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A possible way of connecting the Grassmann variables and the number of generations

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We construct a left-right symmetric model in which the number of generations is related to Grassmann variables. We introduce two sets of complex Grassmann variables (θ_q^1, θ_q^2), (θ_l^1, θ_l^2) and associate each variable with left- and right-handed quark and lepton fields, respectively. Expanding quark and lepton fields in powers of the Grassmann variables, we find that there are exactly three generations of quarks and leptons. Integrating out the Grassmann variables, we obtain phenomenologically acceptable fermion mass matrices.

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How many generations of quarks and leptons are there in nature is one of the outstanding problems of particle physics today. Considerations from nuclear synthesis [1] and experimental data on the Z decay width from the CERN e^+e^- collider LEP [2] both indicate that there are only three generations of light neutrinos, but they do not provide information about the number of heavy generations. On the theoretical side the situation is not any better. The standard model does not have the answer to the problem. To answer this question one needs to go beyond the standard model. Much theoretical effort has been made, ranging from the topological properties of compactified spacetimes in string theory to considerations from anomaly cancellation and composite models to determine the number of generations [3]. But the problem is still far from being solved. In this Brief Report we will study an interesting approach to the generation problem which relates the particle spectrum with Grassmann variables (GV's) [4,5]. It is well known that Taylor expansions in the GV's will have finite terms. This terminating nature of the GV's suggests a very interesting way to classify the particle spectrum when the connection between particle fields and the GV's is made. Extensive work has been done with models based on the SU(5) grand unification group [5]. In Ref. [5] the particle spectra, including the gauge transformation properties, are all specified by the GV's. In the following we will study another way of relating the GV's and the number of generations, and to construct a low energy model.

The gauge group of our model is the $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_{B-L}$ left-right symmetric group. Under this group the quarks q and leptons l tranform as

$$q_L:(3,2,1,1/3)\ ,\quad q_R:(3,1,2,1/3)\ ,$$
$$l_L:(1,2,1,-1)\ ,\quad l_R:(1,1,2,-1)\ . \tag{1}$$

To make connections between the GV's and the number of generations we introduce two sets of GV's: $\theta_q=(\theta_q^1,\theta_q^2)$ and $\theta_l=(\theta_l^1,\theta_l^2)$ which transform under a global group $G=\mathrm{SU}(2)_q\times\mathrm{SU}(2)_l\times\mathrm{U}(1)_f$ as (2,1,α) and (1,2,α), respectively. We also group $q_{L,R}$ and $l_{L,R}$ into $Q=(q_L,q_R^c)$ and $L=(l_L,l_R^c)$, and let them transform as $(2,1,0)$ and $(1,2,0)$, respectively. This way of grouping the quarks and leptons suggests that the global symmetry is in some way related to the helicity of fermions. The fermion and boson fields are component fields of some super bosonic fields E_i expanded in powers of the GV's. There are two classes of expansions of the superfields. One has even powers of the GV's and the other has odd powers. Since the overall superfields E_i are bosonic, it is clear that the component fields with even powers of the GV's in the expansion are boson fields and the ones with odd powers are fermion fields. This expansion does not constrain the gauge transformation properties of the component fields. The superfields can have nontrivial transformation properties under the gauge group.

The Lagrangian density L in ordinary space-time is obtained by first using the available superfields E_i to form terms $L(E_i)$ which are singlets under both the gauge and global symmetries, and then integrating out the GV's; that is,

$$L=\int d^2\theta_q d^2\theta_l d^2\bar\theta_q d^2\bar\theta_l L(E_i)\ . \tag{2}$$

This procedure will select certain terms in $L(E_i)$ because only the terms with proper θ powers will survive the integral.

Let us consider the fermion fields first. The terms with the lowest power in the GV's for fermion fields are

$$E_{1Q}=\bar\theta_q Q_1\ ,\quad E_{1L}=\bar\theta_l L_1\ . \tag{3}$$

Multiplying E_{1i} by $(\bar\theta_q\theta_q)^a(\bar\theta_l\theta_l)^b$, we generate all allowed fermion fields with the same quantum numbers under the global symmetry G in this theory. We have

$$\begin{gathered}E_{1Q}=\bar\theta_q Q_1\ ,\quad E_{2Q}=(\bar\theta_q\theta_q)\bar\theta_q Q_2\ ,\\ E_{3Q}=(\bar\theta_l\theta_l)\bar\theta_q Q_3\ ,\quad E_{4Q}=(\bar\theta_l\theta_l)^2\bar\theta_q Q_4\ ,\\ E_{5Q}=(\bar\theta_q\theta_q)(\bar\theta_l\theta_l)\bar\theta_q Q_5\ ,\quad E_{6Q}=(\bar\theta_q\theta_q)(\bar\theta_l\theta_l)^2\bar\theta_q Q_6\ ,\\ E_{1L}=\bar\theta_l L_1\ ,\quad E_{2L}=(\bar\theta_l\theta_l)\bar\theta_l L_2\ ,\\ E_{3L}=(\bar\theta_q\theta_q)\bar\theta_l L_3\ ,\quad E_{4L}=(\bar\theta_q\theta_q)^2\bar\theta_l L_4\ ,\\ E_{5L}=(\bar\theta_q\theta_q)(\bar\theta_l\theta_l)\bar\theta_l L_5\ ,\quad E_{6L}=(\bar\theta_l\theta_l)(\bar\theta_q\theta_q)^2\bar\theta_l L_6\ .\end{gathered} \tag{4}$$

This set of superfields carries $-\alpha$ of the $\mathrm{U}(1)_f$ charge. One can generate other fields with different expansions which will have different global symmetry transformation properties.

Naively, Eq. (4) contains six generations of quarks

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and leptons. This is, however, not true. Some of the fields are actually the same. To see this let us define the dual fields $\tilde{E}_i$. The dual of a superfield is defined by replacing the powers of the GV's $(\bar{\theta}_q)^a(\theta_q)^b(\bar{\theta}_l)^c(\theta_l)^d$ with $(\bar{\theta}_q)^{2-b}(\theta_q)^{2-a}(\bar{\theta}_l)^{2-d}(\theta_l)^{2-c}$ [5]. We notice that the correct kinetic term in the Lagrangian L_k for the fermion fields can be generated by

$$L_k = \int d^2\bar{\theta}_q d^2\bar{\theta}_l d^2\theta_q d^2\theta_l \bar{E}_i \gamma_\mu D^\mu \tilde{E}_i \ , \qquad (5)$$

if the component field of E_i is the same as the component field of its dual. We therefore require that the component field be the same as the component field of its dual. It is easy to see that the following fields are dual pairs:

$$\begin{aligned} &E_{1Q} \leftrightarrow E_{6Q} \ , \quad E_{2Q} \leftrightarrow E_{4Q} \ , \quad E_{3Q} \leftrightarrow E_{5Q} \ . \\ &E_{1L} \leftrightarrow E_{6L} \ , \quad E_{2L} \leftrightarrow E_{4L} \ , \quad E_{3L} \leftrightarrow E_{5L} \ . \end{aligned} \qquad (6)$$

We therefore have only three generations of quarks and leptons.

We now turn to possible Higgs scalars H_i which may generate fermion masses through Yukawa terms. The Yukawa terms will have the form $E_iE_jH_k$. It is clear that in order to couple the Higgs scalars H_i to fermions, H_i should carry 2 times the $\mathrm{U}(1)_f$ charge as E_i but with opposite signs. We have

$$\begin{aligned} &H_1 = \theta_q\theta_q h_1 \ , \quad H_2 = \bar{\theta}_l\theta_l\theta_q\theta_q h_2 \ , \quad H_3 = (\bar{\theta}_l\theta_l)^2\theta_q\theta_q h_3 \ , \\ &H_4 = \theta_l\theta_l h_4 \ , \quad H_5 = \bar{\theta}_q\theta_q\theta_l\theta_l h_5 \ , \quad H_6 = (\bar{\theta}_q\theta_q)^2\theta_l\theta_l h_6 \ . \end{aligned} \qquad (7)$$

From our previous definition for duals, we find H_2 and H_5 are self-dual, and H_3, H_6 are the duals of H_1, H_4, respectively. h_i are singlets under the global symmetry G. The gauge transformation properties are not specified. In order to form gauge singlets with fermions to generate masses, we assign h_i to transform as $(1,2,2,0)$ under the gauge group. Notice that since h_i are singlets under the global symmetry G, we can also expand H_i by replacing h_i with $\tilde{h}_i = \tau_2 h_i^* \tau_2$ without changing the overall transformation properties of H_i under the gauge and the global symmetries. Therefore $\tilde{h}_i$ should also be included in Eq. (7). With the fields in Eq. (7), we find that only the following Yukawa terms will survive the GV integration:

$$\begin{aligned} &E_{1Q}E_{1Q}\tilde{H}_1 \ , \quad E_{3Q}E_{3Q}H_1 \ , \quad E_{1Q}\tilde{E}_{2Q}H_1 \ , \quad E_{1Q}E_{3Q}H_2 \ , \\ &E_{1L}E_{1L}\tilde{H}_4 \ , \quad E_{3L}E_{3L}H_4 \ , \quad E_{1L}\tilde{E}_{2L}H_4 \ , \quad E_{1L}E_{3L}H_5 \ . \end{aligned} \qquad (8)$$

Because both h_i and $\tilde{h}_i$ are available to form gauge singlets with fermions, each term in Eq. (8) contains two terms. For quarks we have

$$E_{iQ}E_{jQ}H_k \Rightarrow \lambda_{ij}\bar{q}_{iL}h_k q_{jR} + \lambda'_{ij}\bar{q}_{iL}\tilde{h}_k q_{jR} \ , \qquad (9)$$

where the λ's are constants and similarly for leptons. When h_i develop vacuum expectation values, the quarks and leptons obtain their masses. If we now identify

$$\begin{aligned} &q_1 = (c,s) \ , \quad q_2 = (u,d) \ , \quad q_3 = (t,b) \ , \\ &l_1 = (\nu_\mu,\mu), \quad l_2 = (\nu_e,e) \ , \quad l_3 = (\nu_\tau,\tau) \ , \end{aligned} \qquad (10)$$

we obtain the following form for the fermion mass matrices:

$$M_f = \begin{pmatrix} 0 & a & 0 \\ a^* & b & c \\ 0 & c^* & d \end{pmatrix} \ . \qquad (11)$$

One can have different ways to identify the first, the second, and the third generations as suggested in Eq. (10). However, for quarks the identification in Eq. (10) is the only one which is consistent with experimental data. For leptons there is more flexibility. In this model neutrinos only have Dirac masses.

In the above discussions only certain superfields have been studied. The complete expansion of the superfields in powers of the GV's will generate more particles. There will be several different classes of component fields with different quantum numbers under the global symmetry G. We have no control of the gauge transformation properties for each class of the fields. However, for a given class of the superfields we can follow the same procedure discussed before to determine the number of independent component fields. In this sense the number of generation or particle spectra is connected with the GV's.

In conclusion we have suggested a possible way to connect the GV's and the number of generation. We constructed a model with three generations of quarks and leptons and phenomenologically acceptable fermion mass matrices. We must say that more studies are needed to understand the relation betwteen the GV's and the number of generation if they are really connected. The possibility suggested in this paper is only one example.

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