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Angular distribution functions for the decay of charmonium states directly produced by polarized proton-antiproton collisions

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We calculate the combined angular distribution functions in the cascade process $p\bar{p}\rightarrow\chi_J\rightarrow\psi\gamma\rightarrow(e^+e^-)\gamma$ $(J=0,1,2)$ in terms of the complex angular momentum helicity amplitudes or equivalently in terms of the multipole transition amplitudes in $\chi_J\rightarrow\psi\gamma$, when the proton (p) and the antiproton $(\bar{p})$ are both arbitrarily polarized. The results are expressed as a sum over products of linearly independent spherical harmonics. We discuss the results for the special cases when neither particle is polarized, a single particle is polarized, and both particles are polarized. In general, by measuring the angular distributions in polarized $p\bar{p}$ collisions we are not only able to get the magnitudes, but also the relative phases of the decay amplitudes in the process $\chi_J\rightarrow\psi\gamma$ and of the production amplitudes in $p\bar{p}\rightarrow\chi_J$.

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I. INTRODUCTION

There are ongoing experiments [1] at Fermilab to determine the angular distribution of the decay products of the charmonium χ_J states formed directly in the collisions of unpolarized proton (p) and antiproton $(\bar{p})$. Expressions for the combined angular distribution functions of the electron and of the photon in the cascade process $p\bar{p}\rightarrow\chi_J\rightarrow\psi+\gamma\rightarrow(e^+e^-)+\gamma$ $(J=0,1,2)$ when both p and $\bar{p}$ are unpolarized have been derived before [2,3]. These expressions are given in terms of the ordinary trigonometric functions and are bilinear functions of the angular momentum helicity amplitudes in the individual processes $p\bar{p}\rightarrow\chi_J$, $\chi_J\rightarrow\psi+\gamma$, and $\psi\rightarrow e^+e^-$. If we assume that these helicity amplitudes are real they can be obtained from the measured angular distribution functions [1]. But there is no good reason to believe that the helicity amplitudes are real. In fact, the potential model calculations [4] indicate that they may be complex. In that case, one needs to measure the angular distributions of the decay products of the charmonium χ_J states when p and $\bar{p}$ are polarized to obtain the magnitudes and the relative phases of these helicity amplitudes. In this paper we calculate the combined angular distribution functions of the electron and of the photon in the process $p\bar{p}\rightarrow\chi_J\rightarrow\psi+\gamma\rightarrow(e^+e^-)+\gamma$ when p and $\bar{p}$ are arbitrarily polarized. Our final result for the angular distribution function is written as a linear combination of terms involving linearly independent products of spherical harmonics which are functions of the angles giving the directions of the final electron and of the photon with respect to the incoming $\bar{p}$ direction. The coefficients in this expansion are functions of the complex angular momentum helicity amplitudes in the different sequential processes mentioned above, as well as functions of the longitudinal and transverse components of the polarization vectors of p and $\bar{p}$ in their respective rest frames. With zero polarization for both p and $\bar{p}$, the angular distribution function depends only on $|B_i|^2$ $(i=0,1)$, where B_i are the angular momentum helicity amplitudes in the process $p\bar{p}\rightarrow\chi_J$ $(J=0,1,2)$, and on $\mathrm{Re}(A_iA_j^*)$ where the A_i are the angular momentum helicity amplitudes in the process $\chi_J\rightarrow\psi+\gamma$. Even though in this case our results reduce to those of the previous works [2,3] when B_i and A_i are real, as far as we know, this is the first time the results have been expressed in terms of linearly independent and orthonormal spherical harmonic functions. This feature of our results will probably make it easier to extract the helicity amplitudes from the measured angular distribution function. It should also be noted that one of the previous works [2] assumed without any real justification that all the helicity amplitudes were real. When the rest-frame polarization vectors of p and $\bar{p}$ are not zero, in our expression the terms in the coefficients that are linear in the longitudinal and the transverse components of the polarization vectors depend on both the real and the imaginary parts of the helicity amplitudes in the processes $p\bar{p}\rightarrow\chi_J$ and $\chi_J\rightarrow\psi+\gamma$. So by studying the angular distribution function in polarized $p\bar{p}$ collisions we are able to determine not only the magnitudes but also the relative phases of the complex angular momentum helicity amplitudes in both processes. First we consider the general case when both p and $\bar{p}$ are arbitrarily polarized. The expression for the combined angular distribution of the electron and of the photon in the final state is derived by means of the density matrix formalism and is given in terms of the polarization vectors defined for the stationary p and $\bar{p}$. We also discuss the results for the special cases when neither particle is polarized, a single particle is polarized, and both particles are polarized. The following cases prove to be of particular interest: a single particle polarized with both transverse and longitudinal components of the polarization vector being nonzero, and both particles polarized, one with a nonzero transverse component and the other with a nonzero longitudinal component.

The format of the rest of the paper is as follows. In Sec. II, we define the polarization vectors and the density matrix of spin-$\frac{1}{2}$ particles. We then express the density matrix elements for arbitrary values of p and $\bar{p}$ momenta in terms of their rest-frame polarization vectors. We also express the transition amplitude for the process $p\bar{p}\to\chi_J\to\psi+\gamma\to(e^+e^-)+\gamma$ in terms of the Wigner D^J functions and the angular momentum helicity amplitudes in the processes $p\bar{p}\to\chi_J$, $\chi_J\to\psi+\gamma$ and $\psi\to(e^+e^-)$. We then give a formal expression for the angular distribution function of the electron and of the photon in terms of the transition amplitude and the density matrix elements of the initial arbitrarily polarized $p\bar{p}$ system. In Sec. III, we reexpress this angular distribution as a sum of products of spherical harmonics which are functions of the angles defining the directions of the final photon and of the final electron. The coefficients of the spherical harmonics in this sum are functions of the angular momentum helicity amplitudes as well as the longitudinal and transverse components of the rest-frame polarization vectors of p band $\bar{p}$. These coefficients are listed in Appendix A. In Sec. IV we discuss our results. Finally in Sec. V we make some concluding remarks.

II. POLARIZATION VECTORS, DENSITY MATRICES, AND FORMULATION OF THE PROBLEM

Consider a beam of spin-$\frac{1}{2}$ particles consisting of an incoherent mixture of pure spin states, denoted by i, each occurring with probability f_i. If N_i is the number of particles in the pure state $|i\rangle$ and N is the total number of particles,

$$f_i=\frac{N_i}{N}\ . \tag{1}$$

In the rest frame of the beam, the polarization vector is then given by

$$\mathbf{P}=\sum_i f_i\hat{\imath}_i\ , \tag{2}$$

where $\hat{\imath}_i$ is a unit vector pointing in the direction of the filter axis that allows complete transmission of all particles in the state $|i\rangle$. The magnitude of the polarization vector ranges from zero for an unpolarized mixture to 1 for all particles in a pure state or a completely polarized beam. The density matrix for such a mixture in its rest frame can be written as

$$\rho=\tfrac{1}{2}(1+\mathbf{P}\cdot\boldsymbol{\sigma})\ , \tag{3}$$

where σ_x, σ_y, and σ_z are the three (2×2) Pauli matrices.

In an arbitrary Lorentz frame where either p or $\bar{p}$ has the four-momentum $q^\mu=(\varepsilon,\mathbf{q})$, we can define a polarization four-vector s^μ which is related to its rest-frame value of $(0,\mathbf{P})$ by equation

$$s^\mu=\left[\frac{\mathbf{q}\cdot\mathbf{P}}{m},\mathbf{P}+\frac{\mathbf{q}(\mathbf{q}\cdot\mathbf{P})}{m(\varepsilon+m)}\right]\ , \tag{4}$$

where m is the rest mass of the particle. In the arbitrary Lorentz frame, the (4×4) density matrix can be written as

$$\rho_\pm=\frac{\not{q}\pm m}{2m}\frac{1+\gamma_5\not{s}}{2}\ . \tag{5}$$

In Eq. (5), the positive sign refers to the proton and the negative sign refers to the antiproton. The density matrix elements are connected to their rest-frame values in the following way. Let χ_λ be a two-component Pauli spinor for either particle at rest quantized along the direction given by the unit vector $\hat{\mathbf{e}}$. That is,

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{e}}\chi_\lambda=\lambda\chi_\lambda\ . \tag{6}$$

Let e^μ be a four-vector whose value in the rest frame of the particle is $(0,\hat{\mathbf{e}})$. Then we write four-component Dirac spinor $W_\pm(q,\lambda)$ for the proton or the anitproton in an arbitrary Lorentz frame with the properties:

$$\begin{aligned}&\not{q}W_\pm(q,\lambda)=\pm mW_\pm(q,\lambda)\ ,\\&\gamma_5\not{e}W_\pm(q,\lambda)=\lambda W_\pm(q,\lambda)\ .\end{aligned} \tag{7}$$

As before, the plus sign in Eq. (7) refers to the proton and the minus sign refers to the antiproton and q is the four-momentum of either particle. The density matrix elements are then connected to their rest-frame values by the equation

$$\overline{W}_\pm(q,\lambda')\left[\frac{\not{q}\pm m}{2m}\frac{1+\gamma_5\not{s}}{2}\right]W_\pm(q,\lambda)=\chi_{\lambda'}^+\left[\frac{1+\boldsymbol{\sigma}\cdot\mathbf{P}}{2}\right]\chi_\lambda\ . \tag{8}$$

We now give a derivation of our formal result for the angular distribution of e^- and γ when both p and $\bar{p}$ are arbitrarily polarized.

The probability amplitude for the process

$$\bar{p}(\lambda_1)p(\lambda_2)\to\chi_{J,\nu}\to\psi_\sigma+\gamma_\mu\to e^-(\kappa_1)e^+(\kappa_2)+\gamma_\mu\ ,$$

where λ_1, λ_2, ν, σ, μ, κ_1, and κ_2 are the particle helicities, can be written as the product of the amplitudes of three sequential events:

$$\bar{p}(\lambda_1)+p(\lambda_2)\to\chi_{J,\nu'}\chi_{J,\nu}\to\psi_\sigma+\gamma_{\mu'}$$

and

$$\psi_\sigma\to e^-(\kappa_1)+e^+(\kappa_2)\ .$$

If $|p,\theta,\phi;\lambda_1\lambda_2\rangle$ represents a two-particle helicity state in the zero-momentum (c.m.) frame, where p is the magnitude of either particle's momentum and the angles (θ,ϕ) represent the first particle's momentum and λ_1,λ_2 the helicities of the two particles, then following Jacob and Wick [5] and Martin and Spearman [6], we can write an expansion in terms of the angular momentum states as

$$|p,\theta,\phi;\lambda_1\lambda_2\rangle = \sum_{J,M}\left[\frac{2J+1}{4\pi}\right]^{1/2} D^J_{M\lambda}(\phi,\theta,-\phi)|pJM;\lambda_1\lambda_2\rangle\ , \tag{9a}$$

where

$$\lambda=\lambda_1-\lambda_2 \tag{9b}$$

and ϕ, θ, and $-\phi$ are the three Euler angles. The Wigner functions $D^J_{M\lambda}$ are the (M,λ) matrix elements of the $(2J+1)$-dimensional representation of the three-dimensional rotation matrices.

We will work in the χ_J rest frame with the z axis taken to be in the direction of motion of ψ. The momentum of $\bar{p}$, namely, $\mathbf{p}$, makes an angle θ with the z axis. Then,

$$\hat{\mathbf{j}}=\frac{\hat{\mathbf{k}}\times\mathbf{p}}{|\hat{\mathbf{k}}\times\mathbf{p}|} \text{ and } \hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{k}}\ . \tag{10}$$

Using Eq. (9), the helicity amplitude for the process $\bar{p}(\lambda_1)p(\lambda_2)\rightarrow\chi_{J,\nu}$ can be written as

$$\langle J\nu|B|\theta 0;\lambda_1\lambda_2\rangle = \left[\frac{2J+1}{4\pi}\right]^{1/2} B^J_{\lambda_1\lambda_2} d^J_{\nu\lambda}(\theta)\ , \tag{11a}$$

where λ is given by Eq. (9b),

$$d^J_{\nu\lambda}(\theta)=D^J_{\nu\lambda}(0,\theta,0)\ , \tag{11b}$$

and B is a transition operator. Depending upon the author, the amplitudes on both sides of Eq. (11a) are referred to as helicity amplitudes. We shall refer to the amplitude on the left, with the operator between helicity states, as the helicity amplitude. The amplitude $B^J_{\lambda_1\lambda_2}$, which is actually the matrix element of the transition operator between states characterized by the angular momentum and the helicity indices, will be referred to as the angular momentum helicity amplitude. We notice that because of charge-conjugation invariance [2,6], the angular momentum helicity amplitude

$$B^J_{\lambda_1\lambda_2}=(-1)^J B^J_{\lambda_2\lambda_1} \tag{12}$$

and, by parity invariance,

$$B^J_{\lambda_1\lambda_2}=(-1)^J B^J_{-\lambda_1-\lambda_2}\ . \tag{13}$$

The helicity amplitude for the process $\chi_{J,\nu}\rightarrow\psi_\sigma+\gamma_\mu$ with the quarkonium ψ and the photon γ moving along the $+z$ and $-z$ directions, respectively, can be written as

$$\langle 00,\sigma\mu|A|J\nu\rangle = \left[\frac{2J+1}{4\pi}\right]^{1/2} A'^J_{\sigma\mu} D^{J*}_{\nu,\sigma-\mu}(0,0,0)\ , \tag{14a}$$

where

$$D^J_{\nu,\sigma-\mu}(0,0,0)=\delta_{\nu,\sigma-\mu} \tag{14b}$$

and $A'^J_{\sigma\mu}$ are again the angular momentum helicity amplitudes. Charge-conjugation invariance is trivially satisfied in this process. By parity invariance [2,6],

$$A'^J_{\sigma\mu}=(-1)^J A'^J_{-\sigma,-\mu}\ . \tag{14c}$$

In the ψ rest frame, with the direction of the final electron's momentum specified by (θ',ϕ'), the helicity amplitude for the process $\psi_\sigma\rightarrow e^-(\kappa_1)+e^+(\kappa_2)$ becomes

$$\langle\theta',\phi';\kappa_1\kappa_2|C|1\sigma\rangle = \left[\frac{3}{4\pi}\right]^{1/2} C_{\kappa_1\kappa_2} D^{1*}_{\sigma\kappa}(\phi',\theta',-\phi')\ , \tag{15a}$$

where

$$\kappa=\kappa_1-\kappa_2\ . \tag{15b}$$

By charge-conjugation invariance [6], the amplitude $C_{\kappa_1\kappa_2}$ satisfies the equation

$$C_{\kappa_1\kappa_2}=C_{\kappa_2\kappa_1}\ . \tag{15c}$$

By parity invariance [6], we also have

$$C_{\kappa_1\kappa_2}=C_{-\kappa_1-\kappa_2}\ . \tag{15d}$$

If the e^+e^- system is produced by the process $q\bar{q}\rightarrow\gamma\rightarrow e^+e^-$, the helicity-zero amplitude $C_0=\sqrt{2}C_{++}=\sqrt{2}C_{--}$ is of order $m/E\simeq 3.3\times 10^{-4}$ compared to the helicity-1 amplitude $C_{+-}=C_{-+}=C_1$, and should be negligible. But for the sake of an independent experimental determination of C_0 from the measured angular distribution, we keep it in our expressions.

Wick [7] constructs a two-particle helicity state in an arbitrary frame by first constructing it in the zero-momentum frame. The state is then transformed to the arbitrary frame with the angular momentum states of Eq. (9a) transforming as single-particle states of spin J and mass $2E$. Using this technique, the helicity amplitudes in Eq. (15a) have the same values in both the ψ and χ_J rest frames, with no Wigner rotations.

The amplitude $T_{\lambda_1\lambda_2\kappa_1\kappa_2\mu}$ for the process to go from the initial state of $\bar{p}(\lambda_1)p(\lambda_2)$ to the final state of $e^-(\kappa_1)+e^+(\kappa_2)+\gamma_\mu$ through all possible helicity states ν of χ_J and σ of ψ is a sum of linearly independent products of amplitudes of Eqs. (11a), (14a), and (15a) summed over all possible values of ν and σ subject to the constraints $\nu=\sigma-\mu$ and $\mu=\pm1$. We thus get [3]

$$T_{\lambda_1\lambda_2\kappa_1\kappa_2\mu} = \left[\frac{3(2J+1)^2}{(4\pi)^3}\right]^{1/2} B^J_{\lambda_1\lambda_2} C_{\kappa_1\kappa_2} \times\sum_{\nu(\mu)} A'^J_{\nu+\mu,\mu} D^{1*}_{\nu+\mu,\kappa}(\phi',\theta',-\phi') d^J_{\nu\lambda}(\theta)\ , \tag{16}$$

where ν takes the values 0 to $+J$ for $\mu=-1$ and $-J$ to 0 for $\mu=+1$.

The normalized angular distribution function for the cascade process when the initial $\bar{p}$ and p are arbitrarily polarized and the final polarizations of γ, e^-, and e^+ are not observed is given by $W(\theta;\theta',\phi')$:

$$W(\theta;\theta',\phi')=N_J \sum_{\substack{\kappa_1\kappa_2\mu \\ \lambda_1\lambda_2}} T_{\lambda_1\lambda_2\kappa_1\kappa_2\mu}\rho_{1\lambda_1\lambda_1'}\rho_{2\lambda_2\lambda_2'}T^*_{\lambda_1'\lambda_2'\kappa_1\kappa_2\mu}\ . \tag{17}$$

The normalization constant N_J is determined by requiring that, for the unpolarized case, the integral of the distribution function $W(\theta;\theta',\phi')$ over all the directions of γ and e^- or over all the angles θ, ϕ, θ', and ϕ' is 1. In Eq. (17) the symbols $\rho_{1\lambda_1\lambda_1'}$ and $\rho_{2\lambda_2\lambda_2'}$ represent the density matrices of $\bar{p}$ and p, respectively. In the helicity basis states of the particles these matrix elements are

$$\rho_{1\lambda_1\lambda_1'}=\chi_{\lambda_1}^{+}\tfrac{1}{2}(1+\mathbf{P}_1\cdot\boldsymbol{\sigma})\chi_{\lambda_1'}\ , \tag{18}$$

$$\rho_{2\lambda_2\lambda_2'}=\beta_{\lambda_2}^{+}\tfrac{1}{2}(1+\mathbf{P}_2\cdot\boldsymbol{\sigma})\beta_{\lambda_2'}\ . \tag{19}$$

In Eqs. (18) and (19) $\mathbf{P}_1$ and $\mathbf{P}_2$ are the polarization vectors of $\bar{p}$ and p and the helicity basis states χ_{λ_1} of $\bar{p}$ and β_{λ_2} of p are defined by

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi_{\lambda_1}=\lambda_1\chi_{\lambda_1} \tag{20}$$

and

$$\boldsymbol{\sigma}\cdot(-\hat{\mathbf{p}})\beta_{\lambda_2}=\lambda_2\beta_{\lambda_2}\ , \tag{21}$$

where λ_1 and λ_2 can take the values $+1$ or -1. In the coordinate system we defined in the beginning, since $\bar{p}$ and p are in the xz plane with $\phi=0$,

$$\chi_+=\begin{bmatrix}\cos\theta/2\\ \sin\theta/2\end{bmatrix},\quad \chi_-=\begin{bmatrix}-\sin\theta/2\\ \cos\theta/2\end{bmatrix} \tag{22}$$

and the phase of β is such that [7]

$$\beta_\pm=\chi_\mp\ . \tag{23}$$

Equations (18) and (19) can be rewritten as

$$\begin{aligned}\rho_{1\lambda_1\lambda_1'}&=\chi_{\lambda_1}^{+}\frac{1}{2}\begin{bmatrix}1+P_{1z} & P_{1x}-iP_{1y}\\ P_{1x}+iP_{1y} & 1-P_{1z}\end{bmatrix}\chi_{\lambda_1'}\\ &=\frac{1}{2}\begin{bmatrix}(1+P_{1z'}) & (P_{1x'}-iP_{1y'}\\ (P_{1x'}+iP_{1y'} & (1-P_{1z'})\end{bmatrix},\end{aligned} \tag{24}$$

where the unit vectors along the new x', y', and z' axes are related to the corresponding vectors of the xyz coordinate system by

$$\begin{aligned}\hat{\mathbf{i}}'&=\cos\theta\hat{\mathbf{i}}-\sin\theta\hat{\mathbf{k}}=\hat{\mathbf{j}}\times\hat{\mathbf{p}}\ ,\\ \hat{\mathbf{j}}'&=\hat{\mathbf{j}}\ ,\\ \hat{\mathbf{k}}'&=\hat{\mathbf{p}}=\sin\theta\hat{\mathbf{i}}+\cos\theta\hat{\mathbf{k}}\ .\end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned}\rho_{2\lambda_2\lambda_2'}&=\beta_{\lambda_2}^{+}\frac{1}{2}\begin{bmatrix}1+P_{2z} & P_{2x}-iP_{2y}\\ P_{2x}+iP_{2y} & 1-P_{2z}\end{bmatrix}\beta_{\lambda_2'}\\ &=\frac{1}{2}\begin{bmatrix}1-P_{2z'} & P_{2x'}+iP_{2y'}\\ P_{2x'}-iP_{2y'} & 1+P_{2z'}\end{bmatrix}.\end{aligned} \tag{26}$$

In Eqs. (24) and (26), $P_{1z'}$ and $-P_{2z'}$ are the longitudinal components (components along the momenta of the respective particles) and the x' and y' components are the transverse components of the polarization vectors. The angular distribution function $W(\theta,\theta',\phi')$ for arbitrary values of p and $\bar{p}$ momenta is now given in terms of the density matrix elements defined for the stationary proton and antiproton.

In Eq. (17), the angles (θ',ϕ') give the direction of e^- in the ψ rest frame and the angle θ gives that of $\bar{p}$ in the χ_J rest frame. But there is no Lorentz frame where χ_J and ψ are both at rest. In the χ_J rest frame or the $p\bar{p}$ c.m. frame, ψ is moving with a velocity

$$v=\beta\simeq 0.15\ . \tag{27}$$

If the direction of e^- in the χ_J rest frame is given by (θ'',ϕ''), these angles are related to the angles (θ',ϕ') by the relations (to first order in β)

$$\begin{aligned}\cos\theta'&\simeq\cos\theta''-\beta\sin^2\theta''\ ,\\ \sin\theta'&\simeq\sin\theta''+\beta\sin\theta''\cos\theta''\ ,\\ \phi'&=\phi''\ .\end{aligned} \tag{28}$$

In order to get the angular distribution in the $p\bar{p}$ c.m. frame, all we have to do is to reexpress $T_{\lambda_1\lambda_2\kappa_1\kappa_2\mu}$ in terms of (θ'',ϕ'') by making use of Eqs. (28).

III. COMBINED ANGULAR DISTRIBUTION FUNCTION OF THE ELECTRON AND OF THE PHOTON, IN TERMS OF SPHERICAL HARMONICS, WHEN p AND $\bar{p}$ ARE ARBITRARILY POLARIZED

In this section we express the angular distribution function $W(\theta;\theta',\phi')$ of Eq. (17) in terms of the spherical harmonic functions in the variables θ and θ',ϕ'. Substituting Eq. (16) in Eq. (17), we can rewrite Eq. (17) as

$$\begin{aligned}W(\theta;\theta',\phi')=N_J\frac{3(2J+1)^2}{(4\pi)^3}\sum_{\kappa_1\kappa_2}|C_{\kappa_1\kappa_2}|^2\sum_{\substack{\lambda_1\lambda_2\\ \lambda_1'\lambda_2'}}B^J_{\lambda_1\lambda_2}B^{J*}_{\lambda_1'\lambda_2'}\rho_{1\lambda_1\lambda_1'}\rho_{2\lambda_2\lambda_2'}\\ \times\sum_{\mu=\pm1}\sum_{\nu(\mu)\nu'(\mu)}A'^J_{\nu+\mu,\mu}A'^{J*}_{\nu'+\mu,\mu}D^{1*}_{\nu+\mu,\kappa}(\phi',\theta',-\phi')D^1_{\nu'+\mu,\kappa}(\phi',\theta',-\phi')d^J_{\nu\lambda}(\theta)d^J_{\nu'\lambda'}(\theta)\ .\end{aligned} \tag{29}$$

Performing the $\mu=\pm1$ sum gives

$$\sum_{\mu=\pm 1}\sum_{\nu(\mu),\nu'(\mu)} A'^{J}_{\nu+\mu,\mu} A'^{J*}_{\nu'+\mu,\mu} D^{1*}_{\nu+\mu,\kappa} D^{1}_{\nu'+\mu,\kappa} d^{J}_{\nu\lambda} d^{J}_{\nu'\lambda'}$$
$$= \sum_{L'=0,1,2} \langle 11\kappa-\kappa|L'0\rangle(-1)^{\kappa+1} \sum_{\nu\nu'}^{0\to J} A^{J}_{\nu} A^{J*}_{\nu'} (-1)^{\nu'} \langle 11,\nu-1,-(\nu'-1)|L',\nu-\nu'\rangle$$
$$\times \sum_{L=0}^{2J} \langle JJ\lambda\lambda'|L,\lambda+\lambda'\rangle\langle JJ\nu\nu'|L,\nu+\nu'\rangle[D^{L'*}_{\nu-\nu',0} d^{L}_{\nu+\nu',\lambda+\lambda'} + (-1)^{\lambda+\lambda'}(-1)^{L} D^{L'}_{\nu-\nu',0} d^{L'}_{\nu-\nu',0} d^{L}_{\nu+\nu',-(\lambda+\lambda')}] , \tag{30}$$

where we have made the notational replacement

$$A^{J}_{\nu} = A'^{J}_{\nu-1,-1} = (-1)^{J} A'^{J}_{-\nu+1,1} \tag{31}$$

and employed the Clebsch-Gordan series [8]

$$D^{j_1}_{m_1 m'_1} D^{j_2}_{m_2 m'_2} = \sum_{J}^{\Delta(j_1,j_2,J)} \langle j_1 j_2 m_1 m_2|JM\rangle\langle j_1 j_2 m'_1 m'_2|JM'\rangle D^{J}_{MM'} . \tag{32}$$

The $L'=1$ term in Eq. (30) does not contribute to the distribution function $W(\theta;\theta',\phi')$ of Eq. (29) since the distribution function is invariant under the exchange $\kappa_1 \leftrightarrow \kappa_2$. So, substituting Eq. (30) into Eq. (29), we get

$$W(\theta;\theta'\phi') = \frac{3(2J+1)^2}{(4\pi)^3} N_J \sum_{L'}^{0,2} K_{L'} \sum_{L=0}^{2J} \sum_{\lambda\lambda'}^{1,0,1} \Lambda^{L}_{\lambda\lambda'} N^{L'L}_{\lambda\lambda'} . \tag{33}$$

The term $K_{L'}$ is defined as

$$K_{L'} = \sum_{\kappa_1\kappa_2}^{\pm 1/2} |C_{\kappa_1\kappa_2}|^2 \langle 11\kappa-\kappa|L'0\rangle(-1)^{\kappa+1}$$
$$= \sum_{\kappa}^{-1,0,1} |C_{\kappa}|^2 \langle 11\kappa-\kappa|L'0\rangle(-1)^{\kappa+1} , \tag{34a}$$

where

$$C_1 = C_{+-} ,$$
$$C_0 = \sqrt{2} C_{++} = \sqrt{2} C_{--} , \tag{34b}$$
$$C_{-1} = C_{-+} .$$

Because of charge-conjugation or parity invariance [Eqs. (15c) and (15d)] we also have

$$C_1 = C_{-1} . \tag{34c}$$

The symbol $N^{L'L}_{\lambda\lambda'}$ in Eq. (33) stands for

$$N^{L'L}_{\lambda\lambda'} = \sum_{\nu,\nu'}^{0\to J} N^{L'L}_{\nu\nu',\lambda\lambda'} = \sum_{\nu\nu'}^{0\to J} (-1)^{\nu'} \langle 11,\nu-1,-(\nu'-1)|L',\nu-\nu'\rangle\langle JJ\nu\nu'|L,\nu+\nu'\rangle A^{J}_{\nu} A^{J*}_{\nu'}$$
$$\times [D^{L'*}_{\nu-\nu',0} d^{L}_{\nu+\nu',\lambda+\lambda'} + (-1)^{L}(-1)^{\lambda+\lambda'} D^{L'}_{\nu-\nu',0} d^{L}_{\nu+\nu',-(\lambda+\lambda')}] . \tag{35}$$

Since

$$N^{L'L}_{\nu\nu',\lambda\lambda'} = (-1)^{L} N^{L'L*}_{\nu'\nu,\lambda\lambda'} , \tag{36}$$

we get, upon $\nu \leftrightarrow \nu'$ reordering,

$$N^{L'L}_{\lambda\lambda'} = \frac{1}{2} \sum_{\nu\nu'}^{0\to J} [N^{L'L}_{\nu\nu',\lambda\lambda'} + (-1)^{L} N^{L'L*}_{\nu'\nu,\lambda\lambda'}]$$
$$= \frac{1}{4} \sum_{\nu\nu'}^{0\to J} (-1)^{\nu'} \langle 11\nu-1,-(\nu'-1)|L',\nu-\nu'\rangle\langle JJ\nu\nu'|L,\nu+\nu'\rangle$$
$$\times \{ A^{J}_{\nu} A^{J*}_{\nu'} + A^{J*}_{\nu} A^{J}_{\nu'} [D^{L'*}_{\nu-\nu',0} + (-1)^{L} D^{L'}_{\nu-\nu',0}][d^{L}_{\nu+\nu',\lambda+\lambda'} + (-1)^{\lambda+\lambda'} d^{L}_{\nu+\nu',-(\lambda+\lambda')}]$$
$$+ (A^{J}_{\nu} A^{J*}_{\nu'} - A^{J*}_{\nu} A^{J}_{\nu'})[D^{L'*}_{\nu-\nu',0} - (-1)^{L} D^{L'}_{\nu-\nu',0}][d^{L}_{\nu+\nu',\lambda+\lambda'} - (-1)^{\lambda+\lambda'} d^{L}_{\nu+\nu',-(\lambda+\lambda')}]\} . \tag{37}$$

Finally, the symbol $\Lambda^L_{\lambda\lambda'}$ in Eq. (30) denotes

$$\Lambda^L_{\lambda\lambda'}=\sum_{\lambda_1\lambda_1'}^{\pm} B^J_{\lambda_1,\lambda_1-\lambda} B^{J*}_{\lambda_1',\lambda_1'-\lambda'}\rho_{1\lambda_1\lambda_1'} \times\rho_{2\lambda_1-\lambda,\lambda_1'-\lambda'}\langle JJ,\lambda\lambda'|L,\lambda+\lambda'\rangle$$
$$=B_{\lambda_J}B^{J*}_{\lambda'}R_{\lambda\lambda'}\langle JJ\lambda\lambda'|L,\lambda+\lambda'\rangle\ , \quad (38a)$$

where

$$B^J_{-1}=B^J_{-+}\ ,$$
$$B^J_0=B^J_{++}=(-1)^J B^J_{--}\ , \quad (38b)$$
$$B^J_1=B^J_{+-}\ ,$$

and

$$R_{\lambda,\lambda'}=\sum_{\lambda_1,\lambda_1'}^{\pm}\rho_{1\lambda_1\lambda_1'}\rho_{2\lambda_1-\lambda,\lambda_1'-\lambda'}\ . \quad (38c)$$

It should be noted that our present helicity amplitude B_0 differs from that of Ref. [3] by a factor of $\sqrt{2}$, and for $J=1$, B_0 is zero. Since $N^{L'L}_{\lambda\lambda'}=N^{L'L}_{\lambda'\lambda}$ and $\Lambda^L_{\lambda\lambda'}=(-1)^L\Lambda^{L*}_{\lambda'\lambda}$ we have, upon interchanging the indices λ and λ',

$$\sum_{\lambda,\lambda'}^{-1,0,1}\Lambda^L_{\lambda\lambda'}N^{L'L}_{\lambda\lambda'}=\frac{1}{2}\sum_{\lambda,\lambda'}^{-1,0,1}[\Lambda^L_{\lambda\lambda'}+(-1)^L\Lambda^{L*}_{\lambda\lambda'}]N^{L'L}_{\lambda\lambda'}\ . \quad (39)$$

Also since the density matrices are Hermitian, we can write

$$\Lambda^L_{\lambda\lambda'}=\tfrac{1}{4}\langle JJ\lambda\lambda'|L,\lambda+\lambda'\rangle \times\{(B^J_\lambda B^{J*}_{\lambda'}+B^{J*}_\lambda B^J_{\lambda'})[R_{\lambda\lambda'}+(-1)^L R^*_{\lambda\lambda'}] +(B^J_\lambda B^{J*}_{\lambda'}-B^{J*}_\lambda B^J_{\lambda'})[R_{\lambda\lambda'}-(-1)^L R^*_{\lambda\lambda'}]\}\ . \quad (40)$$

In order to calculate $N^{L'L}_{\lambda\lambda'}$, from Eq. (37) we note that

$$N^{L'L}_{\nu\nu',\lambda\lambda'}=N^{L'L}_{\nu'\nu,\lambda\lambda'} \quad (41)$$

and so

$$\sum_{\nu=0}^{J}\sum_{\nu'=0}^{J}N^{L'L}_{\nu\nu',\lambda\lambda'}=\sum_{\nu=0}^{J}\sum_{\nu'=0}^{\nu}(2-\delta\nu\nu')N^{L'L}_{\nu\nu',\lambda\lambda'}$$
$$=\sum_{\delta=0}^{J}\sum_{\sigma=\delta,\delta+2,\ldots}^{2J-\delta}(2-\delta_{\delta_0})N^{L'L}_{\nu\nu',\lambda\lambda'} \quad (42a)$$

where

$$\nu=\tfrac{1}{2}(\sigma+\delta)\ ,\quad \nu'=\tfrac{1}{2}(\sigma-\delta)\ . \quad (42b)$$

Substituting Eqs. (42) into Eq. (37) and then using Eqs. (38) and (39), we can evaluate the right-hand side of Eq. (33) which gives the angular distribution function in terms of the Wigner D^J functions. Finally we want to express the D^J functions in terms of spherical harmonics. Notice that in Eq. (33) all the angular dependence is in $N^{L'L}_{\lambda\lambda'}$ and the polarization dependence is in $\Lambda^L_{\lambda\lambda'}$. There are two kinds of D^J functions in the expression for $N^{L'L}_{\lambda\lambda'}$ given by Eqs. (35) and (37), namely, of the type $D^L_{M0}(\phi',\theta',-\phi')$ and $d^L_{mm'}(\theta)$. Now,

$$D^L_{M0}(\phi',\theta',-\phi')=\left[\frac{4\pi}{2L+1}\right]^{1/2}Y^*_{LM}(\theta;\phi')\ . \quad (43)$$

Expressing $d^L_{mm'}(\theta)$ in terms of spherical harmonic functions is more involved. All the $d^L_{mm'}(\theta)$ in Eq. (37) can be expanded in terms of $d^l_{00}(\theta)$ and $d^l_{10}(\theta)$, which are related to $Y_{l0}(\theta,0)$ and $Y_{l1}(\theta,0)$, respectively. In order to see how this is done, let us give an example. First of all, we note that,

$$d^1_{10}d^1_{11}=d^1_{10}d^1_{-1-1}\ . \quad (44)$$

In writing Eq. (44) we made use of the fact that

$$d^L_{m'm}(\theta)=(-1)^{m'-m}d^L_{-m',-m}(\theta)$$
$$=(-1)^{m'-m}d^L_{mm'}(\theta)\ . \quad (45)$$

Next we expand the left- and the right-hand sides of Eq. (44) by means of Eq. (32). We then get

$$\langle 11,11|22\rangle\langle 11,01|21\rangle d^2_{21}(\theta)$$
$$=\sum_{L=0}^{2}\langle 11,1-1|L0\rangle\langle 11,0-1|L,-1\rangle d^L_{10}(\theta)\ . \quad (46)$$

Equation (46) expresses $d^2_{21}(\theta)$ in terms of spherical harmonics because of Eq. (43). By using suitable combinations as in Eq. (44) and then using Eq. (32), we can express all $d^L_{mm'}(\theta)$ of Eq. (37) in terms of spherical harmonics. We can now express all the angular dependence in Eq. (33) in terms of spherical harmonics as

$$W(\theta;\theta',\phi')=N_J\sum_{L'=0,2}\sum_{M'=0}^{\min(J,L')}\sum_{L=0}^{2J}\sum_{M=0}^{\min(1,L)}C^J_{L'M',LM}[Y_{L'M'}(\theta',\phi')+(-1)^L Y^*_{L'M'}(\theta',\phi')]Y_{LM}(\theta,0)\ . \quad (47)$$

In Eq. (47) the spherical harmonics $Y_{L'M'}(\theta',\phi')$ and the Legendre polynomials and associated Legendre functions $Y_{LM}(\theta,0)$ are linearly independent. The normalization constant N_J which will make the integral of $W(\theta;\theta',\phi')$ over all angles equal to 1 is found to be

$$N_J=\frac{1}{4\pi} \quad (48)$$

for all J. From Eq. (47), using the orthonormality of the spherical harmonics, we can derive an expression for the coefficients $C^J_{L'M',LM}$ as an integral over the angular distribution function. It is given by

$$C^J_{L'M',LM}=\frac{8\pi^2}{1+(-1)^{L+M'}}\times\int_0^{2\pi}d\phi'\int_{-1}^{1}d(\cos\theta')\times\int_{-1}^{1}d(\cos\theta)W(\theta;\theta',\phi')\times Y^*_{L'M'}(\theta',\phi')Y^*_{LM}(\theta,0)\ . \tag{49}$$

If the angular distribution function $W(\theta;\theta',\phi')$ is known experimentally with sufficient completeness, the integral in Eq. (49) can be done numerically and the individual coefficients can be evaluated from the experimental data. It should be clear from Eq. (47) that the coefficients $C^J_{L'M',LM}$ when $M'=0$ and L is odd do not contribute to the sum in that equation. The coefficients of all the contributing terms in Eq. (47) are given in the Appendix.

IV. DISCUSSION OF THE RESULTS

Equations (47) and (28), together with the Appendix giving the expressions for the coefficients $C^J_{L'M',LM}$, are our final results for the angular distribution of the final electron and of the photon in the cascade process produced by arbitrarily polarized $p\bar{p}$ collisions. The results give the angular distribution function in terms of linearly independent spherical harmonics. The coefficients in the expansion $C^J_{L'M',LM}$ are functions of bilinear combinations of the production amplitudes B_0 and B_1 in $p\bar{p}\to\chi_J$ and of the decay amplitudes A^J_1 in $\chi_J\to\psi+\gamma$. They are also functions of the longitudinal ($P_{z'}$) and the transverse ($P_{x'},P_{y'}$) components of the polarization vectors of p and $\bar{p}$. In the case where neither particle is polarized, this result agrees with that of Ref. [3]. An examination of the coefficients in the Appendix yields the following results for the different J values.

$J=0$

If neither particle is polarized then $P_-=1$, $P_A=0$, and $C_{00,00}=1$, $C_{20,00}=C/2\sqrt{5}$. So a measurement of $C_{20,00}$ will determine C. In the standard model, where $c\bar{c}\to\gamma\to e^+e^-$, C is very close to 1. Any significant deviation from 1 will challenge the dominance of this decay mechanism.

For the $J=0$ case only, polarization of p and $\bar{p}$ yields nothing new. However, when the process goes through this channel with polarized p and $\bar{p}$, the results could serve as a check on the values of P_- and P_A.

$J=1$

When neither particle is polarized, $P_+=1$ and $P_B=P_C=0$. So, $C_{00,00}=1$, $C_{20,20}=-C/10$, and a measurement of $C_{20,20}$ will yield C. Both $C_{00,20}$ and $C_{20,00}$ depend upon $|A_0|^2$ and $|A_1|^2$ and, since the normalization is $|A_0|^2+|A_1|^2=1$, we have more than the minimum number of measurables needed to determine the magnitude of the amplitudes. Finally, $\mathrm{Re}(A_1A_0^*)$ can be found from a measurement of $C_{21,21}$.

If a single particle is polarized and it has a nonzero longitudinal component of polarization, then a measurement of $C_{21,11}$ will determine $\mathrm{Im}(A_0A_1^*)$. The term involving the coefficient $C_{21,11}$ is easy to identify in the angular distribution since it is proportional to $\sin\phi'$. So the angular distribution for the $J=1$ case helps us to measure C, $|A_0|$, $|A_1|$, and the relative phase between A_0 and A_1.

When both p and $\bar{p}$ are polarized, nonzero values for P_B and P_C and $P_+\neq1$ are possible. So although there are no remaining quantities to be determined, these measurements will allow a more accurate determination of the amplitudes.

$J=2$

When neither particle is polarized both P_+ and P_- are 1. Then, $C_{00,00}=1$ and $C_{00,20}$, $C_{00,40}$, $C_{20,00}$, $C_{20,20}$, $C_{20,40}$, $C_{21,21}$, $C_{21,41}$, $C_{22,00}$, $C_{22,20}$, and $C_{22,40}$ are the only nonzero coefficients. The five measurable coefficients $C_{00,20}$, $C_{00,40}$, $C_{20,00}$, $C_{20,20}$, and $C_{20,40}$ depend upon C, $|B_0|^2$, $|B_1|^2$, $|A_0|^2$, $|A_1|^2$, and $|A_2|^2$. Because of the normalization conditions, $|B_0|^2+|B_1|^2=1$ and $|A_0|^2+|A_1|^2+|A_2|^2=1$, we have five measurable coefficients to determine four independent quantities. Furthermore, $C_{21,21}$ and $C_{21,41}$ each depend on $\mathrm{Re}(A_1A_0^*)$ and $\mathrm{Re}(A_2A_1^*)$ and $C_{22,00}$, $C_{22,20}$, and $C_{22,40}$ are each proportional to $\mathrm{Re}(A_2A_0^*)$. So, with unpolarized p and $\bar{p}$ beams we can determine the magnitudes of the amplitudes, $\mathrm{Re}(A_1A_0^*)$, $\mathrm{Re}(A_2A_1^*)$, and $\mathrm{Re}(A_2A_0^*)$. The relative phase of B_0 and B_1, $\mathrm{Im}(A_1A_0^*)$, $\mathrm{Im}(A_2A_1^*)$, and $\mathrm{Im}(A_2A_0^*)$ are undetermined.

If a single particle (say particle 2 or p) is polarized, a nearly complete determination is possible. When $P_{2x'}$ is nonzero then $C_{21,11}$ and $C_{21,31}$ each depend on $\mathrm{Im}(A_1A_0^*)$ and $\mathrm{Im}(A_2A_1^*)$ and $C_{22,10}$ and $C_{22,30}$ are each proportional to $\mathrm{Im}(A_2A_0^*)$. Hence the relative phases of A_0, A_1, and A_2 can be determined.

When $(P^2_{2x'}+P^2_{2y'})^{1/2}$ is nonzero, $C_{00,21}$, $C_{00,41}$, $C_{20,21}$, $C_{20,41}$, $C_{21,00}$, $C_{21,20}$, $C_{21,40}$, $C_{22,21}$, $C_{22,41}$, $C_{21,10}$, $C_{21,30}$, $C_{22,11}$, and $C_{22,31}$ are nonzero. Each has terms only of the form $\mathrm{Re}(A_iA_j^*)\mathrm{Im}(B_0B_1^*)$ or $\mathrm{Im}(A_iA_j^*)\mathrm{Re}(B_0B_1^*)$. So $\mathrm{Re}(B_0B_1^*)$ can be found only when $\mathrm{Im}(A_iA_j^*)$ is nonzero. We should also mention that, although it is possible to produce a transverse polarization, it is not possible to control how it is mixed between $P_{2x'}$ and $P_{2y'}$ since only the direction of z is determined by the decay.

If both particles are polarized, nonzero values of P_A, P_B, and P_C and nonunity values of P_+ and P_- serve only to increase the accuracy with which the quantities described above can be determined. However, when P_D or P_E is nonzero, $C_{00,21}$, $C_{00,41}$, $C_{20,21}$, $C_{20,31}$, $C_{21,00}$, $C_{21,20}$, $C_{21,40}$, $C_{22,21}$, $C_{22,41}$, $C_{21,10}$, $C_{21,30}$, $C_{22,11}$, and $C_{22,31}$ contain terms of the form $\mathrm{Re}(A_iA_j^*)\mathrm{Re}(B_0B_1^*)$ and $\mathrm{Im}(A_iA_j^*)\mathrm{Im}(B_0B_1^*)$. So $\mathrm{Re}(B_0B_1^*)$ can be determined even when the A's are real.

V. CONCLUDING REMARKS

We have derived a model-independent expression for the angular distribution of the final electron and the photon in the cascade process $\bar{p}p \rightarrow \chi_J \rightarrow \psi + \gamma \rightarrow e^+e^- + \gamma$ for arbitrarily polarized $\bar{p}$ and p in terms of the angular momentum helicity amplitudes of the individual processes. The derivation is based only on the general principles of quantum mechanics and invariance principles. Equation (47) and Table I can be used to determine the angular momentum helicity amplitudes from the measured angular distribution. By studying the angular distribution for polarized $\bar{p}p$ collisions, we can determine not only the magnitudes of the amplitudes, but also the relative phases among them. The terms involving $\mathrm{Im}(A_i A_j^*)$ and the relative phase of the B's occur in the expression for the angular distribution only when at least one of the beams is polarized. In order to determine the relative phase between any two amplitudes H_i and H_j unambiguously, we need to measure $|H_i|^2$ and $|H_j|^2$, as well as the real and imaginary parts of $H_i H_j^*$. It is not possible to determine the relative phase between the production amplitudes B_0 and B_1 in the process $\bar{p}p \rightarrow \chi_2$ unless $\bar{p}$ or p is polarized.

It is also possible to test the charge-conjugation invariance in the production process $\bar{p}p \rightarrow \chi_1$ from a measurement of the angular distribution of the electron and the photon. From C invariance [Eq. (12)], it follows that the amplitude $B_0 = B_{++} = -B_{--}$ is zero when $J=1$. If it were nonzero, the coefficients $C_{L'M',LM}$ would not be zero when $M' \neq M$. The expressions for these coefficients with $M' \neq M$ involve transverse components of polarization for at least one particle. So if the transverse polarization of at least one particle is not zero and if we see terms involving $C_{L'M',LM}$ when $M' \neq M$ in the angular distribution for the $J=1$ case, C invariance is violated in the process $\bar{p}p \rightarrow \chi_1$.

Finally we should point out the relationship between the angular momentum helicity amplitudes A_i $(i=0 \rightarrow J)$ in the process $\chi_J \rightarrow \psi + \gamma$ and the multipole amplitudes a_k $(k=1 \rightarrow J+1)$. It is given by [9]

$$A_\nu = \sum_{k=1}^{J+1} a_k \left[\frac{2k+1}{2J+1} \right]^{1/2} \langle k1;1,\nu-1|J\nu\rangle , \qquad (50)$$

where a_1 is the $E1$ amplitude, a_2 is the $M2$ amplitude, and a_3 is the $E3$ amplitude. The coefficients of transformation in Eq. (50) form a real orthogonal matrix and so

$$\sum_{k=1}^{J+1} |a_k|^2 = \sum_{\nu=0}^{J} |A_\nu|^2 = 1 . \qquad (51)$$

Using Eq. (50) we can calculate the multipole amplitudes once the angular momentum helicity amplitudes are determined.

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APPENDIX

Expressions for the nonvanishing coefficients $C^J_{L'M',LM}$ in terms of angular momentum helicity amplitudes and components of polarization vectors of $\bar{p}$ and p are given.

The amplitudes A^J_ν in the process $\chi_J \rightarrow \psi + \gamma$ $(J=0,1,2)$ are defined by Eqs. (14) and (31). They also satisfy the normalization condition

$$\sum_{\nu=0}^{J} |A^J_\nu|^2 = 1 .$$

The amplitudes B_0 and B_1 in the process $\bar{p}p \rightarrow \chi_J$ are defined as

$$B^J_0 = B^J_{++} = (-1)^J B^J_{--} ,$$

$$B^J_1 = B^J_{+-} = (-1)^J B^J_{-+} .$$

They satisfy the normalization condition

$$\sum_{\lambda_1\lambda_2} |B^J_{\lambda_1\lambda_2}|^2 = 1$$

or

$$2[|B^J_0|^2 + |B^J_1|^2] = 1 .$$

The amplitudes $C_{\kappa_1\kappa_2}$ in the process $\psi \rightarrow e^+e^-$ satisfy the normalization condition

$$|C_{+-}|^2 + |C_{++}|^2 = 1 ,$$

$$C = \frac{|C_{+-}|^2 - 2|C_{++}|^2}{|C_{+-}|^2 + |C_{++}|^2} = 1 - 3|C_{++}|^2 .$$

We also use the following abbreviations in our expressions:

$$P_\pm = 1 \pm P_{1z'} P_{2z'} ,$$

$$P_A = P_{1x'} P_{2x'} + P_{1y'} P_{2y'} ,$$

$$P_B = P_{1x'} P_{2x'} - P_{1y'} P_{2y'} ,$$

$$P_C = P_{1x'} P_{2y'} + P_{1y'} P_{2x'} ,$$

$$P_D = P_{1y'} P_{2z'} + P_{1z'} P_{2y'} ,$$

$$P_E = P_{1z'} P_{2x'} + P_{1x'} P_{2z'} .$$

1. $J=0$

Here only A_0 and B_0 are nonvanishing. So normalization gives $|A_0|^2 = 1$ and $|B_0|^2 = \frac{1}{2}$:

$$C_{00,00} = \tfrac{1}{2}(P_- + P_A)$$

$$C_{20,00} = \frac{1}{4\sqrt{5}} C(P_- + P_A) = \frac{C}{2\sqrt{5}} C_{00,00}$$

2. $J=1$

In this case, $C_{L'M',LM}=0$ if $M'\neq M$ since $B_{++}=0$:

$$C_{00,00}=|A_0|^2|B_1|^2(P_+ +P_B)+|A_1|^2|B_1|^2(P_+ -\tfrac{1}{2}P_B)\ ,$$

$$C_{00,20}=\frac{1}{2\sqrt{5}}[-2|A_0|^2+A_1|^2]|B_1|^2(P_+ +P_B)\ ,$$

$$C_{20,00}=\frac{C}{2\sqrt{5}}|B_1|^2[|A_0|^2(P_+ +P_B)-2|A_1|^2(P_+ -\tfrac{1}{2}P_B)]\ ,$$

$$C_{21,11}=i\frac{3C}{2\sqrt{5}}|B_1|^2[\mathrm{Re}(A_1A_0^*)P_C-\mathrm{Im}(A_1A_0^*)(P_{1z'}+P_{2z'})]\ ,$$

$$C_{20,20}=-\frac{C}{10}|B_1|^2(|A_0|^2+|A_1|^2)(P_+ +P_B)\ ,$$

$$C_{21,21}=-\frac{3C}{10}|B_1|^2\mathrm{Re}(A_1A_0^*)(P_+ +P_B)\ .$$

3. $J=2$

$$C_{00,00}=|B_0|^2(|A_0|^2+|A_1|^2+|A_2|^2)(P_- +P_A)+|B_1|^2[|A_0|^2(P_+ -P_B)+|A_1|^2(P_+ -\tfrac{1}{6}P_B)+|A_2|^2(P_+ +\tfrac{2}{3}P_B)]\ ,$$

$$C_{00,20}=\frac{\sqrt{5}}{7}\{|B_0|^2(2|A_0|^2+|A_1|^2-2|A_2|^2)(P_- +P_A)$$
$$\times|B_0|^2[|A_0|^2(P_+ -P_B)+|A_1|^2(\tfrac{1}{2}P_+ +\tfrac{11}{6}P_B)-|A_2|^2(P_+ +\tfrac{4}{3}P_B)]\}\ ,$$

$$C_{00,21}=\frac{\sqrt{5}}{7}[2|A_0|^2+|A_1|^2-2|A_2|^2][-P_E\mathrm{Re}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]\ ,$$

$$C_{00,40}=\frac{6}{7}[|A_0|^2-\tfrac{2}{3}|A_1|^2+\tfrac{1}{6}|A_2|^2][|B_0|^2(P_- +P_A)-\tfrac{2}{3}|B_1|^2(P_+ -P_B)]\ ,$$

$$C_{00,41}=\frac{6}{7}\sqrt{\tfrac{10}{3}}(|A_0|^2-\tfrac{2}{3}|A_1|^2+\tfrac{1}{6}|A_2|^2)[-P_E\mathrm{Re}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]\ ,$$

$$C_{20,00}=\frac{C}{2\sqrt{5}}\{(|A_0|^2-2|A_1|^2+|A_2|^2)|B_0|^2(P_- +P_A)+|B_1|^2$$
$$\times[|A_0|^2(P_+ -P_B)-2|A_1|^2(P_+ -\tfrac{1}{6}P_B)+|A_2|^2(P_+ +\tfrac{2}{3}P_B)]\}\ ,$$

$$C_{20,20}=\frac{C}{7}\{|B_0|^2(|A_0|^2-|A_1|^2-|A_2|^2)(P_- +P_A)$$
$$+|B_1|^2[\tfrac{1}{2}|A_0|^2(P_+ -P_B)-|A_1|^2(\tfrac{1}{2}P_+ +\tfrac{11}{6}P_B)-|A_2|^2\tfrac{1}{2}(P_+ +\tfrac{4}{3}P_B)]\}\ ,$$

$$C_{20,21}=\frac{C}{7}[|A_0|^2-|A_1|^2-|A_2|^2][-P_E\mathrm{Re}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]\ ,$$

$$C_{20,40}=\frac{3C}{7\sqrt{5}}[|A_0|^2+\tfrac{4}{3}|A_1|^2+\tfrac{1}{6}|A_2|^2][|B_0|^2(P_- +P_A)-\tfrac{2}{3}|B_1|^2(P_+ -P_B)]\ ,$$

$$C_{20,41}=\frac{\sqrt{6}C}{7}[|A_0|^2+\tfrac{4}{3}|A_1|^2+\tfrac{1}{6}|A_2|^2][-P_E\mathrm{Re}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]\ ,$$

$$C_{21,00}=\frac{C}{2\sqrt{10}}\{[-2\sqrt{\tfrac{2}{3}}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)][P_E\mathrm{Re}(B_0B_1^*)-(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]$$
$$+5[2\sqrt{\tfrac{2}{3}}\mathrm{Im}(A_1A_0^*)-\mathrm{Im}(A_2A_1^*)][P_E\mathrm{Im}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{21,10}=i\frac{C}{2}\sqrt{\tfrac{3}{10}}\{[-2\sqrt{\tfrac{2}{3}}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)][P_D\mathrm{Re}(B_0B_1^*)+(P_{1x'}+P_{2x'})\mathrm{Im}(B_0B_1^*)]$$
$$+[2\sqrt{\tfrac{2}{3}}\mathrm{Im}(A_1A_0^*)-3\,\mathrm{Im}(A_2A_1^*)][-P_D\mathrm{Im}(B_0B_1^*)+(P_{1x'}+P_{2x'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{21,11}=i\frac{C}{2\sqrt{5}}\{[-\sqrt{3}\mathrm{Re}(A_1A_0^*)+2\sqrt{2}\mathrm{Re}(A_2A_1^*)]|B_0|^2P_C-[\sqrt{3}\mathrm{Im}(A_1A_0^*)-\sqrt{2}\mathrm{Im}(A_2A_1^*)]|B_0|^2(P_{1z'}+P_{2z'})\}\ ,$$

$$C_{21,20}=\frac{\sqrt{2}C}{4}\{[\tfrac{5}{7}\sqrt{\tfrac{2}{3}}\mathrm{Re}(A_1A_0^*)+\tfrac{11}{7}\mathrm{Re}(A_2A_1^*)][-P_E\mathrm{Re}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]$$
$$+[\sqrt{\tfrac{2}{3}}\mathrm{Im}(A_1A_0^*)+\mathrm{Im}(A_2A_1^*)][P_E\mathrm{Im}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{21,21}=\frac{\sqrt{3}C}{7}\{[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\mathrm{Re}(A_2A_1^*)]|B_0|^2(P_-+P_A)$$
$$+|B_1|^2[\mathrm{Re}(A_1A_0^*)\tfrac{1}{2}(P_+-P_B)-\sqrt{6}\mathrm{Re}(A_2A_1^*)\tfrac{1}{2}(P_++\tfrac{4}{3}P_B)]\}\ ,$$

$$C_{21,30}=i\left[\frac{3C}{2\sqrt{20}}\right]\{[\sqrt{6}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)][-P_D\mathrm{Re}(B_0B_1^*)-(P_{1x'}+P_{2x'})\mathrm{Im}(B_0B_1^*)]$$
$$+[\sqrt{6}\mathrm{Im}(A_1A_0^*)-3\,\mathrm{Im}(A_2A_1^*)][-P_D\mathrm{Im}(B_0B_1^*)+(P_{1x'}+P_{2x'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{21,31}=-iC\sqrt{\tfrac{1}{35}}\{[\sqrt{6}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)]|B_1|^2P_C+[\sqrt{6}\mathrm{Im}(A_1A_0^*)+3\,\mathrm{Im}(A_2A_1^*)]|B_1|^2(P_{iz'}+P_{2z'})\}\ ,$$

$$C_{21,40}=\frac{8C}{7\sqrt{10}}\{[\sqrt{6}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)][P_E\mathrm{Re}(B_0B_1^*)-(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]\}\ ,$$

$$C_{21,41}=\frac{\sqrt{3C}}{7}\{[\sqrt{6}\mathrm{Re}(a_1A_0^*)+\mathrm{Re}(A_2A_1^*)][|B_0|^2(P_-+P_A)-\tfrac{2}{3}|B_1|^2(P_+-P_B)]\}\ ,$$

$$C_{22,00}=-\frac{C}{\sqrt{5}}\mathrm{Re}(A_2A_0^*)[|B_0|^2(P_-+P_A)+|B_1|^2(P_+-P_B)]\ ,$$

$$C_{22,10}=-iC\sqrt{\tfrac{3}{5}}[\mathrm{Re}(A_2A_0^*)|B_1|^2P_C+\mathrm{Im}(A_2A_0^*)|B_1|^2(P_{iz'}+P_{2z'})]\ ,$$

$$C_{22,11}=i\frac{C}{\sqrt{5}}\{\mathrm{Re}(A_2A_0^*)[P_D\mathrm{Re}(B_0B_1^*)+(P_{1x'}+P_{2x'})\mathrm{Im}(B_0B_1^*)]$$
$$+\mathrm{Im}(A_2A_0^*)[P_D\mathrm{Im}(B_0B_1^*)-(P_{1x'}+P_{2x'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{22,20}=\frac{5C}{7}\mathrm{Re}(A_2A_0^*)[|B_0|^2(P_-+P_A)-\tfrac{1}{5}|B_1|^2(P_+-P_B)]\ ,$$

$$C_{22,21}=C\{\tfrac{5}{7}\mathrm{Re}(A_2A_0^*)[-P_E\mathrm{Re}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]$$
$$+\mathrm{Im}(A_2A_0^*)[P_E\mathrm{Im}(B_0B_1^*)+(P_{1y'}+P_{2y'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{22,30}=i\frac{3C}{\sqrt{35}}[\mathrm{Re}(A_2A_0^*)|B_1|^2P_C+\mathrm{Im}(A_2A_0^*)|B_1|^2(P_{1z'}+P_{2z'})]\ ,$$

$$C_{22,31}=-i3C\sqrt{\tfrac{2}{35}}\{\mathrm{Re}(A_2A_0^*)[P_D\mathrm{Re}(B_0B_1^*)+(P_{1x'}+P_{2x'})\mathrm{Im}(B_0B_1^*)]$$
$$+\mathrm{Im}(A_2A_0^*)[P_D\mathrm{Im}(B_0B_1^*)-(P_{1x'}+P_{2x'})\mathrm{Re}(B_0B_1^*)]\}\ ,$$

$$C_{22,40}=-\frac{6C}{7\sqrt{5}}\mathrm{Re}(A_2A_0^*)[|B_0|^2(P_-+P_A)-\tfrac{2}{3}|B_1|^2(P_+-P_B)]\ ,$$

$$C_{22,41}=\frac{2\sqrt{6}}{7}C\,\mathrm{Re}(A_2A_0^*)[P_E\mathrm{Re}(B_0B_1^*)-(P_{1y'}+P_{2y'})\mathrm{Im}(B_0B_1^*)]\ .$$

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