACD, AND THE PRIMITIVE WORLD

The question about the "true" QCD (or any other non-Abelian Yang-Mills theory) vacuum can be answered by analyzing the stability of the class of gauge field configuration $S^n_{\rm u}(B)$, that we call Savvidy states, characterized by quantum fluctuations around the special solution of classical equations that reduces, in a particular gauge, to a constant chromomagnetic field $\mathbf{B}^n = B\eta^n\mathbf{u}$, where η^n and $\mathbf{u}$ are the direction, respectively, in color and three-dimensional space.

This problem was analyzed by one of us (G.P.) in a series of papers [2], whose results are summarized in Ref. [5], where it is shown that the energy density of $S_{\mathrm{u}}^n(B)$ with respect to the perturbative ground state $B=0$ is given by (note that the classical term $B^2/2$ has disappeared)

$$\Delta E(B)=E(B)-E(0)$$
$$=\frac{11g^2B^2}{32\pi^2}\left[a-\ln\left[\frac{\Lambda^2}{gB}\right]\right]+O\left[g^4B^2\ln\frac{\Lambda^2}{gB}\right], \quad (2.1)$$

where the constant a has been determined in a subsequent lattice calculation[1] [7] to have the value

$$a=18.1\pm 0.2 . \quad (2.2)$$

This result shows that the perturbative vacuum $S_{\mathrm{u}}^n(0)$ is *essentially* unstable against the condensation of a constant chromomagnetic field: Indeed, the minimum of (2.1) is attained for (Λ is the ultraviolet cutoff)

$$gB^* = e^{-(1/2+a)}\Lambda^2 , \quad (2.3)$$

with the value of the energy density

$$\Delta E(B^*) = -\frac{11}{64\pi^2}e^{-[1+(24/11)a]}\Lambda^4 . \quad (2.4)$$

The state of minimum energy $S_{\mathrm{u}}^n(B^*)$ turns out to be characterized by a divergent expectation value of the gauge covariant field strength, $\langle F_{ik}^n\rangle \propto \epsilon_{ikr}u^r\delta^{mn}g^2B^*$; this enormous background chromomagnetic field, which is obtained by "freezing" in a highly correlated state some of the gluon modes that we call "longitudinal" gluons, makes the dynamics of the Savvidy state isomorphic to that of a $(1+1)$-dimensional theory, thus realizing, as is well known, color confinement.

Finally, on the Savvidy state the other gluon modes have the energy spectrum typical of a relativistic particle with a (finite) mass

$$m_g^2 = 4gB^* e^{-8\pi^2/g^2} . \quad (2.5)$$

The crucial point [4], now, is that in order to obtain these very promising results the background field needs to be constant only in tubelike domains with a correlation length

$$d^2 = \frac{1}{8m_g^2} \quad (2.6a)$$

and vanishing (in the limit of infinite ultraviolet cutoff) section

$$A_0 = \frac{2\pi}{gB^*} . \quad (2.6b)$$

One can now gain extra energy by depolarizing these domains and allowing them to be set in rotary motion; in this way, we also restore all the invariances that the vacuum state must possess and that were obviously lost in the Savvidy state.

We are thus led to the following picture of the QCD vacuum: a totally disordered, stochastic ensemble of needle-shaped magnetic domains with divergent (in the limit $\Lambda\to\infty$) background field $|B|\propto\Lambda^2$, which by its stochasticity recovers both rotational and Lorentz invariance; we call this state the chromomagnetic liquid (CML). Please note that the picture we obtain is not unsimilar to the well-known "spaghetti vacuum" of Ref. [3]. The basic difference is that our "needles" have a much smaller cross section [see (2.6b)]. We should also note that Λ, the ultraviolet cutoff, cannot exceed the Planck mass $m_P \simeq 10^{19}$ GeV for at that scale the neglected gravitational quantum fluctuations will severely modify the flat space-time in which we work.

Now we must set up the dynamics of the matter-field excitations upon this state and write down the effective Lagrangian of the theory. Following Ref. [4], we introduce the fundamental field $q_{\mathrm{u},a}(\xi,t)$ (q is any matter field, but the same construction also applies, with the obvious modifications, to the gluon fields), describing quantum fluctuations upon the single "tube" (the needle-shaped domain that for simplicity we take as infinitely long) polarized around the direction $\mathbf{u}$ and centered at the transverse coordinate $\mathbf{a}$, $\xi=\mathbf{x}\cdot\mathbf{u}$ being the coordinate along the tube. This field obeys the equal-time commutation relation

$$\{q_{\mathrm{u,a}}(\xi), q^{\dagger}_{\mathrm{u}',\mathrm{a}'}(\xi')\} = \delta_{\mathrm{u,u}'}\delta_{\mathrm{a,a}'}\delta(\xi-\xi') . \quad (2.7)$$

On the other hand, because of the liquid structure of the CML, these fundamental fields are not directly observable, and the dynamics and space-time properties of hadrons (i.e., the finite energy states) must be fully described in terms of the "collective," observable fields

$$q(\mathbf{x},t) = \frac{1}{\sqrt{N_{\mathrm{u}}A_0}}\sum_{\mathrm{u,a}}\delta_{\mathrm{a}}(\mathbf{x}_{\perp})q_{\mathrm{u,a}}(\mathbf{x}\cdot\mathbf{u},t) ; \quad (2.8)$$

here, $\delta_{\mathrm{a}}(\mathbf{x}_\perp)$ is the characteristic function of the tube, N_{u} is the number of independent directions in the solid angle, and the normalization factor is chosen so as to ensure canonical commutation relations:

$$\{q(\mathbf{x}), q^{\dagger}(\mathbf{x}')\} = \delta^{(3)}(\mathbf{x}-\mathbf{x}') . \quad (2.9)$$

N_{u} can be evaluated by observing that $N_{\mathrm{u}}A_0$ is nothing but the surface of the sphere with radius one-half of the correlation length d; then,

$$N_{\mathrm{u}} = \frac{4\pi d^2}{4A_0} = \frac{\pi}{8m_g^2A_0} . \quad (2.10)$$

As explained in Ref. [4], it is not too difficult to construct the Lagrangian of the theory and perform explicitly the Gaussian functional integration. One crucial point here is that the "longitudinal" gluon fields, which do not

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[1]Recently, there have appeared in the literature other lattice calculations [6], which are seen to corroborate the results of Ref. [7].

carry any autonomous dynamics, are completely integrated out, resulting in the very peculiar [(1+1)-dimensional] dynamics of color charges inside a tubelike domain.

From the effective action we can extract the Hamiltonian

$$H=H_D+\sum_{\mathrm{u,a}} H_{\mathrm{u,a}}+H_{\mathrm{GT}} \; ; \tag{2.11}$$

H_D is simply the usual Dirac kinetic Hamiltonian of the quark fields,

$$H_D=\int d^3\mathbf{x} \, :\bar{q}(\mathbf{x})(i\nabla\cdot\gamma+m)q(\mathbf{x}): \, , \tag{2.12}$$

which can be written directly in terms of collective fields.

As for the interaction term, we have

$$H_{\mathrm{u,a}}=\frac{g^2}{2}\frac{1}{N_{\mathrm{u}}A_0}\int_{-\infty}^{\infty} d\xi \, d\eta \, j^{\mu}_{m,\mathrm{u,a}}(\xi) \times G^{mn}_{\mu\nu}(\xi-\eta) j^{\nu}_{n,\mathrm{u,a}}(\eta) \, , \tag{2.13}$$

where

$$j^{\mu}_{m,\mathrm{u,a}}(\xi)=:\bar{q}_{\mathrm{u,a}}(\xi)\gamma^{\mu}\frac{\lambda_m}{2}q_{\mathrm{u,a}}(\xi): \tag{2.14}$$

is the (normal ordered) color quark current inside the tube and $[n_\mu=(0,\mathbf{u})]$

$$G^{mn}_{\mu\nu}(\xi-\eta)=\delta^{mn}(g_{\mu\nu}+n_\mu n_\nu)\lim_{\lambda\to 0}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{e^{iq(\xi-\eta)}}{q^2+\lambda^2} \tag{2.15}$$

is the longitudinal gluon "propagator" (in fact the interaction is instantaneous and there is no propagation at all). Please note the infrared singularity of $G^{mn}_{\mu\nu}$, leading to a divergent self-energy for any physical system with nonzero color charge: This is nothing but color confinement.

The last piece in the Hamiltonian (2.11) describes the (short-distance) interaction between quarks and massive, propagating, gluons:

$$H_{\mathrm{GT}}=\frac{g^2}{2}\int d^3\mathbf{x}\, d^3\mathbf{y}\, j^{\mu}_m(\mathbf{x})\Delta^{mn}_{\mu\nu}(\mathbf{x}-\mathbf{y})j^{\nu}_n(\mathbf{y}) \, , \tag{2.16}$$

$j^{\mu}_m(\mathbf{x})=:\bar{q}(\mathbf{x})\gamma^{\mu}(\lambda_m/2)q(\mathbf{x}):$ being the usual Dirac current.

In writing down the Hamiltonian term (2.16), we are requested to make a well-defined choice for the time component of the gluon momentum q_0 in the propagator:

$$\Delta^{mn}_{\mu\nu}(\mathbf{x}-\mathbf{y})=\delta^{mn}g_{\mu\nu}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{-q_0^2+\mathbf{q}^2+m_g^2} \, . \tag{2.17}$$

We shall come back later to this point.

We note that the full Hamiltonian of the theory contains other terms, namely, the kinetic term of the massive gluons and their confining interaction, having the same structure as (2.13), but they are not relevant in discussing the meson spectrum.

The next point is to set up a new kind of perturbation theory and outline a strategy to approximately (but completely) solve the problem of hadron dynamics. We start by fixing the timelike hypersurface defining the Hamiltonian and all global operators, and by writing the decomposition

$$H=H^{(0)}+H^{(1)} \, , \tag{2.18}$$

where $H^{(0)}$ is obtained by replacing the currents $j^{\mu}_m(\mathbf{x})$ and $j^{\mu}_{m,\mathrm{u,a}}(\xi)$ with their truncated partners $j^{(0)\mu}_m(\mathbf{x})$ and $j^{(0)\mu}_{m,\mathrm{u,a}}(\xi)$, which by definition contain only terms of the type $b^\dagger b, d^\dagger d$ in their usual expansion, while $H^{(1)}$ contains all the other terms. We make two fundamental observations.

(i) $H^{(0)}_{\mathrm{u,a}}$ is the only term in the Hamiltonian that is potentially divergent, $H^{(1)}_{\mathrm{u,a}}$ having no component containing the color charge operator. Thus $H^{(0)}_{\mathrm{u,a}}$ is the only term of the interaction Hamiltonian that must (and shall) be treated nonperturbatively.

(ii) $H^{(0)}$ commutes separately with all the field number operators, its eigenstates being thus labeled by a fixed number of quark, antiquark, and gluons $(N_q,N_{\bar{q}},N_g)$. This is no longer true if we take into account the full Hamiltonian $H^{(1)}$ that contains pair creation and annihilation.

Thus we first diagonalize $H^{(0)}$, this diagonalization being carried out separately in each subspace of Fock space with fixed $(N_q,N_{\bar{q}},N_g)$, obtaining what we call the "primitive world" (PW): the Fock space of stable, noninteracting hadrons (i.e., color singlet), the right asymptotic Hilbert space for perturbation theory. Next we evaluate hadronic interactions by taking into account, perturbatively, this time, the effect of pair creation induced by $H^{(1)}$.

As a first step in our research program, in the next sections we carry out the calculation of the meson spectrum, i.e., the diagonalization of $H^{(0)}$ in the ($N_q=1$, $N_{\bar{q}}=1$, $N_g=0$) sector.

III. RELATIVISTIC SPECTRUM EQUATION

Our main goal in this section is to solve the eigenvalue equation for the meson spectrum. Let us choose the rest frame and denote by $|M_J\rangle$ the generic mesonic state of mass M_J, the label J denoting the collection of all quantum number of the particular state being studied. Then such equation reads

$$\left[H_D+\sum_{\mathrm{u,a}}H_{\mathrm{u,a}}\right]|M_J\rangle=M_J|M_J\rangle \, . \tag{3.1}$$

In writing (3.1) we have neglected the last term of the Hamiltonian (2.11), H_{GT}, which we shall take into account in the next section (recall that H_{GT} does not contain any infrared divergence).

Let us define now the plane-wave color-singlet tubelike meson (in the rest frame)

$$|p,\mathbf{u},rs\rangle=\frac{\delta^{\alpha\beta}}{\sqrt{3}}b^{\dagger}_{\mathrm{u},0}(p,r,\alpha)d^{\dagger}_{\mathrm{u},0}(-p,s,\beta)|\Omega\rangle \tag{3.2}$$

and the collective one

$$|\mathbf{p},rs\rangle=\frac{\delta^{\alpha\beta}}{\sqrt{3}}b^{\dagger}(\mathbf{p},r,\alpha)d^{\dagger}(-\mathbf{p},s,\beta)|\Omega\rangle \; ; \tag{3.3}$$

they are eigenstates, respectively, of the relative longitudinal and collective momentum operator, so that (3.3) is also an eigenstate of the kinetic Hamiltonian:

$$H_D|\mathbf{p},rs\rangle=(\sqrt{\mathbf{p}^2+m_a^2}+\sqrt{\mathbf{p}^2+m_b^2})|\mathbf{p},rs\rangle \ . \quad (3.4)$$

It is useful to introduce also the (collective) state $|p;l,s,j\rangle$, which is a simultaneous eigenstate of momentum squared $\hat{\mathbf{P}}^2$, orbital angular momentum $\hat{\mathbf{L}}^2$, spin $\hat{\mathbf{S}}^2$, and total angular momentum $\hat{\mathbf{J}}^2$ operators. Obviously, the $|p;l,s,j\rangle$ states too are eigenstates of H_D, and their relation with the basis (3.3) is given by

$$|p;l,s,j\rangle=\frac{1}{\sqrt{2j+1}}\sum_{m_l,m_s,m_j}C(j,l,s;m_l,m_s,m_j)\sum_{r,r'}\int d\Omega_{\hat{\mathbf{p}}}Y_{l,m_l}(\hat{\mathbf{p}})\chi^{rr'}(s,m_s)|\mathbf{p},rr'\rangle \ , \quad (3.5)$$

where $\chi^{rr'}(s,m_s)$, $Y_{l,m_l}(\hat{\mathbf{p}})$, and $C(j,l,s;m_l,m_s,m_j)$ are, respectively, the spin wave function, the spherical harmonics, and the Clebsch-Gordan coefficients. We therefore expand the state $|M_J\rangle$ onto the basis we have just introduced (for a moment we restore the flavor indices a,b and Q is the flavor wave function):

$$|M_J\rangle=(2\pi)^{3/2}\sqrt{2M_J}Q^{ba}\int_0^\infty dp\, p\, u_J(p)|p;l,s,j;ab\rangle \ ; \quad (3.6)$$

$u_J(p)$ is the (longitudinal) reduced wave function and the factor $(16\pi^3M_J)^{1/2}$ is chosen in such a way that

$$\int_0^\infty dp|u_J(p)|^2=1 \quad (3.7)$$

ensures the correct relativistic normalization of the state:

$$\langle M_J|M_J\rangle=(2\pi)^3 2M_J \ . \quad (3.8)$$

Actually, our Hamiltonian (2.11) commutes with $\hat{\mathbf{J}}^2,\hat{P},\hat{C}$, but not with $\hat{\mathbf{J}}^2,\hat{\mathbf{L}}^2,\hat{\mathbf{S}}^2$, so that the "good" quantum numbers of meson states are[2] j^{PC}, not $^{2s+1}l_j$, as implied by our expansion (3.6). However, the basis $|l,s,j\rangle$ is not very different from $|j,PC\rangle$, the only mixing occurring in $1^{--},2^{++},3^{--},\ldots$ sectors between $^3l_{l+1}$ and $^3(l+2)_{l+1}$ states and, only for the unequal mass case, in the $1^+,2^-,3^+,\ldots$ sectors between ^1l_l and ^3l_l states.

In fact, it is easier to solve the eigenvalue equation by expanding onto the basis $|l,s,j\rangle$, the expressions of the kernel V_J being simpler, and then carry out the complete diagonalization of the Hamiltonian evaluating perturbatively the effect of the mixing, which we know in advance to be rather small.

By projecting Eq. (3.1) on the basis (3.5), we obtain the one-dimensional Schrödinger-like equation with relativistic kinematics ($E_a=\sqrt{p^2+m_a^2}$, $E_b=\sqrt{p^2+m_b^2}$):

$$(E_a+E_b-M_J)u_J(p)+\int_0^\infty dp'\,pp'V_J(p,p')u_J(p')=0 \ ; \quad (3.9)$$

the potential entering our integral problem is given by

$$V_J(p,p')=\left\langle p;l,s,j\left|\sum_{\mathbf{u},\mathbf{a}}H_{\mathbf{u},\mathbf{a}}\right|p';l,s,j\right\rangle \quad (3.10)$$

and can be extracted, using (3.5), from the matrix element

$$V(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=\left\langle \mathbf{p},rs\left|\sum_{\mathbf{u},\mathbf{a}}H_{\mathbf{u},\mathbf{a}}\right|\mathbf{p}',r's'\right\rangle \ . \quad (3.11)$$

The matrix element (3.11) cannot be evaluated directly, for the Hamiltonian is expressed in terms of tubelike operators, whereas the states are written in terms of the collective ones.

Let us therefore use the completeness of the eigenstates of the relative position operator $|\mathbf{x},rs\rangle$ to get

$$V(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=\int d^3\mathbf{x}\,d^3\mathbf{x}'\langle \mathbf{p},rs|\mathbf{x},rs\rangle \times V(\mathbf{x},\mathbf{x}')_{rs}^{r's'}\langle \mathbf{x}',r's'|\mathbf{p}',r's'\rangle \ , \quad (3.12)$$

where we have introduced the (nonlocal) potential in coordinate space,

$$V(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=\left\langle \mathbf{x},rs\left|\sum_{\mathbf{u},\mathbf{a}}H_{\mathbf{u},\mathbf{a}}\right|\mathbf{x}',r's'\right\rangle \ , \quad (3.13)$$

and the projector $\langle \mathbf{x}|\mathbf{p}\rangle$ is simply given by

$$\langle \mathbf{x},rs|\mathbf{p},r's'\rangle=\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\delta_{rr'}\delta_{ss'} \ . \quad (3.14)$$

Finally, we insert the completeness of the states (3.2),

$$\sum_{\mathbf{u}}\sum_{rs}\int_{-\infty}^{\infty}dp|p,\mathbf{u},rs\rangle\langle p,\mathbf{u},rs|=\mathbf{1} \ , \quad (3.15)$$

to obtain

[2] Please recall that $P=(-1)^{l+1}$, $C=(-1)^{l+s}$, and that the charge conjugation is defined only for the equal-mass case.

$$V(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=\sum_{\mathbf{v},\mathbf{v}'}\int_{-\infty}^{\infty}dp\,dp'\langle\mathbf{x},rs|p,\mathbf{v},rs\rangle\sum_{\mathbf{u},\mathbf{a}}\langle p,\mathbf{v},rs|H_{\mathbf{u},\mathbf{a}}|p',\mathbf{v}',r's'\rangle\langle p',\mathbf{v}',r's'|\mathbf{x}',r's'\rangle\ . \tag{3.16}$$

The evaluation of the matrix element of the Hamiltonian $H_{\mathbf{u},\mathbf{a}}$ is now straightforward, and to complete the calculation we only need the scalar product between the collective position and the tubelike momentum eigenstates, which is given by [8] ($r=|\mathbf{x}|$)

$$\langle\mathbf{x},rs|p,\mathbf{u},r's'\rangle=\left[\frac{N_{\mathbf{u}}}{4\pi A_0}\right]^{1/2}\delta_{\mathbf{u}}(\hat{\mathbf{x}})\frac{e^{ipr}}{\sqrt{2\pi}\,r}\delta_{rr'}\delta_{ss'}\ . \tag{3.17}$$

The result of the calculation, which is sketched in Appendix A, is that the potential can be written as the sum of a long-range and a short-range part:

$$V(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=V_{lr}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}+V_{sr}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}\ , \tag{3.18}$$

where

$$V_{lr}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=\mu^2 r\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta_{rr'}\delta_{ss'}\ , \tag{3.19a}$$

$$V_{sr}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=-\frac{\mu^2}{\pi^2}\frac{\delta^{(2)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')}{rr'}\left\{\frac{3}{2}\delta_{rr'}\delta_{ss'}[K_0(m_a|r-r'|)+K_0(m_b|r-r'|)]-[\delta_{rr'}\delta_{ss'}+\boldsymbol{\sigma}_{rr'}\cdot\boldsymbol{\sigma}_{ss'}-(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})_{rr'}(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}')_{ss'}]\right.$$
$$\left.\times K_0\left[\frac{m_a+m_b}{2}|r+r'|\right]+V_{\text{res}}(r,r')_{rs}^{r's'}\right\}\ . \tag{3.19b}$$

We would like to make here a few remarks.

(i) The long-range, confining, linear potential is both flavor and spin independent; it stems from the cancellation of the infrared divergence that occurs only if the state is a color singlet.

(ii) The short-range, nonlocal terms containing the K_0 Bessel function depend drastically on the quark masses and cause (3.18) to have a logarithmic singularity for vanishing masses; this is, of course, a good signal for chiral symmetry breaking.

The "residual" interaction $V_{\text{res}}(r,r')_{rs}^{r's'}$, whose explicit expression can be found in Appendix A, has a highly nonlocal structure, and no simple formula can be given for its contribution to the potential kernel $V_J(p,p')$; we therefore discard this term from the eigenvalue equation (3.9) and shall treat it perturbatively, controlling at the end the consistency of such procedure. The contribution of the other terms of (3.19) to $V_J(p,p')$ turns out to be, according to the decomposition (3.18),

$$pp'V_{lr,J}(p,p')=\frac{\mu^2}{\pi}\frac{1}{pp'}Q_l'\left[\frac{p^2+p'^2}{2pp'}\right]\ , \tag{3.20a}$$

$$pp'V_{sr,J}(p,p')=-\frac{3\mu^2}{\pi}\frac{\delta(p-p')}{\sqrt{p^2+m^2}}-\frac{\mu^2}{\pi^3}\left\{3O(p,p',m_a)+3O(p,p',m_b)-2(1+S)O\left[p,p',\frac{m_a,m_b}{2}\right]\right\}\ ; \tag{3.20b}$$

in (3.20b), we have introduced

$$O(p,q,m)=\frac{1}{q^2-p^2}\left[\frac{q}{\sqrt{p^2+m^2}}\ln\frac{p+\sqrt{p^2+m^2}}{m}-\frac{p}{\sqrt{q^2+m^2}}\ln\frac{q+\sqrt{q^2+m^2}}{m}\right] \tag{3.21}$$

and the spin-dependent factor

$$S=\frac{4}{3}s(s+1)-2-\delta_{s1}\frac{11l(l+1)-3j(j+1)+6}{3(2l-1)(2l+3)}\ . \tag{3.22}$$

The simple pole of the Legendre functions $Q_l'(z)$ at $z=1$ requires the integral appearing in (3.9) to be interpreted as a principal value integral, thus rendering our integral equation singular. A well-suited technique for dealing with the numerical solution of such equations is Multhopp's method, which we shall briefly describe in Appendix B.

After having solved numerically (3.9) with the kernel (3.20), we can evaluate perturbatively the contribution of the term $V_{\text{res}}(r,r')_{rs}^{r's'}$; to first order, this results in the mass shift

$$\delta M_J=\frac{\langle M_J|V_{\text{res}}|M_J\rangle}{\langle M_J|M_J\rangle}$$
$$=-\frac{\mu^2}{\pi^2}\int_0^{\infty}dr\,dr'\,u_J(r)V_{\text{res},J}(r,r')u_J(r')\ , \tag{3.23}$$

with $u_J(r)$ the Fourier transform of the reduced momentum wave function:

$$u_J(r)=\left[\frac{2}{\pi}\right]^{1/2}\int_0^\infty dp\, pr\, j_l(pr)u_J(p) \ . \tag{3.24}$$

It is a gratifying observation to check that the contribution δM_J turns out to be, *a posteriori*, not very important: It is completely negligible for heavy flavor quarks and gives a correction of the order of 10% for the light ones.

IV. "COULOMB" INTERACTION

We must now take into account the effect of $H_{\rm GT}$, the one-gluon exchange Hamiltonian, Eq. (2.17), that until now has been neglected.

As mentioned in Sec. II, the first question we wish to ask concerns the ambiguity in propagator (2.18): Actually, the interaction among quarks and massive gluons is not instantaneous, so that we must give some prescription in order to perform the integration over the relative time variable x_0-y_0 in the effective action; this is equivalent to the choice of the time component of the gluon momentum $q_0=p_0-p_0'$ in the propagator of the instantaneous Hamiltonian.

The most natural choice [9] seems to take both the quark energies p_0 and p_0' on the mass shell, setting

$$q_0=E_a-E_a'=E_b'-E_b \ . \tag{4.1}$$

This choice is different from the usual static approximation $q_0=0$, which completely neglects retardation effects, and amounts in approximating the true time evolution of the quark field in the Hamiltonian with the free one.

The next point about the Coulomb interaction is to set up the computational strategy: The relevant matrix element turns out to have the form

$$U(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=-\frac{4}{3}\frac{\alpha_S}{\pi}\frac{1}{2\pi}\frac{1}{-(E_a-E_a')^2+(\mathbf{p}-\mathbf{p}')^2+m_g^2}$$
$$\times C(p,p')\left\{f_{\rm SI}(\mathbf{p},\mathbf{p}')+f_{\rm SS}(\mathbf{p},\mathbf{p}')\boldsymbol{\sigma}_{rr'}\cdot\boldsymbol{\sigma}_{ss'}+i[f_{\rm SO}^{(1)}(\mathbf{p},\mathbf{p}')\delta_{ss'}\boldsymbol{\sigma}_{rr'}+f_{\rm SO}^{(2)}(\mathbf{p},\mathbf{p}')\delta_{rr'}\boldsymbol{\sigma}_{ss'}]\cdot(\mathbf{p}\wedge\mathbf{p}')\right.$$
$$\left.+[(\sigma_i)_{rr'}(\sigma_j)_{ss'}-\frac{1}{3}(\boldsymbol{\sigma})_{rr'}\cdot(\boldsymbol{\sigma})_{ss'}][f_{\rm ST}^{(1)}(p,p')p^ip^j+f_{\rm ST}^{(2)}(p,p')p'^ip'^j+f_{\rm ST}^{(3)}(\mathbf{p},\mathbf{p}')p^ip'^j]\right\} \ . \tag{4.2}$$

The calculation that leads to (4.2) is reported in Appendix C, together with the explicit expressions of the coefficients f and of the normalization factor C. Here we want only to stress that we have not performed any nonrelativistic expansion of our kernel to get a local Hamiltonian of the Breit-Fermi type: Equation (4.2) takes fully into account the effects of relativistic kinematics as well as, albeit in first approximation, the retardation effects.

From (4.2) we can extract, via (3.5), the kernel

$$U_J(p,p')=\langle p;l,s,j|H_{\rm GT}|p';l,s,j\rangle \ , \tag{4.3}$$

which contains the full action of $H_{\rm GT}$ on a definite meson state.

Unfortunately, if we try to solve Eq. (3.9) with the potential kernel V_J+U_J, we find that the eigenvalue equation becomes singular. This is due to the fact that we have extracted our Hamiltonian from the one-loop effective action and, in order to make contact with the Schrödinger formalism, we have systematically dropped out all the effects of pair creation. On the other hand, we know that the quark-gluon vertex has to be renormalized by including higher-order graphs that remove the singularity at $r=0$ of the Coulomb potential.

One possibility [8] to overcome this problem, which is a consequence of the approximations we have made in our treatment, is to insert in the eigenvalue equation for the meson spectrum the kernel $U_J^0(p,p')$, obtained in the static approximation of the propagator ($q_0=0$), and then evaluate perturbatively the retardation effects induced by $U_J(p,p')-U_J^0(p,p')$. Actually, $U_J^0(p,p')$ is less singular than $U_J(p,p')$ and makes the eigenvalue problem well defined; however, it gives a bad asymptotic behavior ($p\to\infty$) to the wave function $u_J(p)$ and causes some of the matrix elements that we shall consider in the next section to diverge.

For these reasons we treat the effects of $U_J(p,p')$ with standard perturbation theory, to get the second-order correction to the eigenvalues and the first-order correction to the eigenstates, whose zeroth-order expression $M_n^{(0)}$ and $|M^{(0)},n\rangle$ are obtained by solving (3.9). We have

$$M_n^{(1)}+M_n^{(2)}=\langle M^{(0)},n|H_{\rm GT}|M^{(0)},n\rangle+\sum_{m\neq n}\frac{M_n^{(0)}|\langle M^{(0)},m|H_{\rm GT}|M^{(0)},n\rangle|^2}{M_m^{(0)}(M_n^{(0)}-M_m^{(0)})} \ , \tag{4.4a}$$

$$|M^{(1)},n\rangle=\sum_{m\neq n}\frac{M_n^{(0)}\langle M^{(0)},m|H_{\rm GT}|M^{(0)},n\rangle}{M_m^{(0)}(M_n^{(0)}-M_m^{(0)})}|M^{(0)},m\rangle \ ; \tag{4.4b}$$

here n,m are radial excitation quantum numbers and we have omitted the subscript J; the slightly modified energy denominator we have used in (4.4) is a simple way to take into account vertex renormalization.

The matrix elements appearing in (4.4) turn out to be simply

$$\langle M^{(0)},m|H_{\rm GT}|M^{(0)},n\rangle = \int_0^\infty dp\, dp'\, pp'\, u_m(p) U_J(p,p') u_n(p') \ . \quad (4.5)$$

Since the Multhopp method is accurate only for few radial excitations ($n<n_0$ with $n_0 \simeq 10$), for higher radial excitations we take the zeroth-order masses and wave functions obtained with a WKB approximation [10] to the problem (3.9).

V. RELATION AMONG COUPLING CONSTANTS

The question we want to address in this section concerns the relation among the coupling constants. Actually, the basic QCD Lagrangian contains, in addition to the quark masses, a single coupling constant g, whereas our potential is given in terms of three parameters: μ, the "string tension," which has been defined in Appendix A to be

$$\mu^2 = \frac{4}{3}\frac{g^2(gB^*)}{2N_u A_0} \ , \quad (5.1)$$

where the coupling g is evaluated at the typical energy scale of the "longitudinal" gluons, the gluon mass m_g, which from the variational calculation turns out to be given by

$$m_g^2 = 4\Lambda^2 \exp\left[-\left[\frac{1}{2}+a+\frac{8\pi^2}{g^2(\Lambda^2)}\right]\right] \ , \quad (5.2)$$

and the usual running α_S entering the one-gluon-exchange potential, which is simply

$$\alpha_S(Q^2) = \frac{g^2(Q^2)}{4\pi} \ . \quad (5.3)$$

As g appears in (5.1), (5.2), and (5.3) at different scales, in order to achieve our goal, i.e., to relate μ, m_g, and α_S to each other, we need the evolution law of the running coupling constant; what can we say about the form of the function $g^2(Q^2)$? For energy scales much larger than gB^* [Eq. (2.3)], which is the typical energy scale of the magnetic condensation of the vacuum, the CML is practically indistinguishable from the perturbative vacuum, and the relevant dynamics is well described perturbatively by the basic QCD Lagrangian, so that for $Q^2 \gg gB^*$ we come back to the usual evolution law of asymptotic freedom. On the other hand, when $Q^2 \ll gB^*$ in the CML the gluonic modes are "frozen," so that at such energy scales the coupling constant has no evolution at all.

According to these considerations, we simply modify the first-order evolution law for the QCD running coupling constant in order to reproduce the limiting behaviors that we have just sketched, and for the evolution law we write down the expression

$$g^2(Q^2) = \frac{g^2(gB^*)}{1+(21/48\pi^2)g^2(gB^*)\ln[(Q^2+gB^*)/2gB^*]} \ ; \quad (5.4)$$

putting $Q^2=\Lambda^2$ and $Q^2=0$ and inverting (5.4), we obtain, respectively,

$$g^2(gB^*) = \frac{g^2(\Lambda^2)}{1-(21/48\pi^2)(a+\frac{1}{2}-\ln 2)g^2(\Lambda^2)} \ , \quad (5.5a)$$

$$g^2(0) = \frac{g^2(\Lambda^2)}{1-(21/48\pi^2)(a+\frac{1}{2})g^2(\Lambda^2)} \ . \quad (5.5b)$$

From (5.1), (5.2), (2.7), and (5.5a), we have

$$\mu^2 = \frac{64}{3\pi}\frac{g^2(\Lambda^2)\Lambda^2}{1-(21/48\pi^2)(a+\frac{1}{2}-\ln 2)g^2(\Lambda^2)} \times \exp\left[-\left[\frac{1}{2}+a+\frac{8\pi^2}{g^2(\Lambda^2)}\right]\right] \ , \quad (5.6)$$

whereas to obtain α_S we are interested to the region of low Q^2, where the coupling evolves very slowly; so we can take $\alpha_S(Q^2)\simeq\alpha_S(0)$ and obtain

$$\alpha_S = \frac{1}{4\pi}\frac{g^2(\Lambda^2)}{1-(21/48\pi^2)(a+\frac{1}{2})g^2(\Lambda^2)} \ . \quad (5.7)$$

In this way we have related the three parameters that enter in the determination of the meson spectrum to a single constant, namely, $g^2(\Lambda^2)$. We can now solve (5.6) for $g^2(\Lambda^2)$ and then, using (5.4) and (5.7), obtain the values of m_g and α_S as a function of μ. Unfortunately, we do not know precisely either the value of the cutoff, which we take as

$$\Lambda = 1.22 e^{19\pm1}\ \text{GeV} \ , \quad (5.8)$$

or that of the constant a, whose value is reported in Eq. (2.2). By taking

$$\mu = (0.48\pm0.02)\ \text{GeV} \ , \quad (5.9)$$

a typical value that well describes the meson spectrum, we obtain

$$g^2(\Lambda^2) = 1.06\pm0.06 \ , \quad (5.10a)$$

$$m_g = (0.15\pm0.04)\ \text{GeV} \ , \quad (5.10b)$$

$$0.3 \lesssim \alpha_S \lesssim 1.5 \ . \quad (5.10c)$$

We see that, once we fix the string tension, that is, the most relevant parameter for the spectrum, we have a good determination of the gluon mass and a reasonable bound on the values of α_S. Nevertheless, it is a very remarkable fact that we obtain the right order of magnitude for m_g and α_S to correctly reproduce the spectrum, which is a good signal that we are on the right way to understand hadron dynamics.

VI. DECAY CONSTANTS

The simple expression (3.6) of the mesonic state and the knowledge of the corresponding wave function allows us to obtain simple expressions for the pseudoscalar and vector decay constants f_P and f_V that parametrize, respectively, the $P\to l\bar{\nu}_l$ and $V\to e^+e^-$ processes. Let us define f_P and f_V through

$$\langle\Omega|A^{\mu}_{ab}(x)|P(p)\rangle = iQ^{ba}\sqrt{2}f_P p^{\mu}e^{-ipx}\,, \qquad (6.1a)$$

$$\langle\Omega|J^{\mu}_{ab}(x)|V(p,\epsilon)\rangle = Q^{ba}f_V\epsilon^{\mu}e^{-ipx} \qquad (6.1b)$$

[note the $\sqrt{2}$ in (6.1a)], and, for later use,

$$\langle\Omega|A^{5}_{ab}(x)|P(p)\rangle = Q^{ba}\sqrt{2}f_5 M_P e^{-ipx}\,. \qquad (6.1c)$$

Here $|P(p)\rangle$ and $|V(p,\epsilon)\rangle$ are the pseudoscalar and vector meson state with momentum p^{μ} and polarization ϵ^{μ}, while the expression of the currents are (the i in the last expression is to ensure Hermiticity)

$$A^{\mu}_{ab}(x) = \delta_{\alpha\beta}\bar{q}^{\beta}_{b}(x)\gamma^{\mu}\gamma^{5}q^{\alpha}_{a}(x)\,, \qquad (6.2a)$$

$$J^{\mu}_{ab}(x) = \delta_{\alpha\beta}\bar{q}^{\beta}_{b}(x)\gamma^{\mu}q^{\alpha}_{a}(x)\,, \qquad (6.2b)$$

$$A^{5}_{ab}(x) = i\delta_{\alpha\beta}\bar{q}^{\beta}_{b}(x)\gamma^{5}q^{\alpha}_{a}(x)\,. \qquad (6.2c)$$

In the rest frame ($\mathbf{p}=0$) the nonzero components of (6.1) become

$$\langle\Omega|A^{0}_{ab}(0)|P\rangle = iQ^{ba}\sqrt{2}f_P M_P\,, \qquad (6.3a)$$

$$\langle\Omega|\mathbf{J}_{ab}(0)|V(\epsilon)\rangle = Q^{ba}f_V\epsilon\,, \qquad (6.3b)$$

$$\langle\Omega|A^{5}_{ab}(0)|P\rangle = Q^{ba}\sqrt{2}f_5 M_P\,, \qquad (6.3c)$$

while, according to (3.5), the expression of the normalized states are

$$|P\rangle = i2\pi\sqrt{M_P}\frac{\delta^{\alpha\beta}}{\sqrt{3}}Q^{ba}\frac{\delta^{rs}}{\sqrt{2}}\int d^3\mathbf{p}\,p u_P(p)b^{\dagger}(\mathbf{p},r,a,\alpha)d^{\dagger}(-\mathbf{p},s,b,\beta)|\Omega\rangle\,, \qquad (6.4a)$$

$$|V(\epsilon)\rangle = 2\pi\sqrt{M_V}\frac{\delta^{\alpha\beta}}{\sqrt{3}}Q^{ba}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon})^{rs}}{\sqrt{2}}\int d^3\mathbf{p}\,p u_V(p)b^{\dagger}(\mathbf{p},r,a,\alpha)d^{\dagger}(-\mathbf{p},s,b,\beta)|\Omega\rangle\,. \qquad (6.4b)$$

Finally, we obtain (see also [11] for a careful derivation)

$$f_P = \frac{\sqrt{3}}{2\pi}\frac{1}{\sqrt{M_P}}\int_0^{\infty}dp\,pN_{ab}(p)\left[1-\frac{E_a-m_a}{E_b+m_b}\right]u_P(p)\,, \qquad (6.5a)$$

$$f_V = \frac{\sqrt{6M_V}}{2\pi}\int_0^{\infty}dp\,pN_{ab}(p)\left[1+\frac{1}{3}\frac{E_a-m_a}{E_b+m_b}\right]u_V(p)\,, \qquad (6.5b)$$

$$f_5 = \frac{\sqrt{3}}{2\pi}\frac{1}{\sqrt{M_P}}\int_0^{\infty}dp\,pN_{ab}(p)\left[1+\frac{E_a-m_a}{E_b+m_b}\right]u_P(p)\,, \qquad (6.5c)$$

where we have introduced the factor

$$N_{ab}(p) = \left[\frac{(E_a+m_a)(E_b+m_b)}{E_aE_b}\right]^{1/2}\,. \qquad (6.6)$$

VII. BREAKING OF CHIRAL SYMMETRY

We wish now to address the problem of what happens in the chiral limit of QCD.

We recall that our q-$\bar{q}$ potential [Eq. (3.19)] has a logarithmic singularity in the chiral limit that originates from the (effective) two-dimensionality of the dynamics. In fact, when we solve (3.9) for vanishing quark masses, we obtain the remarkable situation depicted in Fig. 1: For $m_a < m^*$ (with $m^* \simeq 85$ MeV), the pseudoscalar state acquires a negative energy. This means that $|\Omega\rangle$ cannot be the true ground state of the theory, which therefore must be quantized in terms of a new ground state $|\omega\rangle$ and new creation and annihilation operators. This can be achieved, in general, via a Bogoliubov transformation (see, e.g., [12]): The "true" vacuum $|\omega\rangle$ can be easily seen to be a superposition of the $|\Omega\rangle$ state and of a condensed state of quark-antiquark pairs. Matter field excitations on this vacuum are described in terms of an "effective" mass that depends on the momentum, which

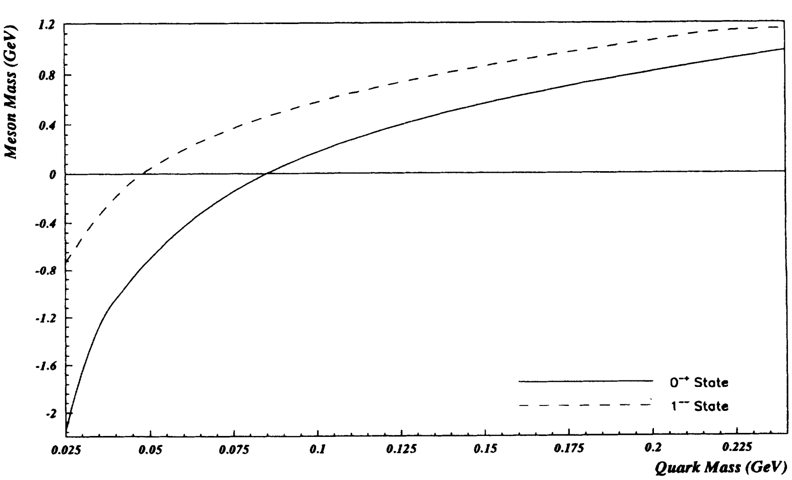

FIG. 1. Quarkonium vs quark mass. The critical mass is $m^* \simeq 85$ MeV.

We assume in the following that the constituent mass can be taken as momentum independent, in such a way that the vacuum rearrangement process can be described only by new mass parameters $m_a^{(c)}$, which simply replace in the Hamiltonian the "current" masses m_a.

We would like now to find the relation between constituent and current mass without analyzing explicitly the Bogoliubov transformation and the consequent gap equation. To do this let us take the divergence of the axial vector current (6.2a); recalling (6.2c), we write

$$\partial_\mu A_{ab}^\mu(x)=(m_a+m_b)A_{ab}^5(x) \ . \qquad (7.1)$$

We now take the matrix element of this equation between the vacuum and a pseudoscalar state, and observe that [see Eq. (6.1)]

$$\langle \omega|\partial_\mu A_{ab}^\mu(0)|P\rangle = Q^{ba}\sqrt{2}f_P M_P^2 \ ; \qquad (7.2)$$

thus, we finally obtain

$$M_P=(m_a+m_b)f_5 f_P^{-1} \ , \qquad (7.3)$$

which is the relation that we needed, for M_P, f_5, and f_P depend now on the constituent masses $m_a^{(c)}$ and $m_b^{(c)}$.

We are now in the position to prove that the vacuum rearrangement process that we have just sketched is nothing but the spontaneous chiral symmetry breaking. In fact, Eq. (7.1) tells us that in the chiral limit ($m_a \to 0$, $m_b \to 0$) the axial vector current is conserved.

To meet the hypotheses of the Goldstone theorem [13], we only need to show that the state $|\omega\rangle$ breaks chiral invariance; a standard way to do this is to evaluate the vacuum expectation value of the non-zero-chirality operator $\bar{q}(0)q(0)$ in the chiral limit. The result of the simple calculation is

$$\lim_{m_a \to 0} \langle \omega|\delta_{\alpha\beta}\bar{q}_a^\beta(0)q_a^\alpha(0)|\omega\rangle = -\frac{3}{\pi^2}\int_0^\Lambda dp\, p^2 \frac{m^{(c)*}}{\sqrt{p^2+m^{(c)*2}}} \ , \qquad (7.4a)$$

which is manifestly different from zero, being

$$m^{(c)*}=\lim_{m_a \to 0} m_a^{(c)}[m_a] \simeq 85 \text{ MeV} \ , \qquad (7.4b)$$

the chiral limit of the constituent mass.

In Fig. 2 we show the relation between current and constituent mass that we have obtained solving (7.3), whereas in Table I we report the values of the quark masses that we used to get the meson spectrum (at this level we have no need to take into account any mass difference between the u and d quarks).

The smallness of light quark mass shows that chiral symmetry is nearly conserved in this sector and that the pion, while conserving its q-$\bar{q}$ bound state character, plays the *role* of the Nambu-Goldstone boson in the spontaneous chiral symmetry-breaking process.

As explained above, the main dynamical consequence of chiral symmetry breaking is that the constituent masses replace everywhere the current ones in the Hamiltonian. But when we take into account the action of massive gluons, a new effect arises: The negligible quark self-energy diagrams due to the massive gluon exchange, which could anyhow be incorporated in the definition of the current masses, after the vacuum rearrangement due to the (almost) spontaneous chiral symmetry violation, are no more negligible for a non-negligible dynamical constituent mass now arises. Thus, in calculating the shift in the masses of the meson states induced to second order by the massive gluon-quark coupling, we must consider terms of the form

$$\delta m_a^{(c)}=\frac{4}{3}\frac{\alpha_S}{\pi}m_a^{(c)}\Delta\left[\frac{M^2}{m_a^{(c)2}},\frac{m_g^2}{m_a^{(c)2}}\right] \qquad (7.5)$$

(the function Δ is reported in Appendix C), where the "cutoff" M must be of the order of the typical energy scale at which the chiral symmetry breaking occurs.

This mass shift modifies the kinetic Hamiltonian and hence the dispersion relation of quarks and antiquarks. To be consistent with our treatment of the effects of the Coulomb interaction, we expand the kinetic Hamiltonian to second order in α_S to get

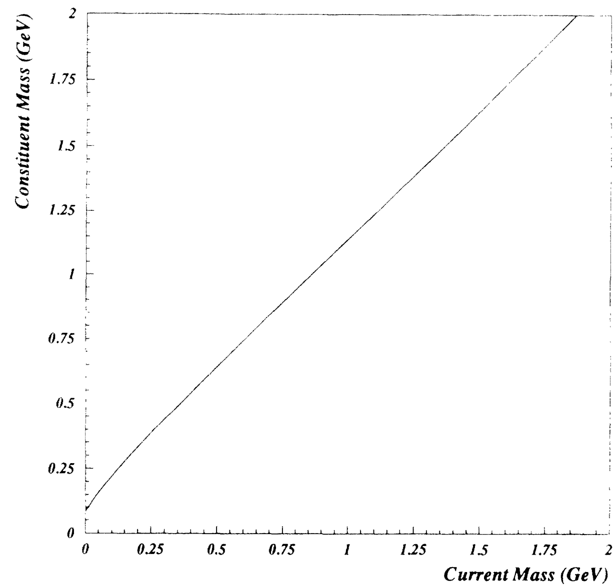

FIG. 2. Constituent vs current quark mass. For high masses the relation is (almost) linear.

TABLE I. Quark masses used in this paper.

Quark flavor	Current mass (GeV)	Constituent mass (GeV)
Up	0.019	0.113
Down	0.019	0.113
Strange	0.074	0.189
Charm	1.248	1.385
Bottom	4.625	4.736

$$\langle \mathbf{p},rs|H_D|\mathbf{p}',r's'\rangle=\delta^{(3)}(\mathbf{0})\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rr'}\delta_{ss'}[E_a+E_b+\delta E_a(p)+\delta E_b(p)]\ , \tag{7.6}$$

where

$$\delta E_a(p)=\frac{4}{3}\frac{\alpha_S}{\pi}\frac{m_a^{(c)2}}{E_a}\Delta\left[\frac{M^2}{m_a^{(c)2}},\frac{m_g^2}{m_a^{(c)2}}\right]+\frac{1}{2}\left[\frac{4}{3}\frac{\alpha_S}{\pi}\right]^2\left[\frac{1}{E_a}-\frac{m_a^{(c)2}}{E_a^2}\right]m_a^{(c)2}\Delta\left[\frac{M^2}{m_a^{(c)2}},\frac{m_g^2}{m_a^{(c)2}}\right]+o(\alpha_S^2)\ . \tag{7.7}$$

Thus, in (7.6), in addition to the usual kinetic term, we have the potential (we omit, as usual, the rest-frame δ function)

$$U^{\mathrm{CSB}}(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rr'}\delta_{ss'}[\delta E_a(p)+\delta E_b(p)]\ , \tag{7.8}$$

and to evaluate the effects of $U^{\mathrm{CSB}}(\mathbf{p},\mathbf{p}')_{rs}^{r's'}$ on meson eigenvalues and eigenvectors, instead of (4.5) we must apply (4.4) with

$$\langle M^{(0)},m|H_{\mathrm{GT}}|M^{(0)},n\rangle=\int_0^\infty dp\,dp'\,pp'\,u_m(p)[U_J(p,p')+U_J^{\mathrm{CSB}}(p,p')]u_n(p')\ ; \tag{7.9}$$

the projection of $U^{\mathrm{CSB}}(\mathbf{p},\mathbf{p}')_{rs}^{r's'}$ on the $|l,s,j\rangle$ basis can be immediately carried out to give

$$U_J^{\mathrm{CSB}}(p,p')=\frac{\delta(p-p')}{pp'}[\delta E_a(p)+\delta E_b(p)]\ . \tag{7.10}$$

VIII. RESULTS AND COMPARISON WITH EXPERIMENTS

We open this section by recalling that the structure of the meson spectrum that stems from our approximation is

$$M_J=M_J^{(0)}+\delta M_J^{(0)}+M_J^{(1)}+M_J^{(2)}\ , \tag{8.1}$$

where the first term is the eigenvalue of the Schrödinger equation (3.9), obtained by Multhopp's method, while the other terms are perturbative correction evaluated by means of (3.23) and (4.4a); as for the wave function, this is given by

$$u_J(p)=u_J^{(0)}(p)+u_J^{(1)}(p)\ , \tag{8.2}$$

where the correction $u_J^{(1)}(p)$ can be extracted in an obvious way from (4.4b). We have fixed the relevant parameters (without any best fitting procedure) at the values

$$\mu=0.48\ \mathrm{GeV}\ , \tag{8.3a}$$

$$\alpha_S=0.40\ , \tag{8.3b}$$

$$m_g=0.14\ \mathrm{GeV}\ , \tag{8.3c}$$

whereas the quark masses we have used are reported in Table I.

Figures 3–9 show our results for the meson spectrum. For heavy quarkonia, our results are in very good agreement with available experimental data, with the only exceptions of high radial excitations in the 1^{--} sector: In our opinion this is due to the fact that, over the threshold for charm-anticharm production (the same kind of considerations, of course, applies to b-$\bar{b}$ states), we expect a considerable amount of mixing and mass shift induced by the "unitarity" effects on the self-energy function of the state; this is a very interesting problem that belongs to the perturbative effects associated to the pair creation Hamiltonian $H^{(1)}$.

Heavy flavored mesons also are in satisfactory agreement with the data.

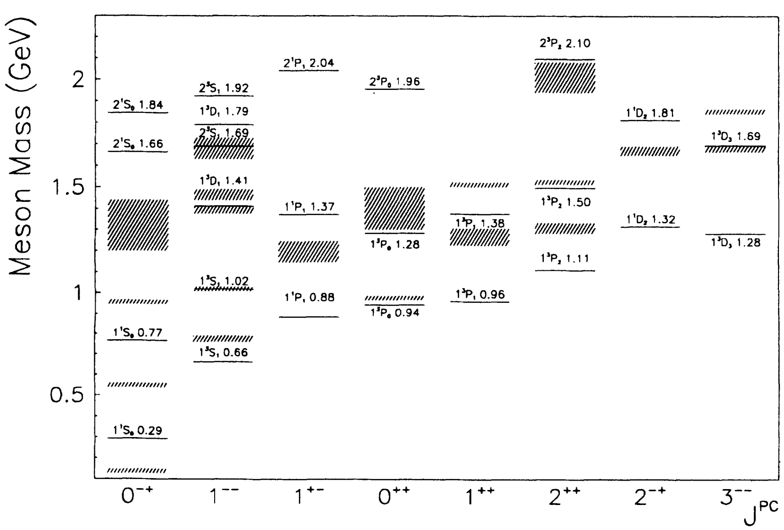

FIG. 3. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the light unflavored mesons. Units are in GeV.

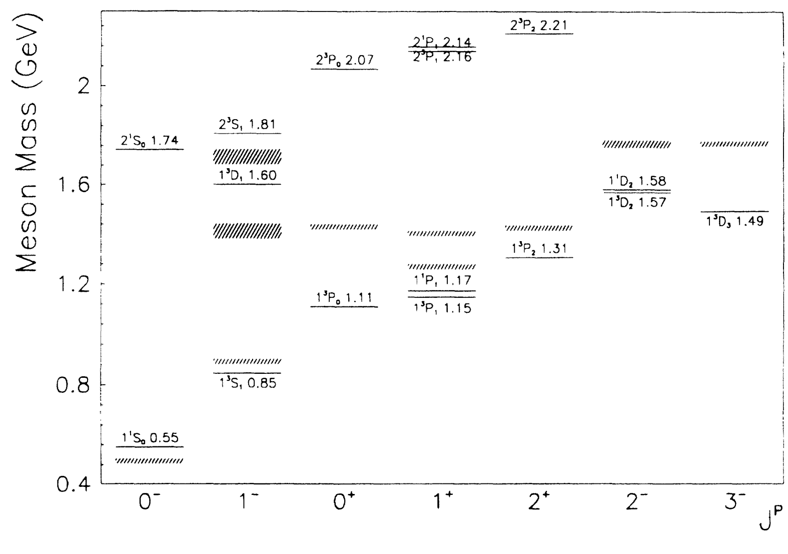

FIG. 4. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the strange mesons. Units are in GeV.

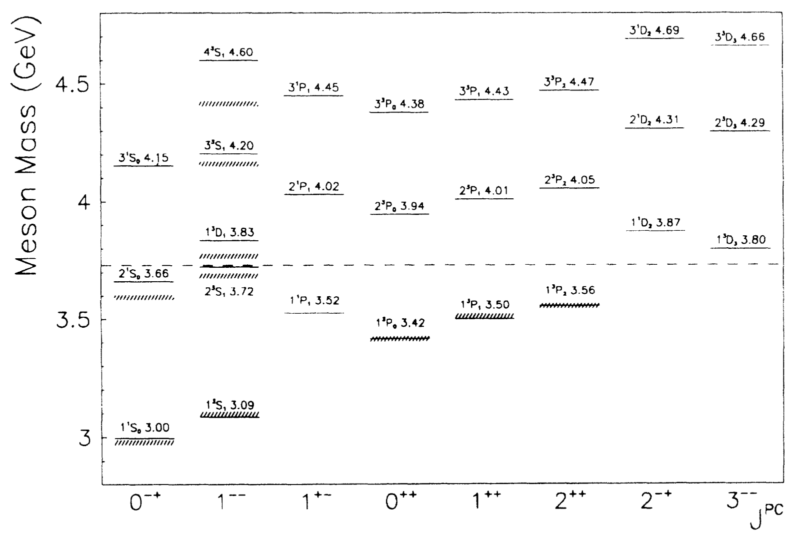

FIG. 5. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the c-$\bar{c}$ mesons (charmonium). The dashed line is the threshold. Units are in GeV.

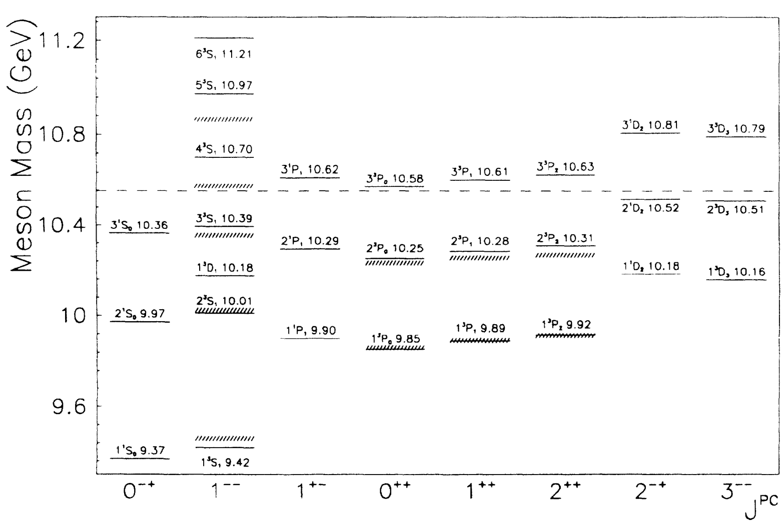

FIG. 6. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the b-$\bar{b}$ mesons (bottomonium). The dashed line is the threshold. Units are in GeV.

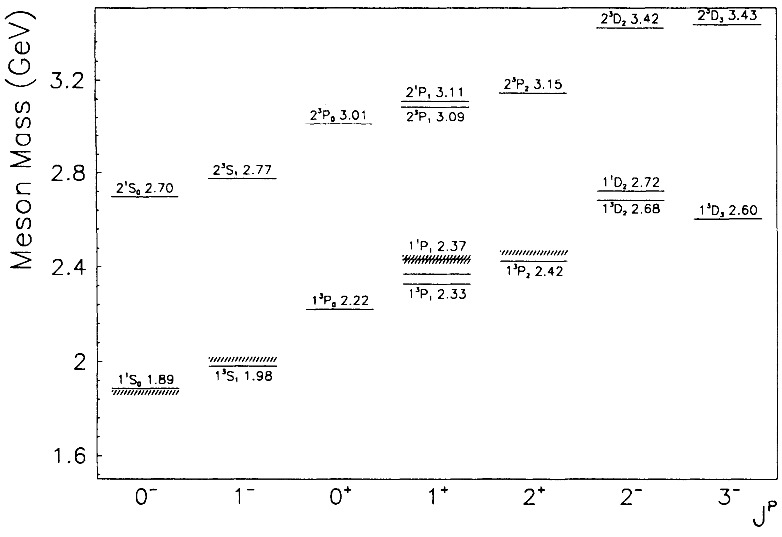

FIG. 7. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the charmed mesons. Units are in GeV.

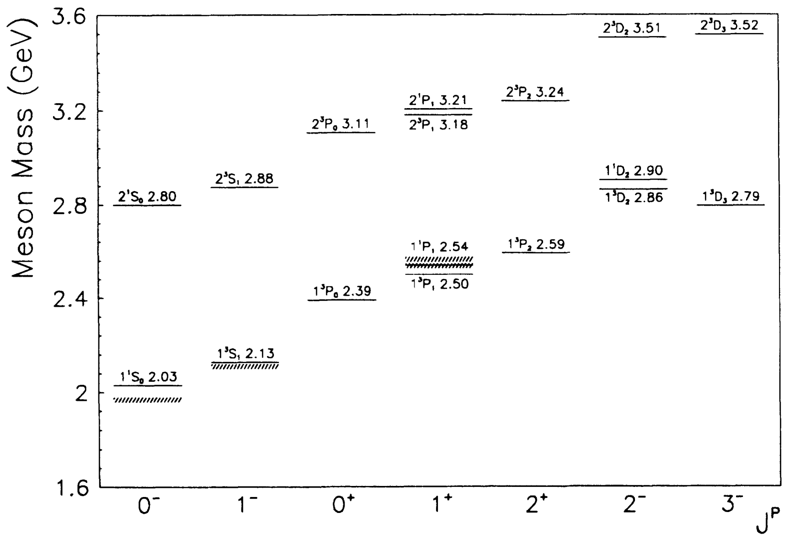

FIG. 8. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the charmed strange mesons. Units are in GeV.

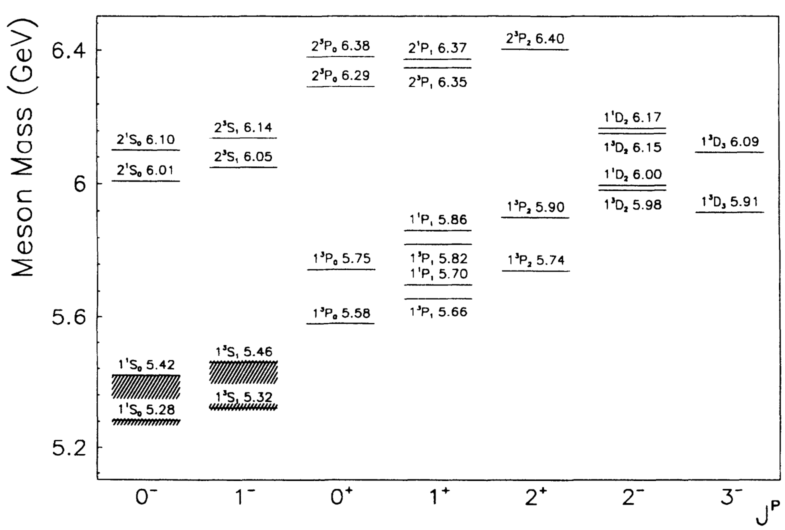

FIG. 9. Comparison between our theoretical calculation (solid lines) and the available experimental data (dashed areas) for the bottom and bottom strange mesons. Units are in GeV.

TABLE II. Theoretical pseudoscalar and vector decay constants. The experimental values, where available, are shown for comparison.

Pseudoscalar state	f_P (MeV)		Vector state	f_V (GeV^2)	
π	104	(92.5±0.2)	ρ	0.14	(0.117±0.003)
			ω	0.047	(0.036±0.001)
K	101	(113.0±1.0)			
η	79	(92±5)	Φ	0.10	(0.081±0.001)
η'	78	(83±5)			
D	170	(<220)			
D_s	186				
B	164				
B_s	180				
η_c	311		J/Ψ	0.97	(0.84±0.02)
			$\Psi(2S)$	1.00	(0.69±0.03)
η_b	492		Υ	2.17	(2.26±0.03)
			$\Upsilon(2S)$	1.87	(1.63±0.04)

For the light states, we make the following remarks.

(i) There is nothing in our Hamiltonian that distinguishes between isovector and isoscalar states. Thus in the unflavored sector we have no mixing between u-$\bar{u}$ and s-$\bar{s}$ states; in particular, this explains why in the 0^{-+} sector we have not a good description of η and η' mesons that, as is well known, are strongly mixed. This is another interesting problem whose solution has recently been found in the effects of $H^{(1)}$ [14].

(ii) The pseudoscalar states lie systematically above the experimental data; this is probably due to our "crude" approximation in treating chiral symmetry breaking.

(iii) For the P-wave states, one must repeat the consideration made above on the heavy quarkonia: The states are strongly coupled to the open channels, so that one expects unitarity mass shifts of the order of the widths.

Table II shows our results for the pseudoscalar and vector decay constants: The overall agreement with data is good; we observe that the value of the decay constants depends strongly on the wave functions. It thus appears that our approach gives us a good approximation of the "true" wave function.

IX. CONCLUSIONS

What has been presented in the preceding sections is the first comprehensive analysis of a key problem of nonperturbative QCD. When computed in the framework of ACD, the theory of finite energy (in the limit $\Lambda \to \infty$) quantum fluctuations of QCD, the hadrons, around the (most likely) ground state, the CML, the meson (q-$\bar{q}$) spectrum and the associated $q\bar{q}$ wave functions show a remarkable likeness to the real world. This is all the more comforting, since our calculation is, within the limitations and approximations that have been clearly spelled out, really a "first principles" calculation, based on a *minimal* set of inputs: $g(\Lambda^2)$ (the bare coupling constants) and the quark masses (the constituent masses, whose unique connection to the bare masses has been elucidated in Sec. VII).

The fact that we have been able to obtain *all* masses within 100–150 MeV, in terms of a (relatively) simple analysis, is, in our opinion, remarkable for two reasons. First, the agreement we obtain is within what we *do* expect for the approximate calculation that we have carried out here, neglecting all "unitarity" effects, which we know are governed by the quark pair creation Hamiltonian $H^{(1)}$, which does not belong to the primitive world. Indeed, the discrepancies we register are of the same size of the typical hadrons' widths, which are induced by the same dynamics. As for the second reason, we must stress that obtaining a complete set of eigenvalues and eigenfunctions belonging to the diagonalization of the PW Hamiltonian has cost relatively little in terms of calculation, with the very definite advantage that, differently from the costly and complicated Monte Carlo simulations of LGT's, we do get a simple understanding of what confinement is and where it comes from.

Finally, at least in the case of heavy quarkonia, there is quite a lot of overlap with other "QCD-inspired" calculations: We modestly acknowledge that the progress we have achieved there may not look impressive; however, we note that for the first time we have a *unified* approach of heavy quark (short-distance) and light quark (long-distance) spectroscopy, and that this has not required any methodological shift.

The road ahead is certainly very long and steep, but this work convinces us that it is well worthwhile to proceed along it; we do hope that this is also the conviction of the reader who has had the endurance to get this far.

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APPENDIX A: CALCULATION OF THE $q-\bar{q}$ POTENTIAL

We start the calculation of the "potential" writing down the expansion of the Dirac field at $t=0$:

$$q^{\alpha}_{\mathbf{u},\mathbf{a}}(\xi)=\sum_r \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \{ b_{\mathbf{u},\mathbf{a}}(p,r,\alpha) u(p\mathbf{u},r) e^{ip\xi} + d^{\dagger}_{\mathbf{u},\mathbf{a}}(p,r,\alpha) v(p\mathbf{u},r) e^{-ip\xi} \} ; \tag{A1}$$

to ensure (2.7) the commutation relations for the creation-annihilation operators turn out to be [spinors are normalized according to $u^{\dagger}(\mathbf{p},r)u(\mathbf{p},s)=\delta_{rs}$ and so on]

$$\{ b_{\mathbf{u},\mathbf{a}}(p,r,\alpha), b^{\dagger}_{\mathbf{u}',\mathbf{a}'}(p',r',\alpha') \} = \delta_{\mathbf{u},\mathbf{u}'}\delta_{\mathbf{a},\mathbf{a}'}\delta_{rr'}\delta_{\alpha\alpha}\delta(p-p') . \tag{A2}$$

The normal-ordered (0) current is therefore given by

$$j^{(0)\mu}_{m,\mathbf{u},\mathbf{a}}(\xi)=\sum_{t_1,t_2}\sum_{\alpha_1,\alpha_2}\int_{-\infty}^{\infty}\frac{dh_1 dh_2}{2\pi}\left\{ \bar{u}(h_1\mathbf{u},t_1)\gamma^{\mu}\left[\frac{\lambda^m}{2}\right]_{\alpha_1\alpha_2} u(h_2\mathbf{u},t_2)e^{-i(h_1-h_2)\xi} b^{\dagger}_{\mathbf{u},\mathbf{a}}(h_1,t_1,\alpha_1) b_{\mathbf{u},\mathbf{a}}(h_2,t_2,\alpha_2) \right.$$
$$\left. -\bar{v}(h_1\mathbf{u},t_1)\gamma^{\mu}\left[\frac{\lambda^m}{2}\right]_{\alpha_1\alpha_2} v(h_2\mathbf{u},t_2)e^{i(h_1-h_2)\xi} d^{\dagger}_{\mathbf{u},\mathbf{a}}(h_2,t_2,\alpha_2) d_{\mathbf{u},\mathbf{a}}(h_1,t_1,\alpha_1) \right\} . \tag{A3}$$

Let us now concentrate on the typical term entering the matrix elements of the Hamiltonian appearing in (3.16):

$$\langle p,\mathbf{v},rs|H_{\mathbf{u},\mathbf{a}}|p',\mathbf{v}',r's'\rangle^{(1)}$$
$$=\frac{g^2}{2N_{\mathbf{u}}A_0}\lim_{\lambda\to 0}\sum_{t_1,t_2,t_3,t_4}\int_{-\infty}^{\infty} d\xi\, d\eta \frac{dq}{2\pi}\frac{dh_1 dh_2 dh_3 dh_4}{(2\pi)^2}\delta_{mn}(g_{\mu\nu}+n_{\mu}n_{\nu})\frac{\delta^{\alpha\beta}}{\sqrt{3}}\frac{\delta^{\alpha'\beta'}}{\sqrt{3}}$$
$$\times \frac{e^{iq(\xi-\eta)}}{q^2+\lambda^2} e^{i(h_2-h_1)\xi} e^{i(h_4-h_3)\eta} \bar{u}(h_1\mathbf{u},t_1)\gamma^{\mu}\left[\frac{\lambda^m}{2}\right]_{\alpha_1\alpha_2} u(h_2\mathbf{u},t_2)\bar{u}(h_3\mathbf{u},t_3)$$
$$\times \gamma^{\nu}\left[\frac{\lambda^n}{2}\right]_{\alpha_3\alpha_4} u(h_4\mathbf{u},t_4)\langle\Omega| d_{\mathbf{v},0}(-p,s,\beta) b_{\mathbf{v},0}(p,r,\alpha) b^{\dagger}_{\mathbf{u},\mathbf{a}}(h_1,t_1,\alpha_1) b_{\mathbf{u},\mathbf{a}}(h_2,t_2,\alpha_2)$$
$$\times b^{\dagger}_{\mathbf{u},\mathbf{a}}(h_3,t_3,\alpha_3) b_{\mathbf{u},\mathbf{a}}(h_4,t_4,\alpha_4) b^{\dagger}_{\mathbf{v}',0}(p',r',\alpha') d^{\dagger}_{\mathbf{v}',0}(-p',s',\beta')|\Omega\rangle . \tag{A4}$$

Evaluating the operator string by means of (A2), we obtain the following.

(i) From the color part of the expression we have the factor

$$\delta_{mn}\frac{\delta^{\alpha\beta}}{\sqrt{3}}\frac{\delta^{\alpha'\beta'}}{\sqrt{3}}\left[\frac{\lambda^m}{2}\right]_{\alpha\alpha_2}\left[\frac{\lambda^n}{2}\right]_{\alpha_2\alpha'}\delta_{\beta\beta'}=\frac{4}{3} . \tag{A5}$$

(ii) The "tube" (transverse) part of the diagram reduces simply to

$$\delta_{\mathbf{a},0}\delta_{\mathbf{v},\mathbf{u}}\delta_{\mathbf{v}',\mathbf{u}} . \tag{A6}$$

(iii) The integration over ξ and η leaves us with

$$(2\pi)^2\delta(0)\delta(q-h_1+h_2) . \tag{A7}$$

Taking into account (A5)–(A7) and (3.17), (3.16) becomes

$$V(\mathbf{x},\mathbf{x}')^{r's'}_{rs}=\frac{1}{A_0}\delta(0)\delta^{(2)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')\frac{4}{3}\frac{g^2}{2N_{\mathbf{u}}A_0}\int_{-\infty}^{\infty}\frac{dp\,dp'}{2\pi}\frac{e^{i(px-p'x')}}{xx'}K(p,p';\hat{\mathbf{x}})^{r's'}_{rs} ; \tag{A8}$$

in this expression, the factor $(1/A_0)\delta(0)$ is the rest-frame three-dimensional δ function (we systematically omit this term, as it factorizes from all terms of eigenvalue equation), and we have used

$$\lim_{N_{\mathbf{u}}\to\infty}\frac{N_{\mathbf{u}}}{4\pi}\sum_{\mathbf{u}}\delta_{\mathbf{u}}(\hat{\mathbf{x}})\delta_{\mathbf{u}}(\hat{\mathbf{x}}')=\delta^{(2)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}') \tag{A9}$$

(this is exactly the same limit as $\Lambda\to\infty$).

$K(p,p';\hat{\mathbf{x}})^{r's'}_{rs}$ is simply the longitudinal spinor part of the matrix element (A4) and for the full Hamiltonian turns out to be the sum of two contributions, the first of which is to be interpreted as due to "self-mass" diagrams (the $b^{\dagger}bb^{\dagger}b$ and $d^{\dagger}dd^{\dagger}d$ terms in the product of the currents), while the second one is due to "exchange" diagrams (the two $b^{\dagger}bd^{\dagger}d$ combinations):

$$K^{(\mathrm{sm})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=\delta(p-p')\lim_{\lambda\to 0}\sum_t\int_{-\infty}^{\infty}\frac{dh}{2\pi}\frac{1}{(p-h)^2+\lambda^2}$$
$$\times\{\delta_{ss'}\bar{u}(p\hat{\mathbf{x}},r)\gamma_\perp^\mu u(h\hat{\mathbf{x}},t)\bar{u}(h\hat{\mathbf{x}},t)\gamma_{\perp\mu}u(p\hat{\mathbf{x}},r')$$
$$+\delta_{rr'}\bar{v}(h\hat{\mathbf{x}},t)\gamma_\perp^\mu v(-p\hat{\mathbf{x}},s)\bar{v}(-p\hat{\mathbf{x}},s')\gamma_{\perp\mu}v(h\hat{\mathbf{x}},t)\}\ , \tag{A10a}$$

$$K^{(\mathrm{ex})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=-2\lim_{\lambda\to 0}\frac{1}{2\pi}\frac{\bar{u}(p\hat{\mathbf{x}},r)\gamma_\perp^\mu u(p'\hat{\mathbf{x}},r')\bar{v}(-p'\hat{\mathbf{x}},s')\gamma_{\perp\mu}v(-p\hat{\mathbf{x}},s)}{(p-p')^2+\lambda^2}\ , \tag{A10b}$$

where we have introduced the notation

$$\gamma_\perp^\mu=\gamma_\nu(g^{\mu\nu}+n^\mu n^\nu)\ . \tag{A11}$$

To evaluate (A10) let us define the functions

$$a(p,q;m)=\frac{(E_p+m)(E_q+m)+pq}{[4E_pE_q(E_p+m)(E_q+m)]^{1/2}}\ , \tag{A12a}$$

$$b(p,q;m)=\frac{(E_q+m)p-(E_p+m)q}{[4E_pE_q(E_p+m)(E_q+m)]^{1/2}}\ , \tag{A12b}$$

which have the properties

$$a^2(p,q;m)+b^2(p,q;m)=1\ ; \tag{A13}$$

we easily obtain

$$\bar{u}(p\hat{\mathbf{x}},r)\gamma_0u(q\hat{\mathbf{x}},s)=a(p,q;m_a)\delta_{rs}\ , \tag{A14a}$$

$$\bar{v}(-p\hat{\mathbf{x}},r)\gamma_0v(-q\hat{\mathbf{x}},s)=a(p,q;m_b)\delta_{rs}\ , \tag{A14b}$$

$$\bar{u}(p\hat{\mathbf{x}},r)\gamma_\perp u(q\hat{\mathbf{x}},s)=ib(p,q;m_a)(\sigma_\perp\wedge\hat{\mathbf{x}})_{rs}\ , \tag{A14c}$$

$$\bar{v}(-p\hat{\mathbf{x}},r)\gamma_\perp v(-q\hat{\mathbf{x}},s)=-ib(p,q;m_b)(\sigma_\perp\wedge\hat{\mathbf{x}})_{rs}\ . \tag{A14d}$$

We finally have

$$K^{(\mathrm{sm})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=\delta(p-p')\delta_{rr'}\delta_{ss'}\lim_{\lambda\to 0}\int_{-\infty}^{\infty}\frac{dh}{2\pi}\frac{1}{(p-h)^2+\lambda^2}[2-3b^2(p,h;m_a)-3b^2(p,h;m_b)]\ , \tag{A15a}$$

$$K^{(\mathrm{ex})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=-2\lim_{\lambda\to 0}\frac{1}{2\pi}\frac{1}{(p-p')^2+\lambda^2}[a(p,p';m_a)a(p,p';m_b)\delta_{rr'}\delta_{ss'}+b(p,p';m_a)b(p,p';m_b)(\sigma_\perp)_{rr'}\cdot(\sigma_\perp)_{ss'}]\ . \tag{A15b}$$

We can now extract the singular terms when $\lambda\to 0$ and perform the limit in the regular ones; moreover, we use

$$\int_{-\infty}^{\infty}dh\frac{b^2(p,h)}{(p-h)^2}=\frac{1}{E_p} \tag{A16}$$

to rewrite $K(p,p';\hat{\mathbf{x}})_{rs}^{r's'}$ as the sum of a singular and a regular part:

$$K^{(\mathrm{sing})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=\frac{1}{\pi}\lim_{\lambda\to 0}\left[\frac{\pi}{\lambda}\delta(p-p')-\frac{1}{(p-p')^2+\lambda^2}\right]\delta_{rr'}\delta_{ss'}\ , \tag{A17a}$$

$$K^{(\mathrm{sm})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=-\frac{3}{2\pi}\left[\frac{1}{E_a}+\frac{1}{E_b}\right]\delta(p-p')\delta_{rr'}\delta_{ss'}\ , \tag{A17b}$$

$$K^{(\mathrm{ex})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}=+\frac{1}{\pi}\frac{[1-a(p,p';m_a)a(p,p';m_b)]}{(p-p')^2}\delta_{rr'}\delta_{ss'}-\frac{1}{\pi}\frac{b(p,p';m_a)b(p,p';m_b)}{(p-p')^2}(\sigma_\perp)_{rr'}\cdot(\sigma_\perp)_{ss'}\ . \tag{A17c}$$

By inserting the expression of $K^{(\mathrm{sing})}(p,p';\hat{\mathbf{x}})_{rs}^{r's'}$, (A17a), into (A8) and performing the integrations, we obtain the long-range part of the potential:

$$V_{\mathrm{lr}}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=\frac{4}{3}\frac{g^2}{2N_{\mathrm{u}}A_0}\frac{\delta^{(2)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')}{xx'}\delta(x-x')\delta_{rr'}\delta_{ss'}\lim_{\lambda\to 0}\frac{1}{\lambda}[1-e^{-\lambda x}]$$
$$=\mu^2\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta_{rr'}\delta_{ss'}x\ , \tag{A18}$$

where we have defined the "string tension" μ as

$$\mu=\left[\frac{4}{3}\frac{g^2}{2N_u A_0}\right]^{1/2} . \tag{A19}$$

Please note that in order to obtain the cancellation of the $O(1/\lambda)$ term in (A18) the color factors coming from the self-mass and exchange graphs need to be equal to each other, and this happens only if the color wave function of the meson is the singlet one; i.e., color is confined.

By using the integral representation of the K_0 modified Bessel function

$$K_0(x)=\int_0^\infty dt\frac{e^{itx}}{\sqrt{1+t^2}} , \tag{A20}$$

the contribution of the regular part of the self-mass diagrams to the potential can be cast in the form

$$V_{\rm sm}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=-\frac{3\mu^2}{2\pi^2}\frac{\delta^{(2)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')}{xx'}\delta_{rr'}\delta_{ss'}[K_0(m_a|r-r'|)+K_0(m_b|r-r'|)] ; \tag{A21}$$

as for the contribution of the regular part of the exchange graphs to $V(\mathbf{x},\mathbf{x}')_{rs}^{r's'}$, we cannot give a simple formula for the general case; however, in the equal-mass case ($m_a=m_b=m$) we are able to obtain

$$V_{\rm ex}(\mathbf{x},\mathbf{x}')_{rs}^{r's'}=\frac{\mu^2}{2\pi^2}\frac{\delta^{(2)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')}{xx'}[\delta_{rr'}\delta_{ss'}+(\sigma_\perp)_{rr'}\cdot(\sigma_\perp)_{ss'}]\int_{-\infty}^{\infty}\frac{dt}{\cosh^2 t}K_0[m\sqrt{x^2+x'^2+2xx'\cosh(2t)}] ; \tag{A22}$$

this is why, in order to perform explicit calculations, we choose to write this last term as in (3.19b), with $V_{\rm res}(r,r')_{rs}^{r's'}$ given by

$$V_{\rm res}(r,r')_{rs}^{r's'}=[\delta_{rr'}\delta_{ss'}+(\sigma_\perp)_{rr'}\cdot(\sigma_\perp)_{ss'}]K_0\left[\frac{m_a+m_b}{2}|r+r'|\right]$$
$$-2\int_{-\infty}^{\infty}dp\,dp'e^{i(px-p'x')}\left\{\frac{[1-a(p,p';m_a)a(p,p';m_b)]}{(p-p')^2}\delta_{rr'}\delta_{ss'}-\frac{b(p,p';m_a)b(p,p';m_b)}{(p-p')^2}(\sigma_\perp)_{rr'}\cdot(\sigma_\perp)_{ss'}\right\} . \tag{A23}$$

APPENDIX B: MULTHOPP'S METHOD

Multhopp's technique is a method for finding eigenstates and eigenfunctions of bound state equations that is very appropriate when dealing with singular integral equations.

Let us consider the integral equation

$$[g(p)-E]+\int_0^\infty dp'\,V(p,p')u(p')=0 \tag{B1}$$

and make the change of variables $p=-h\cot\theta$, $p'=-h\cot\theta'$, with h a positive, arbitrary constant and $\theta\in[\pi/2,\pi]$. Then the above equation reads

$$[g(\theta)-E]+\int_{\pi/2}^{\pi}\frac{h\,d\theta'}{\sin^2\theta'}V(\theta,\theta')u(\theta')=0 , \tag{B2}$$

where, for simplicity, we have written $u(-h\cot\theta)$ as $u(\theta)$, $g(-h\cot\theta)$ as $g(\theta)$ and $V(\theta,\theta')$ instead of $V(-h\cot\theta,-h\cot\theta')$.

As $\{(\sqrt{4/\pi},\sin(2n\theta)\}_{n\in\mathbf{N}}$ is an orthonormal basis on $[\pi/2,\pi]$, we can write the expansion

$$u(\theta)=\sum_{n=1}^{\infty}c_n\sin(2n\theta) . \tag{B3}$$

We now truncate the series at a maximal value N and introduce N discrete values ("Multhopp's angles"):

$$\theta_k=\frac{\pi}{2}\left[1+\frac{k}{N+1}\right] , \quad k=1,\ldots,N ; \tag{B4}$$

for these special values, the expansion (B3) becomes

$$u_k=\sum_{n=1}^{N}c_n\sin(2n\theta_k) , \tag{B5}$$

where $u_k\equiv u(\theta_k)$.

By using the orthonormality relation

$$\frac{2}{N+1}\sum_{k=1}^{N}\sin(2n\theta_k)\sin(2m\theta_k)=\delta_{mn} , \tag{B6}$$

we obtain

$$u(\theta')=\sum_{n=1}^{N}\sum_{m=1}^{N}c_n\delta_{nm}\sin(2m\theta')$$
$$=\sum_{j=1}^{N}\frac{2}{N+1}\sum_{m=1}^{N}\sin(2m\theta_j)\sin(2m\theta')u_j , \tag{B7}$$

and the integral equation (B2) reduces to the simple algebraic problem

$$\sum_{j=1}^{N}M_{kj}u_j=Eu_k , \quad k=1,\ldots,N , \tag{B8}$$

with the $N\times N$ matrix M_{kj} given by

$$M_{kj}=g(\theta_k)\delta_{kj}+\frac{2}{N+1}\sum_{m=1}^{N}\sin(2m\theta_j)\times\int_{\pi/2}^{\pi}\frac{h\,d\theta'}{\sin^2\theta'}V(\theta_k,\theta')\sin(2m\theta')\ . \tag{B9}$$

Once the eigenvalue problem (B8) has been solved, we can extract from (B5) [use (B6)] the expansion coefficients as

$$c_n=\frac{2}{N+1}\sum_{k=1}^{N}\sin(2n\theta_k)u_k\ ,\quad k=1,\ldots,N\ . \tag{B10}$$

The accuracy and stability of the method can be checked by iterating the procedure with increasing values of N. We have found that the solution converges quite rapidly and that N of the order of 80 gives us very good results for the first low-lying eigenvalues and eigenvectors.[3] Also, the convergence of the method depends critically on the value of h, which must be chosen in such a way that the eigenfunctions are well centered around the values θ_k; this means that h must be of the order of the mean value of p; on the other hand, small variations of h around this value have no relevant effect on the solutions.

In fact, Multhopp's technique provides us with excellent approximations of the wave function $u(p)$ only far from the asymptotic regions $p\to 0$ and $p\to\infty$ (that correspond to $\theta\to\pi/2$ and $\theta\to\pi$), the reason being that, when we discretize the variable θ (and hence p), only a few points fall into these regions.

Moreover, it is easy to see that Multhopp's solution has the behaviors

$$u(p)\sim\begin{cases}p\ , & p\to 0\ ,\\ 1/p\ , & p\to\infty\ ,\end{cases} \tag{B11}$$

to compare with the analytic asymptotic behavior of the solutions of our eigenvalues problem

$$u_J(p)\sim\begin{cases}p^{l+1}\ , & p\to 0\ ,\\ 1/p^{l+4}\ , & p\to\infty\ .\end{cases} \tag{B12}$$

Hence we take as the reduced wave function of the meson the expression

$$u_J(p)=\begin{cases}a\left[\dfrac{p}{p+b}\right]^{l+1}\ , & p<p_{\rm inf}\ ,\\ \displaystyle\sum_{n=1}^{N}c_n\sin\left[2n\left[\pi-\arctan\frac{h}{p}\right]\right]\ , & p_{\rm inf}\le p\le p_{\rm sup}\ ,\\ \dfrac{c}{(p-d)^{l+4}}\ , & p>p_{\rm sup}\ ,\end{cases} \tag{B13}$$

which has the correct asymptotic behaviors; the coefficients a, b, c, and d are chosen in such a way that $u\in C^1(\mathbb{R})$, and the continuation points are chosen to be in the region of the first and last Multhopp angle, respectively; we have also checked that any variation of $p_{\rm inf}$ and $p_{\rm sup}$ in these regions has no notable effects on the matrix elements that we evaluate with the functions $u(p)$.

APPENDIX C: CALCULATION OF THE COULOMB POTENTIAL

The calculation of the relevant matrix elements of the Hamiltonian term $H_{\rm GT}$ is much less involved than that sketched in Sec. III and Appendix A for the confining potential; for now, the Hamiltonian is written directly in terms of the collective operators, those appearing also in (3.3).

Writing the usual decomposition for the Dirac field at $t=0$ [remember that spinors are normalized according to $u^\dagger(\mathbf{p},r)u(\mathbf{p},s)=\delta_{rs}$],

$$q^\alpha(\mathbf{x})=\sum_r\int_{-\infty}^{\infty}\frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\{b(\mathbf{p},r,\alpha)u(\mathbf{p},r)e^{i\mathbf{p}\cdot\mathbf{x}}+d^\dagger(\mathbf{p},r,\alpha)v(\mathbf{p},r)e^{-i\mathbf{p}\cdot\mathbf{x}}\} \tag{C1}$$

and expanding the Hamiltonian on the creation and annihilation operators, we obtain, as usual, four terms: two corresponding to quark self-mass diagrams and two corresponding to exchange diagrams.

We start by evaluating the contribution of exchange terms, which results in the kernel

$$\begin{aligned}U(\mathbf{p},\mathbf{p}')_{rs}^{r's'}&=\langle\mathbf{p},rs|H_{\rm GT}|\mathbf{p}',r's'\rangle^{\rm ex}\\&=2\frac{g^2}{2}\sum_{t_1,t_2,t_3,t_4}\int d^3\mathbf{x}d^3\mathbf{y}\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{d^3\mathbf{h}_1d^3\mathbf{h}_2d^3\mathbf{h}_3d^3\mathbf{h}_4}{(2\pi)^6}\delta_{mn}g_{\mu\nu}\frac{\delta^{\alpha\beta}}{\sqrt{3}}\frac{\delta^{\alpha'\beta'}}{\sqrt{3}}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{-q_0^2+\mathbf{q}^2+m_g^2}\\&\quad\times e^{i(\mathbf{h}_2-\mathbf{h}_1)\cdot\mathbf{x}}e^{i(\mathbf{h}_4-\mathbf{h}_3)\cdot\mathbf{y}}\bar{u}(\mathbf{h}_1,t_1)\gamma^\mu\left[\frac{\lambda^m}{2}\right]_{\alpha_1\alpha_2}u(\mathbf{h}_2,t_2)\bar{u}(\mathbf{h}_3,t_3)\gamma^\nu\left[\frac{\lambda^n}{2}\right]_{\alpha_3\alpha_4}u(\mathbf{h}_4,t_4)\\&\quad\times\langle\Omega|d(-\mathbf{p},s,\beta)b(\mathbf{p},r,\alpha)b^\dagger(\mathbf{h}_1,t_1,\alpha_1)b(\mathbf{h}_2,t_2,\alpha_2)\\&\quad\times d^\dagger(\mathbf{h}_3,t_3,\alpha_3)d(\mathbf{h}_4,t_4,\alpha_4)b^\dagger(\mathbf{p}',r',\alpha')d^\dagger(-\mathbf{p}',s',\beta')|\Omega\rangle\ ,\end{aligned} \tag{C2}$$

[3]We have tested Multhopp's technique by solving eigenvalue problems whose analytical solutions are known.

where the factor 2 takes into account the existence of two identical contributions. Carrying out the evaluation of the operator string and the **x** and **y** integrations, we are left with

$$U(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=-\frac{4}{3}2\frac{g^2}{2}\frac{\delta^{(3)}(\mathbf{0})}{(2\pi)^3}\frac{\bar{u}(\mathbf{p},r)\gamma^\mu u(\mathbf{p}',r')\bar{v}(-\mathbf{p}',s')\gamma_\mu u(-\mathbf{p},s)}{-(E_a-E_a')^2+(\mathbf{p}-\mathbf{p}')^2+m_g^2}\,, \tag{C3}$$

the factor $\frac{4}{3}$ coming from the color part of the diagram; we discard, as in the other terms of our equations, the factor $\delta^{(3)}(\mathbf{0})$, which is the center-of-mass momentum conservation, and rewrite (C3) as

$$U(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=-\frac{4}{3}\frac{\alpha_S}{\pi}\frac{1}{2\pi}\frac{1}{-(E_a-E_a')^2+(\mathbf{p}-\mathbf{p}')^2+m_g^2}B(\mathbf{p},\mathbf{p}')_{rs}^{r's'}\,, \tag{C4}$$

where

$$\alpha_S=\frac{g^2}{4\pi} \tag{C5}$$

and

$$B(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=\bar{u}(\mathbf{p},r)\gamma^\mu u(\mathbf{p}',r')\bar{v}(-\mathbf{p}',s')\gamma_\mu u(-\mathbf{p},s) \tag{C6}$$

is simply the product of the current. After a little algebra we obtain

$$\begin{aligned}B(\mathbf{p},\mathbf{p}')_{rs}^{r's'}=C(p,p')\{&f_{\rm SI}(\mathbf{p},\mathbf{p}')+f_{\rm SS}(\mathbf{p},\mathbf{p}')\sigma_{rr'}\cdot\sigma_{ss'}+i[f_{\rm SO}^{(1)}(\mathbf{p},\mathbf{p}')\delta_{ss'}\sigma_{rr'}+f_{\rm SO}^{(2)}(\mathbf{p},\mathbf{p}')\delta_{rr'}\sigma_{ss'}]\cdot(\mathbf{p}\wedge\mathbf{p}')\\&+[(\sigma_i)_{rr'}(\sigma_j)_{ss'}-\tfrac{1}{3}(\sigma)_{rr'}\cdot(\sigma)_{ss'}][f_{\rm ST}^{(1)}(p,p')p^ip^j+f_{\rm ST}^{(2)}(p,p')p'^ip'^j+f_{\rm ST}^{(3)}(\mathbf{p},\mathbf{p}')p^ip'^j]\\&+f_{\rm MX}(p,p')(\sigma_{rr'}\wedge\sigma_{ss'})\cdot(\mathbf{p}\wedge\mathbf{p}')\}\;;\end{aligned} \tag{C7}$$

we thus have a structure comprising spin-independent, spin-spin, spin-orbit, and tensor terms; the last term, in particular, is different from zero only in the unequal mass case and, moreover, has no diagonal matrix elements (in the $|l,s,j\rangle$ space); it is responsible for the mixing in the $1^+,2^-,3^+,\ldots$ sectors for the flavored mesons, so that we take into account this term only in the final step of our calculations.

If we define the quantities

$$\begin{aligned}h_a&=\sqrt{E_a+m_a}\,,\\h_b&=\sqrt{E_b+m_b}\,,\\h_a'&=\sqrt{E_a'+m_a}\,,\\h_b'&=\sqrt{E_b'+m_b}\,,\end{aligned} \tag{C8}$$

the functions appearing in (C7) are given by

$$\begin{aligned}f_{\rm SI}(\mathbf{p},\mathbf{p}')=&h_ah_bh_a'h_b'+\frac{h_ah_b}{h_a'h_b'}p'^2+\frac{h_a'h_b'}{h_ah_b}p^2\\&+\left[\frac{h_ah_a'}{h_bh_b'}+\frac{h_bh_b'}{h_ah_a'}+\frac{h_ah_b'}{h_a'h_b}+\frac{h_a'h_b}{h_ah_b'}\right]\mathbf{p}\cdot\mathbf{p}'\\&+\frac{(\mathbf{p}\cdot\mathbf{p}')^2}{h_ah_bh_a'h_b'}\,,\end{aligned} \tag{C9a}$$

$$\begin{aligned}f_{\rm SS}(\mathbf{p},\mathbf{p}')=-\frac{4}{3}\Bigg[&\frac{p^2p'^2}{h_ah_bh_a'h_b'}+2\frac{h_ah_b}{h_a'h_b'}p'^2+2\frac{h_a'h_b'}{h_ah_b}p^2\\&+2\left[\frac{h_ah_b'}{h_a'h_b}+\frac{h_a'h_b}{h_ah_b'}\right]\mathbf{p}\cdot\mathbf{p}'-\frac{(\mathbf{p}\cdot\mathbf{p}')^2}{h_ah_bh_a'h_b'}\Bigg]\,,\end{aligned} \tag{C9b}$$

$$f_{\rm SO}^{(1)}(\mathbf{p},\mathbf{p}')=2\left[\frac{h_bh_b'}{h_ah_a'}+\frac{h_a'h_b}{h_ah_b'}+\frac{h_ah_b'}{h_a'h_b}+\frac{\mathbf{p}\cdot\mathbf{p}'}{h_ah_bh_a'h_b'}\right]\,, \tag{C9c}$$

$$f_{\rm SO}^{(2)}(\mathbf{p},\mathbf{p}')=2\left[\frac{h_ah_a'}{h_bh_b'}+\frac{h_a'h_b}{h_ah_b'}+\frac{h_ah_b'}{h_a'h_b}+\frac{\mathbf{p}\cdot\mathbf{p}'}{h_ah_bh_a'h_b'}\right]\,, \tag{C9d}$$

$$f_{\rm ST}^{(1)}(p,p')=4\left[\frac{h_a'h_b'}{h_ah_b}+\frac{p'^2}{h_ah_bh_a'h_b'}\right]\,, \tag{C9e}$$

$$f_{\rm ST}^{(2)}(p,p')=4\left[\frac{h_ah_b}{h_a'h_b'}+\frac{p^2}{h_ah_bh_a'h_b'}\right]\,, \tag{C9f}$$

$$f_{\rm ST}^{(3)}(\mathbf{p},\mathbf{p}')=-4\left[\frac{h_ah_b'}{h_a'h_b}+\frac{h_a'h_b}{h_ah_b'}+\frac{2\mathbf{p}\cdot\mathbf{p}'}{h_ah_bh_a'h_b'}\right]\,, \tag{C9g}$$

$$f_{\rm MX}(p,p')=2\left[\frac{h_a'h_b}{h_ah_b'}-\frac{h_ah_b'}{h_a'h_b}\right]\,, \tag{C9h}$$

whereas the normalization factor is

$$C(p,p')=\frac{1}{4\sqrt{E_aE_bE_a'E_b'}}\,. \tag{C10}$$

As for the self-mass terms, they are best evaluated without making any approximation in the propagator, coming back to the original Lagrangian and evaluating the self-energy function $\Sigma(p)$; this is a completely standard calculation, except for the fact that we have massive gluons; the result is

$$\Sigma_a(p)=\frac{4}{3}\frac{\alpha_s}{\pi}\frac{1}{2}\int_0^1 dt(2m_a-t\not{p}) \times \ln\left[\frac{tM^2}{(1-t)m_a^2+tm_g^2-t(1-t)p^2}\right], \tag{C11}$$

where M is a momentum cutoff introduced via the Pauli-Villars regularization method in order to avoid ultraviolet divergences. We can now extract the quark mass shift as

$$\delta m_a=\sum_a(p)|_{\not{p}=m_a}; \tag{C12}$$

the integration can be carried out explicitly, and we obtain

$$\delta m_a=\frac{4}{3}\frac{\alpha_S}{\pi}m_a\Delta\left[\frac{M^2}{m_a^2},\frac{m_g^2}{m_a^2}\right], \tag{C13}$$

where the function Δ is

$$\Delta(y,x)=\frac{1}{4}\left\{3\ln y+\frac{3}{2}+x-\frac{1}{2}x^2\ln x +\frac{x+2}{2}\left[\theta(x-4)\frac{\sqrt{x^2-4x}}{2}\ln\frac{x+\sqrt{x^2+4x}}{x-\sqrt{x^2-4x}} -\theta(4-x)\sqrt{4x-x^2}\times\arctan\left[\frac{4-x}{x}\right]^{1/2}\right]\right\}. \tag{C14}$$

Thus the contribution of the self-energy terms results only in a (divergent for $M\to\infty$) mass shift,[4] which is reabsorbed in the process of renormalization of the QCD Lagrangian and in the definition of the current mass, which is a free parameter of the theory, so that the self-energy diagrams have no effects on the dynamics of the bound states.

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[4]Actually, we have made some approximation also in the evaluation of this contribution, putting the quarks on the mass shell, i.e., setting $\not{p}=m_a$. This is the same kind of approximation that we make in order to obtain the instantaneous propagator for the exchange term; namely, we replace the matter-field time evolution in the bound state with the free one.

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