Variational calculation of the phase shifts in the $\lambda\phi^4$ model

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In this paper, we develop Schiff's discretization procedure for calculating the phase shifts in the $\lambda\phi^4$ model in $(D+1)$-dimensional space-time $(D>0)$ with the Gaussian wave-functional approach. In $1+1$ and $2+1$ dimensions, the phase shifts are negative, which indicates the interaction between two pions is repulsive. In $3+1$ dimensions, the phase shifts vanish. We also discuss the dependences of the phase shifts upon the scattering energy in detail.

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The scattering process serves as a major source of information about the properties of various elementary particles, while the phase shift is a crucial quantity characterizing this process. Traditionally, the scattering problem is coped with mainly by the covariant perturbation technique. Recently, however, the development of the Gaussian wave-functional (GWF) approach has hinted that it is possible to analytically calculate the phase shift in quantum field theory. In the last decade, the Gaussian wave-functional approach in the Schrödinger picture field theory which was gradually developed by Schiff [1], Rosen [2], Barnes and Ghandour [3], etc., has become a powerful tool for investigating the vacuum structure [4–10] as well as searching for the bound states in quantum field theory [5,11,12]. Being explicitly computed, the two-particle state energy may be analyzed to extract knowledge on the phase shift. As an attempt, we have investigated the scattering state in the sinh-Gordon and the sine-Gordon models with this approach [13,14]. In this Brief Report, we continue to study the simplest self-coupling model, the $\lambda\phi^4$ model, with the same approach.

The $\lambda\phi^4$ model was originally introduced into quantum field theory to describe the pion-pion interaction [1,15]. By now it has become an important model in quantum field theory (including gauge theory), finite-temperature field theory, quantum cosmology, and condensed-matter physics. This model is also an ideal theoretical laboratory for various new methods. Since being advocated [16], the Gaussian wave-functional approach has been used for many problems in $\lambda\phi^4$ theory, such as spontaneous symmetry breaking, triviality, vacuum stability, the bound-state problem, and so on [4,17–21]. But the scattering problem of this model is an exception. Although the phase shift in $3+1$ dimensions was calculated in Schiff's pioneering work [1], the treatment of the divergence makes for a very lame conclusion [4]. To our understanding, the scattering problem in lower dimensions is also interesting and important. The purpose of this report is to calculate the scattering phase shifts of the two-particle system in this model for different dimensions and discuss the lower-dimensional results in detail.

Consider a scalar field $\phi(x)=\phi_x$ [$x=(x_1,x_2,\ldots,x_D)$ is the coordinate in D-dimensional space] described by the Hamiltonian

$$H=\int_x \mathcal{H}_x=\int_x \{\tfrac{1}{2}\Pi_x^2+\tfrac{1}{2}(\nabla\phi_x)^2+\tfrac{1}{2}m^2\phi_x^2+\lambda\phi_x^4\}\ , \tag{1}$$

where $\int_x=\int dx=\int dx_1 dx_2\cdots dx_D$ and ∇ is the gradient operator in D-dimensional space. m and λ are the bare mass and coupling parameters, respectively.

From [3,4], the Gaussian vacuum state is

$$|\varphi_0\rangle=N_f\exp\left[-\frac{1}{2}\int_{y,z}(\phi_y-\varphi_0)f_{yz}(\phi_z-\varphi_0)\right]\ , \tag{2}$$

where $\int_{y,z}=\int dy\,dz=\int dy_1dy_2\cdots dy_Ddz_1dz_2\cdots dz_D$, N_f is some normalization constant, φ_0 is a classical constant field, and

$$f_{yz}=\int\frac{dp}{(2\pi)^D}f(p)e^{ip(y-z)} \tag{3}$$

with the one-particle variational energy [3,4]

$$f(p)=\sqrt{p^2+\mu^2(\varphi_0)} \tag{4}$$

and $p=(p_1,p_2,\ldots,p_D)$ a vector in D-dimensional p-space. In Eq. (4),

$$\mu^2(\varphi_0)=m^2+6\lambda f_{xx}^{-1}+12\lambda\varphi_0^2\ , \tag{5}$$

where f_{xy}^{-1} is the inverse of f_{xy}.

The functional annihilation operator is [3]

$$A_f(p)=\left[\frac{1}{2(2\pi)^D f(p)}\right]^{1/2}\times\int_x e^{-ipx}\left[f(p)(\phi_x-\varphi_0)+\frac{\delta}{\delta\phi_x}\right]\ , \tag{6}$$

and its adjoint is the functional creation operator

$$A_f^\dagger(p)=\left[\frac{1}{2(2\pi)^D f(p)}\right]^{1/2}\times\int_x e^{ipx}\left[f(p)(\phi_x-\varphi_0)-\frac{\delta}{\delta\phi_z}\right]. \tag{7}$$

Then a trial positroniumlike state of two particles with the center of mass in rest can be constructed as

$$|2\rangle=\int dp\,\Sigma(p)|p,-p\rangle$$
$$=\int dp\,\Sigma(p)A_f^\dagger(p)A_f^\dagger(-p)|\varphi_0\rangle\,, \tag{8}$$

where $\Sigma(p)$ is the Fourier transformaton of an S-wave function of the two particles. Using functional integration techniques, one can calculate the total energy of the two-particle system as

$$m_2=\frac{\int_x\langle 2|\mathcal{H}_x|2\rangle}{\langle 2|2\rangle}-\langle\varphi_0|H|\varphi_0\rangle$$
$$=\frac{2\int dp[\Sigma(p)]^2 f(p)+[3\lambda/(2\pi)^D]\{\int dp[\Sigma(p)/f(p)]\}^2}{\int dp[\Sigma(p)]^2}\,. \tag{9}$$

The two terms in this expression can be regarded as the kinetic energy of two constituent particles and their interacting energy, respectively.

Minimizing m_2 with respect to $\Sigma(p)$, one can have

$$\Sigma(k)=\frac{A}{f(k)[2f(k)-m_2]}\,, \tag{10}$$

where A is a normalization constant, and accordingly Eq. (9) can be read as

$$\int\frac{dp}{f^2(p)[2f(p)-m_2]}=-\frac{(2\pi)^D}{3\lambda}\,. \tag{11}$$

When $m_2>2\mu(\varphi_0)$, the positronium like state with Eq. (10) is a scattering state and Eq. (11) contains the scattering information of the two particles. As a matter of fact, the integrand on the left-hand side (LHS) of Eq. (11) has a singularity $p_0=\sqrt{(m_2/2)^2-\mu^2(\varphi_0)}$. So the integral of Eq. (11) can be regarded as the corresponding principal value integral plus the contribution of this pole to the integral. Through the same discretization procedure as used in Ref. [1], the integral for $1+1$, $2+1$, and $3+1$ dimensions can be written as

$$\int_{-\infty}^{\infty}\frac{dp}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]}=\frac{2\pi}{m_2\sqrt{(m_2/2)^2-\mu^2(\varphi_0)}}\cot\delta$$
$$+\mathcal{P}\int_{-\infty}^{\infty}\frac{dp}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]}\,, \tag{12}$$

$$\int_0^\infty\int_0^{2\pi}\frac{pdpd\varphi}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]}=\frac{2\pi^2}{m_2}\cot\delta+\mathcal{P}\int_0^\infty\int_0^{2\pi}\frac{pdpd\varphi}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]} \tag{13}$$

and

$$\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{p^2dp\,\sin\theta d\theta d\varphi}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]}$$
$$=\frac{4\pi^2[(m_2/2)^2-\mu^2(\varphi_0)]^{1/2}}{m_2}\cot\delta+\mathcal{P}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{p^2dp\,\sin\theta d\theta\,d\varphi}{[p^2+\mu^2(\varphi_0)][2\sqrt{p^2+\mu^2(\varphi_0)}-m_2]}\,, \tag{14}$$

respectively. An analysis on the wave function shows that the quantity δ defined above is indeed the phase shift of the two-particle state at large distances. Thus, Eqs. (12), (13), and (14) dictate the connection between the singular points and the scattering phase shift. As a result, according to Eq. (11) the expression of δ for $1+1$ dimensions has the form

$$\cot\delta=\tfrac{1}{2}\sqrt{\tilde{m}_2^2-1}-\frac{2\tilde{m}_2\sqrt{\tilde{m}_2^2-1}}{3\tilde{\lambda}}$$
$$-\frac{1}{\pi}\ln(\tilde{m}_2-\sqrt{\tilde{m}_2^2-1}) \tag{15}$$

with $\tilde{\lambda}=\lambda/\mu^2$, and, for $(2+1)$ dimensions,

$$\cot(\delta)=-\frac{4\tilde{m}_2}{3\tilde{\lambda}}+\frac{1}{\pi}\ln(\tilde{m}_2-1) \tag{16}$$

with $\tilde{\lambda}=\lambda/\mu$, where $\tilde{m}_2=m_2/2\mu(\varphi_0)$.

According to Eqs. (15) and (16), the dependences of δ on $\tilde{m}_2$ for $1+1$ and $2+1$ dimensions are described graphically in Figs. 1 and 2 separately. In these figures, one can see that in $1+1$ and $2+1$ dimensions, this model has a negative phase shift. This means the interaction is repulsive between two pions. Moreover, in $1+1$ dimensions, when $\tilde{\lambda}>4\pi/(6+3\pi)$ the phase shift has minimum value, while when $\tilde{\lambda}<4\pi/(6+3\pi)$ the phase shift increases monotonously from $-\pi/2$ to zero with the increase of $\tilde{m}_2$. Figure 2 shows that in $2+1$ dimensions the phase shift always has a minimum value for all the values of λ.

For the case of $3+1$ dimensions, the principal value integral is ultraviolet divergent. It seems that the renormalization of the bare coupling constant λ has to be considered. In fact, this has been done in dealing with the vacuum problems of the same model. The relation between λ and the renormalized coupling constant λ_R has been given by Eq. (3.19) in Ref. [4] and can be rewritten, with our notation, as (for $D>0$)

$$\lambda_R=\lambda\frac{1-6\lambda I_2(\mu)}{1+3\lambda I_2(\mu)}\ , \tag{17}$$

where $I_2(\mu)=\int dp(2\pi)^{-D}(p^2+\mu^2)^{-3/2}$. Although $I_2(\mu)$ is ultraviolet divergent for $D=3$, Eq. (17) implies that λ can be still taken as a finite parameter. Therefore, from Eqs. (11) and (14) one can conclude that phase shift δ vanishes for $3+1$ dimensions. However, when λ is finite, the ϕ^4 theory is not a viable one and the corresponding Gaussian effective potential is unbounded below [4]. Thus, for a finite value of λ the phase shift in $3+1$ dimensions has no physical senses. Of course, there is another possibility from Eq. (17) that λ is infinitesimal and equals $-1/3I_2(\mu)$. This possibility has been investigated and a nontrivial precarious theory has been proposed in Ref. [4]. It is clear that Eqs. (11) and (14) still hold for $\lambda=-1/3I_2(\mu)$ (certainly with μ changed). Substituting Eq. (14) and $\lambda=-1/3I_2(\mu)$ into Eq. (11), one can have

$$\cot\delta=-\frac{2\tilde{m}_2}{\pi\sqrt{\tilde{m}_2^2-1}}-\frac{1}{2\sqrt{\tilde{m}_2^2-1}}-\frac{2\tilde{m}_2\ln(\tilde{m}_2-\sqrt{\tilde{m}_2^2-1})}{\pi(\tilde{m}_2^2-1)}+\frac{\tilde{m}_2}{\pi\sqrt{\tilde{m}_2^2-1}}\mathcal{P}\int_0^{\pi/2}\frac{d\theta}{\cos\theta}\ . \tag{18}$$

The last term in Eq. (18) is evidently divergent, and therefore the phase shift δ is zero. In addition, we also have noticed the nontrivial autonomous theory which states that when $\lambda=1/6I_2(\xi)$ with ξ a finite parameter, the $(3+1)$-dimensional $\lambda\phi^4$ theory exhibits a massless symmetric phase and a massive broken-symmetry phase [22,23]. From Eqs. (9)–(11) and (13) in Ref. [22], our Eqs. (11) and (14) also hold for $\lambda=1/6I_2(\xi)$ (λ_B, m_B, and Ω in Ref. [22] are λ, m, and μ in our notation, respectively), and so we obtain

$$\cot\delta=\frac{4\tilde{m}_2}{\pi\sqrt{\tilde{m}_2^2-1}}-\frac{1}{2\sqrt{\tilde{m}_2^2-1}}-\frac{2\tilde{m}_2\ln(\tilde{m}_2-\sqrt{\tilde{m}_2^2-1})}{\pi(\tilde{m}_2^2-1)}-\frac{5\tilde{m}_2}{\pi\sqrt{\tilde{m}_2^2-1}}\mathcal{P}\int_0^{\pi/2}\frac{d\theta}{\cos\theta}\ . \tag{19}$$

Obviously, the phase shift δ equals zero again. In a word, in the $(3+1)$-dimensional $\lambda\phi^4$ field theory scattering phase shifts vanish for both the precarious and autonomous versions, which is consistent with the remarks at the end of Sec. VI in Ref. [4]. As for the case $D>3$,

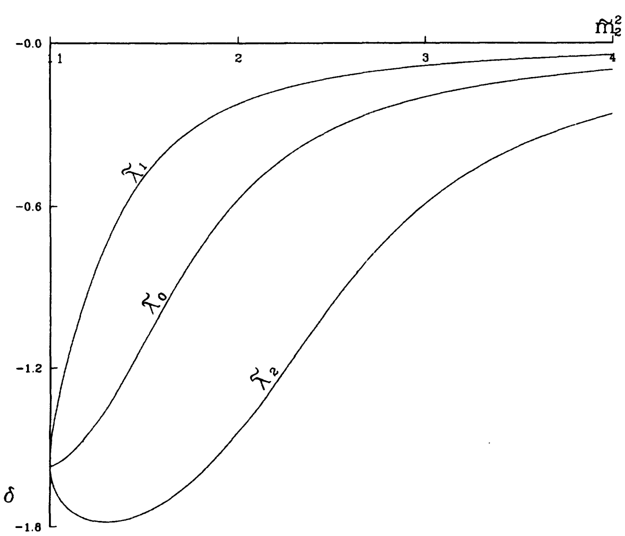

FIG. 1. The phase-shift curves in $1+1$ dimensions for some values of the reduced coupling $\tilde{\lambda}$, see Eq. (15). In the figure, $\tilde{\lambda}_1=2\pi/(6+3\pi)$, $\tilde{\lambda}_0=4\pi/(6+3\pi)$ and $\tilde{\lambda}_2=8\pi/(6+3\pi)$.

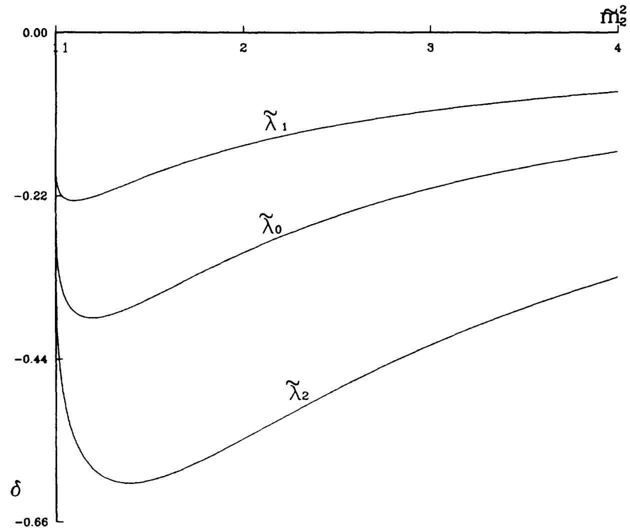

FIG. 2. The phase-shift curves in $2+1$ dimensions for some values of the reduced coupling $\tilde{\lambda}$, see Eq. (16). In the figure, $\tilde{\lambda}_1=2\pi/(6+3\pi)$, $\tilde{\lambda}_0=4\pi/(6+3\pi)$, and $\tilde{\lambda}_2=8\pi/(6+3\pi)$.

phase shifts may be discussed through the analogous treatment with the above. Here we shall not continue to treat it.

Comparing Fig. 2 with Fig. 1, one finds the curves in Fig. 2 are closer to the horizontal axis than those in Fig. 1. Furthermore, in the higher dimensions the curves reduce to the horizontal axis. Therefore it seems that the dimensionality of space tends to cripple the effect of interaction between two quantum particles.

In conclusion we have obtained the phase shifts of the $\lambda\phi^4$ model in various dimensions with the GWF approach. Although it is the discretization procedure that makes it possible to derive the phase shift, the results do not depend upon the discretization at all. Our results are nonperturbative and reasonable, though the degree of approximation of the variational approach is difficult to estimate. The earlier investigation by Stevenson [4] and the researches for the bound state [5,8,9] show that the GWF result is qualitatively correct at least. We hope that the GWF approach can be extended to practical scattering processes such as hydron scattering, etc., which are helpful for the development of quantum field theory.

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