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Path integral approach to two-dimensional QCD in the light-front frame

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Two-dimensional quantum chromodynamics in the light-front frame is studied following Hamiltonian methods. The theory is quantized using the path integral formalism and an effective theory similar to the Nambu–Jona-Lasinio model is obtained. Confinement in two dimensions is derived by analyzing directly the constraints in the path integral.

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I. INTRODUCTION

Quantum field theory quantized in the light-front frame has been extensively studied in the past few years as an alternative way for understanding nonperturbative phenomena [1]. Although this approach is quite old [2], only recently have new techniques of calculation been developed [3,4] that could allow, in principle, the study of phenomena such as confinement or hadronization, which are very difficult to understand through the conventional approach.

Several years ago, 't Hooft studied the solubility of QCD in two dimensions (QCD_2) in the light-front frame, introducing the $1/N$ expansion [5]. In his work he was able to solve the theory in the large-N limit and then show how the bound state spectrum can be obtained by solving a Bethe-Salpeter equation in this limit. However, in spite of the relevance that the 't Hooft results could have in our understanding of QCD in four dimensions, not much further progress was reached at that time in order to understand the perturbative and nonperturbative structure of QCD in the light-front frame.

The revival of light-front quantization has been mainly pioneered by the authors of Ref. [3] and later also by those of Ref. [4]. In these references two different nonperturbative methods for calculating light-front wave functions have been proposed, which although promising still present some technical difficulties despite intense recent research [1].

The canonical structure of QCD in the light-front frame and its subsequent quantization via the path integral method is to our knowledge still an open problem.[1] In the past this approach has been very useful in the understanding of many aspects of gauge theories and we will show that it is very useful in the present context as well. The purpose of this research is the study of the canonical structure of light-front QCD_2 and its quantization following the path integral method.

[1]In Ref. [6] the Hamiltonian formulation of the Schwinger model is studied at the classical level. On the other hand, in Ref. [7] the reader can find recent canonical studies of two-dimensional light-front gauge theories.

II. QCD_2 IN LIGHT-FRONT COORDINATES: HAMILTONIAN ANALYSIS

In this section we study the canonical structure of QCD_2 in light-front coordinates following the Dirac constrained theory [8]. The Lagrangian is

$$\mathcal{L}=-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\bar{\psi}^r(i\not{D}-m)\psi^r \tag{2.1}$$

($\mu=0,1;\ a=1,2,\ldots,N^2;\ r=1,2,\ldots,N$).

In the light-front frame approach one defines the coordinates

$$x^{\pm}=\frac{1}{\sqrt{2}}(x^0\pm x^1) \tag{2.2}$$

and then writes all the quantities involved in the Lagrangian (2.1) in terms of $x^{\pm}$ instead of x^0 and x^1. After doing this the Lagrangian density (2.1) becomes

$$\mathcal{L}=\tfrac{1}{2}(F^a_{+-})^2+\bar{\psi}^r(i\gamma_-D_++i\gamma_+D_--m)\psi^r\ , \tag{2.3}$$

where $\gamma_\pm$ and $D_\pm$ are defined in complete analogy with (2.2), i.e.,

$$\gamma_{\pm}=\frac{1}{\sqrt{2}}(\gamma_0\pm\gamma_1)\ ,\qquad D_{\pm}=\frac{1}{\sqrt{2}}(D_0\pm D_1)\ , \tag{2.4}$$

and the $\gamma_\pm$ matrices satisfy

$$\gamma_\pm^2=0,\quad \{\gamma_+,\gamma_-\}=2\ . \tag{2.5}$$

In order to carry out the Hamiltonian formulation, we are forced to choose a time coordinate, which is usually chosen as x^+. Thus the canonical momenta are

$$\pi^a_+=\frac{\delta L}{\delta\partial_+A^a_+}=0\ , \tag{2.6a}$$

$$\pi^a_-=\frac{\delta L}{\delta\partial_+A^a_-}=F^a_{+-}\ , \tag{2.6b}$$

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$$P^r = \frac{\delta L}{\delta \partial_+ \psi^r} = i\bar{\psi}^r \gamma_- \ , \tag{2.6c}$$

$$\bar{P}^r = \frac{\delta L}{\delta \partial_+ \bar{\psi}^r} = 0 \ , \tag{2.6d}$$

where $L = \int dx_- \mathcal{L}$.

Observing (2.6), one can see that there are three primary constraints, namely,

$$\pi_+^a = 0 \ , \tag{2.7a}$$

$$\chi^r = P^r - i\bar{\psi}^r \gamma_- = 0 \ , \tag{2.7b}$$

$$\bar{\chi}^r = \bar{P}^r = 0 \ , \tag{2.7c}$$

and which must be preserved in time x^+.

The total Hamiltonian can then be computed, with the result

$$H_T = \int dx_- [\tfrac{1}{2}(\pi_-^a)^2 + \pi_-^a \partial_- A_+^a + g\pi_-^a f^{abc} A_-^b A_+^c + g\bar{\psi}^{\gamma_-}\psi^r A_+^a T^a - i\bar{\psi}^r \gamma_+ D_- \psi^r - m\bar{\psi}^r \psi^r + u_0^a \pi_+^a + u_1^r \chi^r + u_2^r \bar{\chi}^r] \ . \tag{2.8}$$

where u_0^a, u_1^r, and u_2^r are Lagrange multipliers.

One can see that the preservation in time of π_+^a implies the secondary constraint

$$G^a = \partial_- \pi_-^a + g\pi_-^c f^{abc} A_-^b + g\bar{\psi}^r \gamma_- \psi^r T^a \ , \tag{2.9}$$

and that the other constraints $(\chi^r, \bar{\chi}^r)$ do not generate new constraints.

A straightforward analysis shows that the constraints G^a, χ^r, and $\bar{\chi}^r$ are second class while π_+^a is first class. On the other hand, a simple inspection also shows that $(G^a, \chi^r, \bar{\chi}^r)$ are not a minimal number of second-class constraints. The minimal set is found by combining appropriately G^a, χ^r, and $\bar{\chi}^r$, and it is straightforward to verify that this set is

$$\Omega_0^a = \pi_+^a \ , \tag{2.10a}$$

$$\Omega_1^a = G^a + i(\bar{\chi}^r T^a \psi^r + \bar{\psi}^r T^a \chi^r) \ , \tag{2.10b}$$

$$\chi^r = P^r - i\bar{\psi}^r \gamma_- \ , \tag{2.10c}$$

$$\bar{\chi}^r = \bar{P}^r \ , \tag{2.10d}$$

where (Ω_0^a, Ω_1^a) and $(\chi^r, \bar{\chi}^r)$ are first- and second-class constraints, respectively.

The first-class constraints satisfy the algebra

$$\{\Omega_\mu^a(x_-), \Omega_\nu^b(x'_-)\} = 0 \ , \tag{2.11}$$

while the nonvanishing Poisson brackets between χ^r and $\bar{\chi}^r$ are

$$\{\chi^r{}_a(x_-), \bar{\chi}_\beta^{r'}(x'_-)\} = -i(\gamma_-)_{\beta\alpha}\delta^{rr'}\delta(x_- - x'_-) \ , \tag{2.12}$$

where α, β are spinorial indices.

In order to eliminate the second-class constraints, we define the usual Dirac brackets. In this case the nonvanishing Dirac brackets between the canonical variables are

$$\{A_+^a(x_-), \pi_+^b(x'_-)\}_{DB} = \delta(x_- - x'_-)\delta^{ab} = \{A_-^a(x_-), \pi_-^b(x'_-)\}_{DB} \ , \tag{2.13a}$$

$$\{\psi^r{}_\alpha(x_-), \bar{\psi}_\beta^{r'}(x'_-)\}_{DB} = (\gamma_-)_{\beta\alpha}^{-1}\delta^{rr'}\delta(x_- - x'_-) \ , \tag{2.13b}$$

$$\{\psi^r{}_\alpha(x_-), P_\beta^{r'}(x'_-)\}_{DB} = \delta^{rr'}\delta^{ab}\delta(x_- - x'_-) \ , \tag{2.13c}$$

$$\{\bar{\psi}^r{}_\alpha(x_-), \bar{P}_\beta^{r'}(x'_-)\}_{DB} = \delta^{rr'}\delta^{ab}\delta(x_- - x'_-) \ . \tag{2.13d}$$

The set of equations (2.8) and (2.10)–(2.13) defines completely the canonical structure of the theory. The next step is to fix the gauge in order to quantize the theory.

III. GAUGE FIXING AND THE PATH INTEGRAL QUANTIZATION

In this section we discuss the quantization of the previous model following the path integral approach. There are several reasons that justify this study: (i) To our knowledge the quantization of QCD_2 in the light-front frame following the path integral approach has never been discussed before, (ii) this study could throw some light into the derivation of the Feynman rules in the light-front quantization method and the influence of the zero modes, and (iii) the path integral approach could allow for the influence of new fields that could simplify the perturbative structure of the theory.

With these facts in mind, in this section we try to clarify the problem of gauge fixing and path integral quantization of QCD_2 in the light-front frame.

A. Gauge fixing

The gauge freedom is reflected from the Hamiltonian point of view in the presence of first-class constraints. In the problem at hand, we have two first-class constraints [Eqs. (2.10a) and (2.10b)], and as a consequence, two conditions are necessary in order to fix completely the gauge freedom. Thus we can start by imposing the following condition as gauge fixing:

$$\Omega_2^a = A_-^a = 0 \ , \tag{3.1}$$

which is known as light-cone gauge and which must be imposed as a new constraint of the theory. Following Dirac's method [8], (3.1) must be preserved in time, i.e.,

$$\partial_+ \Omega_2^a = \{\Omega_2^a, H_T\} = 0 \ , \tag{3.2}$$

Computing (3.2), we find that this consistency condition implies the new constraint

$$\Omega_3^a = \pi_-^a + \partial_- A_+^a + gf^{abc} A_-^b A_+^c = 0 \ . \tag{3.3}$$

Conditions (3.1) and (3.3) fix completely the gauge freedom. In fact, computing the Poisson algebra we find that the nonvanishing brackets between the first-class constraints and the gauge conditions are

$$\{\Omega_0^a(x_-),\Omega_3^b(x'_-)\}=(\delta^{ab}\partial_- - gf^{abc}A_-^c)\delta(x_- - x'_-)\ , \tag{3.4a}$$

$$\{\Omega_1^a(x_-),\Omega_3^b(x'_-)\}=gf^{abc}[A_+^c\partial_-\delta(x_- - x'_-)+\partial_- A_+^c\delta(x_- - x'_-)] + gf^{abc}\pi_-^c\delta(x_- - x'_-)+g^2 f^{afc}f^{bgc}A_-^f A_+^g\delta(x_- - x'_-)\ , \tag{3.4b}$$

which is a second-class constraint algebra.

The question now is, are there other alternative gauge-fixing conditions besides $A_-^a=0$? The answer to this question is, of course, yes, although the correct way to implement other possible gauge-fixing conditions in the light-front frame is not trivial.

One could try to find, for instance, the analogue of the gauge fixing in a covariant gauge theory, but this procedure does not work here. Indeed, this can be verified by constructing the analogue of the Lorentz gauge $\partial_\mu A^{a\mu}=0$ in the light-front frame,

$$\partial_- A_+^a+\partial_+ A_-^a=0\ , \tag{3.5}$$

but a simple analysis shows that this condition does not fix the gauge freedom. In fact, it can be shown that (3.5) is not a true condition because when preservation in time is imposed, we cannot generate a new constraint fixing the remaining gauge symmetry.

The same occurs when we consider the analogue of the axial gauge $n^\mu A_\mu^a=0$ in the light-front frame,

$$n_- A_+^a+n_+ A_-^a=0\ . \tag{3.6}$$

A possible solution to this problem consists in modifying slightly the previous gauge conditions. Using (3.5) and (3.6), one can see that the unique possible choices for the above gauge conditions are[2]

$$\partial_+ A_-^a=0\ \ \text{(Lorentz gauge)}\ , \tag{3.7}$$

$$n_+ A_-^a=0\ \ \text{(axial gauge)}\ . \tag{3.8}$$

The Hamiltonian analysis shows, however, that Eqs. (3.7) and (3.8), although they are two independent gauge conditions that fix completely the gauge freedom, are contained in (3.1). This last point can be shown by computing the analogue of (3.2) using (3.7) and (3.8). This calculation gives

$$\partial_+\Omega_3^a=0=n_+\Omega_3^a\ ,$$

and, as a consequence, to impose (3.7) or (3.8) is formally the same condition (3.1).

This last result means that in two dimensions in the light-front frame, the light-cone gauge (3.1) contains a complete family of gauge conditions that simplify the canonical analysis.[3] This last point is another advantage of the light-front approach.

B. Path integral quantization

The quantization in this case must be performed using the modification introduced by Senjanovic [10] because there are second-class constraints.

The generating functional is

$$Z=\int \mathcal{D}\pi_+^a\mathcal{D}A_+^a\mathcal{D}\pi_-^a\mathcal{D}A_-^a\mathcal{D}\bar\psi^r\mathcal{D}\psi^r\mathcal{D}P^r\mathcal{D}\bar P^r\det\|M^{ab}\|\det\|\{\chi_\alpha,\chi_\beta\}\|^{1/2}\delta(\Omega_0^a)\delta(\Omega_1^a)\delta(\Omega_2^a)\delta(\Omega_3^a)\delta(\chi^r)\delta(\bar\chi^r)$$
$$\times\exp\left[i\int dx_+dx_-[\pi_+^a\partial_+A_+^a+\pi_-^a\partial_+A_-^a+P^r\partial_+\psi^r+\bar P^r\bar\psi^r-\tfrac{1}{2}(\pi_-^a)^2-\pi_-^a\partial_-A_+^a - g\pi_-^a f^{abc}A_-^bA_+^c-g\bar\psi^r\gamma_-T^a\psi^rA_+^a+i\bar\psi^r\gamma_+D_-\psi^r-m\bar\psi^r\psi^r]\right]\ . \tag{3.9}$$

Using (3.4), the first determinant can be explicitly computed,

$$\det\|M^{ab}\|=\det\begin{bmatrix}0 & \kappa^{ab}\\ -\kappa^{ab} & \rho^{ab}\end{bmatrix}\delta(x_- - x'_-)\ , \tag{3.10}$$

where its elements are

$$\kappa^{ab}=\delta^{ab}\partial_- - gf^{abc}A_-^c\ ,$$
$$\rho^{ab}=gf^{abc}\partial_-A_+^c+gf^{abc}A_+^c\partial_-+gf^{abc}\pi_-^c + g^2f^{afc}f^{bgc}A_+^fA_-^g\ ,$$

while the second determinant

$$\det\{\chi_\alpha^r,\bar\chi_\beta^{r'}\}=\det\begin{bmatrix}0 & -i(\gamma_-)_{\beta\alpha}^{-1}\\ -i(\gamma_-)_{\alpha\beta} & 0\end{bmatrix}\times\delta(x_- - x'_-)\delta^{rr'} \tag{3.11}$$

is a c number and can be dropped off the path integral as a normalization factor.

In integration in $\pi_-^a,\pi_+^a,A_-^a,\bar P^r,P^r$ and exponentiating (3.10) in terms of anticommutative ghosts, we find

$$Z=\mathcal{N}\int\mathcal{D}A_+^a\mathcal{D}\bar\psi^r\mathcal{D}\psi^r\mathcal{D}(\text{ghosts})\delta[-\partial_-^2A_+^a+g\bar\psi^r\gamma_-T^a\psi^r]$$
$$\times\exp\left[i\int dx_+dx_-[\tfrac{1}{2}(\partial_-A_+^a)^2+i\bar\psi^r\gamma_+\partial_-\psi^r+i\bar\psi^r\gamma_-\partial_+\psi^r-g\bar\psi^r\gamma_-T^a\psi^rA_+^a-m\bar\psi^r\psi^r+(\text{ghosts})]\right]\ , \tag{3.12}$$

[2]The other possible choices $\partial_-A_+^a=0$ for the Lorentz gauge and $n_-A_+^a=0$ for the axial gauge do not fix the gauge freedom.

[3]Recently, there has been some discussion about the problem of fixing the residual gauge in light-front QCD_4 [9]. In this paper we assume periodic boundary conditions, in which case the zero modes do not have to be considered.

where $\mathcal{N}$ is a normalization factor.

Using the antisymmetry of the structure constants and the anticommutative character of the ghosts, it is easy to see that there is no coupling between the ghosts and the gauge fields and that the integration in the ghosts fields is trivial. This result is a consequence of a general statement that establishes that all axial-like gauges are free of interactions with the ghosts [11]. Having this fact in mind, (3.12) becomes

$$Z=\mathcal{N}'\int \mathcal{D}A_+^a\mathcal{D}\bar{\psi}^r\mathcal{D}\psi^r\delta[-\partial_-^2A_+^a+g\bar{\psi}^r\gamma_-T^a\psi^r]$$
$$\times\exp\left[i\int dx_+dx_-[\tfrac{1}{2}(\partial_-A_+^a)^2+i\bar{\psi}^r\gamma_+\partial_-\psi^r+i\bar{\psi}^r\gamma_-\partial_+\psi^r-g\bar{\psi}^r\gamma_-T^a\psi^rA_+^a-m\bar{\psi}^r\psi^r]\right]. \tag{3.13}$$

This formula gives the path integral version of QCD_2.

The constraint (Gauss's law):

$$\partial_-^2A_+^a-J_-^a=0, \tag{3.14}$$

with $J_-^a=g\bar{\psi}^r\gamma_-T^a\psi^r$, physically can be understood as follows. In two dimensions there only exists an electric field, which is given by $E^a=\partial_-A_-^a$. Using this field and Eq. (3.14), E^a is given by

$$E^a(x_-)=\int dx'_-\epsilon(x_--x'_-)J_-^a(x'_-), \tag{3.15}$$

where $\epsilon(x_--x'_-)$ is the sign function. In order to see what this result means, let us assume for the moment that the quarks are pointlike. Thus $J_-^a(x_-)$ can be written as

$$J_-^a(x_-)=\sum_{\alpha=1}^{N}q_\alpha\delta(x_--x_-^\alpha)T^a, \tag{3.16}$$

where q_α is the quark charge and x_-^α is the place where the αth quark is localized. Using (3.16), the electric field becomes

$$E^a(x_-)=\sum_{\alpha=1}^{N}q_\alpha\epsilon(x_--x_-^\alpha)T^a. \tag{3.17}$$

Therefore the electric field between the particles is unable to spread out and the quarks are confined. The reader should note that confinement is present irrespective of the (non-)Abelian character of the gauge field [12].

The next step is to integrate the A_+^a field. Using the identity

$$\int\mathcal{D}\phi\,\delta(A\phi+B)\exp\left[i\int dx(\phi C\phi+D\phi)\right]$$
$$=\exp\left[i\int dx\left[\frac{B}{A}C\frac{B}{A}-\frac{D}{A}B\right]\right], \tag{3.18}$$

where A,B,C,D are operators (with inverse), Eq. (3.13) becomes

$$Z=\mathcal{N}\int\mathcal{D}\bar{\psi}^r\mathcal{D}\psi^r\exp\left[i\int dx_+dx_-\left[\bar{\psi}^r(i\gamma_+\partial_-+i\gamma_-\partial_+-m)\psi^r-\tfrac{1}{4}\int dx'_-J_-^a(x_-)|x_--x'_-|J_-^a(x'_-)\right]\right], \tag{3.19}$$

where $J_-^a(x_-)=g\bar{\psi}^r(x_-)\gamma_-T^a\psi^r(x_-)$ and $|x_--x'_-|$ is the propagator of the gluon field obtained by inverting the operator ∂_-^2.

Equation (3.19) is the starting point for the perturbative and nonperturbative evaluation of quantum corrections of QCD_2. Nonperturbatively, this formula was used in Ref. [13] in order to derive the 't Hooft equation for the bound states in the large-N limit.

From the canonical point of view, the effective action that appears in Eq. (3.19) was used by the authors of Ref. [14] for the numerical study of light-front quantized QCD_2.

Finally, we should mention that the Higgs mechanism in light-front quantized field theory was also considered in Ref. [15]

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[1] For a review and references, see S. J. Brodsky, G. McCartor, H.-C. Pauli, and S. S. Pinsky, Part. World **3**, 109 (1993).

[2] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949); S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[3] H. C. Pauli and S. J. Brodsky, Phys. Rev. D **32**, 1993 (1985); **32**, 2001 (1985).

[4] R. J. Perry, A. Harindranath, and K. G. Wilson, Phys. Rev. Lett. **65**, 2959 (1990); D. Mustaki, S. Pinsky, J. Shigemitsu, and K. G. Wilson, Phys. Rev. D **43**, 3411 (1991).

[5] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).

[6] D. Mustaki, Phys. Rev. D **42**, 1184 (1990).

[7] F. Lenz, M. Thies, S. Levit, and K. Yazaki, Ann. Phys. (N.Y.) **208**, 1 (1991); T. Heinzl, S. Krusche, and E. Werner, Phys. Lett. B **275**, 410 (1992); K. Hornbostel, Phys. Rev. D **45**, 3781 (1992).

[8] See, for example, K. Sundermeyer, *Constrained Hamiltonian Systems* (Springer Verlag, Berlin, 1984).

[9] W.-M. Zhang and A. Harindranath, Phys. Lett. B **314**, 223 (1993).

[10] P. Senjanovic, Ann. Phys. (N.Y.) **100**, 227 (1976).

[11] See, for example, G. Leibbrandt, Rev. Mod. Phys. **59**, 1067 (1987); W. Konetschny, Phys. Lett. B **90**, 263 (1980).

[12] S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) **93**, 267 (1975); S. Coleman, *ibid.* **101**, 239 (1976).

[13] W. R. Gutiérrez, Nucl. Phys. **B176**, 185 (1980).

[14] K. Hornbostel, S. J. Brodsky, and H. C. Pauli, Phys. Rev. D **41**, 3814 (1990).

[15] P. P. Srivastava, Report No. CBPF-010/92 (unpublished); Report No. CBPF-004/92 (unpublished).