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Head-on collision of n vortices

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We show that the result of the head-on collision on n vortices in the Abelian-Higgs model is the scattering by an angle of $\frac{\pi}{n}$ even when there is another vortex of arbitrary topological index at the center of scattering. This is obtained by analysis of both the geodesic flow on the moduli space and of the string approximation to the short range interactions of vortices. We also provide formal tools for an analysis of more general scattering events with n vortices by diagonalization of the moduli space metrics around the point where the n vortices coincide. Arguments after a certain degree of equivalence of the string and moduli space approaches are presented.

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I. INTRODUCTION

Recently there has been some interest in the head-on scattering of n solitons in $2+1$ dimensions. In [1] some arguments were given which suggest that in all head-on collisions of n indistinguishable objects the scattering angle is expected to be $\frac{\pi}{n}$. The results of [1] are expected to hold in all models in which the solitons are topologically stable and the topology is based on the $S^2 \to S^2$ mappings. In this work we show that the $\frac{\pi}{n}$ scattering is the result of the head-on collision of n vortices in the Abelian-Higgs model in $2+1$ dimensions at the critical value of the coupling constant [2–4]. The vortices are solitons that are stable thanks to the nontrivial topology of the mappings $S^1 \to S^1$.

The collisions in which more than two vortices take part can be expected to be significant statistically in a multivortex system of large density. The thermodynamics of a multivortex system has been studied recently in [5]. We find such a local parametrization of the moduli space [6–8] near the points where the n vortices coincide in which the metrics are diagonal. This enables us to obtain the general solution to the equations of motion near the center of collision. The back transformation from the geodesic coordinates to the coordinates of the zeros of the Higgs field amounts to looking for the complex roots of a certain nth-order polynomial. The case of the head-on collision is so simple that it needs just a toy calculation. When the n vortices collide in the head-on fasion the whole configuration shrinks to the coincident position and then reappears rotated by an angle of $\frac{\pi}{n}$ with respect to the initial one even when there is another vortex of arbitrary topological index in the center of collision.

Next we address the same problem but with the help of the string model for short range interactions of vortices [9], which is an extension of the Ben-Ya'acov model [10] to the case of overlapping vortices. The results for the head-on collision are the same as those obtained in the moduli space approximation. We comment on the identity of the results giving arguments after the equivalence of the two approaches in the case of short range interactions of parallel strings. The equivalence provides us with an alternative tool for investigations of multivortex collisions — the string model.

II. APPROXIMATION BY GEODESIC MOTION

Let us consider the Abelian-Higgs model in $2+1$ dimensions defined by the Lagrangian

$$L = \frac{1}{2}(D_\mu\phi)(D^\mu\phi)^\star - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}\lambda(\phi\phi^\star - 1)^2 \quad , \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$. It is known [3,4] that at the critical value of $\lambda = 1$ the field equations for the static configuration with the positive topological index n reduce to

$$(D_1 + iD_2)\phi = 0 \ \text{ and } F_{12} = \frac{1}{2}(1 - \phi\phi^\star) \ . \tag{2}$$

The n-vortex solution may be written as

$$\phi(r,\theta) = e^{in\theta} f(r) \ , \tag{3}$$

$$A_i(r,\theta) = -\varepsilon_{ij}x_j\frac{n}{r^2}a(r) \ , \tag{4}$$

where f and a satisfy

$$r\frac{df}{dr} - n(1-a)f = \frac{2n}{r}\frac{da}{dr} + (f^2 - 1) = 0 \tag{5}$$

with boundary conditions $f(0) = a(0) = 0$ and at infinity $f = a = 1$. Let us try the following ansatz [4,7] for the fluctuations of ϕ and of $A_+ \equiv A_1 + iA_2$:

$$\delta\phi = nf(r)e^{in\theta}h(r,\theta) \quad , \tag{6}$$

$$a_+ = \frac{n}{r}e^{i\theta}a(r,\theta) \quad , \tag{7}$$

*Electronic address: ufjacekd@ztc386a.if.uj.edu.pl where h and a are complex functions. Following [7,8], to completely determine the fields we impose the background gauge condition

$$\partial_i \dot{A}_i + \dot{\phi}_a \varepsilon_{ab} \phi_b = 0 \ . \tag{8}$$

Substitution of the fluctuations to the field equations (2) and to the above gauge condition and next linearization yields

$$f^2 h - \frac{i}{r}\frac{\partial a}{\partial r} - \frac{1}{r^2}\frac{\partial a}{\partial \theta} = 0 \ , \tag{9}$$

$$a = \frac{\partial h}{\partial \theta} - ir\frac{\partial h}{\partial r} \ . \tag{10}$$

Upon substitution of Eq. (10) to Eq. (9) one obtains a simple equation,

$$\nabla^2 h = f^2 h \ , \tag{11}$$

from which it is easy to obtain the general form of the fluctuation of the Higgs field. Once this fluctuation has been found Eq. (10) enables us to find the corresponding change of the gauge field. The function h can be Fourier expanded in θ:

$$h(r,\theta) = \sum_{k=1}^{n} [\lambda_1^{(k)} + i\lambda_2^{(k)}] H_{(k)}(r) e^{-ik\theta} \ . \tag{12}$$

The series does not contain the zero-order term because the vortex solution (3) and (4) is stable with respect to the monopole fluctuations, they cannot be zero modes. To explain why it is truncated from above let us observe that the profile functions satisfy

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2}\right) H_{(k)}(r) = f^2(r) H_{(k)}(r) \ . \tag{13}$$

For large r H's can be approximated by the modified Bessel functions. To preserve square integrability of the fluctuation (6) we choose them to be exponentially vanishing at infinity. When we go from infinity to the vicinity of zero the function $H_{(k)}$ approaches $\delta_+ r^k + \delta_- r^{-k}$. In the case of nonzero δ_-, which we think to be quite general, the profile functions can be conveniently normalized so that

$$H_{(k)}(r) \sim \frac{-1}{n} r^{-k} \ , \quad r \to 0 \ . \tag{14}$$

When we take into account that in the same limit $f(r) \sim r^n$, we can see that if we want to preserve the regularity of the Higgs field then k cannot be greater that n, see Eq. (6).

The series (12) contains $2n$ real parameters: $\lambda_\alpha^{(k)}$ with $k = 1, \ldots, n$ and $\alpha = 1, 2$. It is just the number of parameters the general solution of Eqs. (2) with the topological index n should contain, as it was shown by Weinberg [4] with a help of his index theorem. Although Weinberg used the Coulomb gauge instead of the background gauge the number of parameters should be a gauge-invariant quantity. In the range of small r and up to linear terms in λ's the perturbed Higgs field looks like

$$\phi(z) \sim f_0(z^n - \lambda^{(n)} - \lambda^{(n-1)} z^1 - \cdots - \lambda^{(1)} z^{n-1}) \ , \tag{15}$$

where $z = x + iy$, $\lambda^{(k)} = \lambda_1^{(k)} + i\lambda_2^{(k)}$, and we have taken into account that, for small r, $f(r) \sim f_0 r^n$. When all the λ's are equal to zero we have n coincident vortices. In the case of nonzero $\lambda^{(p)}$ the asymptotics takes a simple form:

$$\phi(z) \sim f_0(z^p - \lambda^{(p)}) z^m \ , \quad m + p = n \ . \tag{16}$$

We can see that p zeros of the Higgs field have been moved from the original coincident position ($z = 0$) to the pth order roots of $\lambda^{(p)}$. The geodesics on the moduli space in which $+\lambda^{(p)}$ turns to $-\lambda^{(p)}$ corresponds, in more physical terms, to such a head-on collision that the p zeros of the Higgs field shrink to the coincident position and then the whole configuration reappears but rotated by an angle of π/p with respect to the initial configuration. This way of reasoning, originally applied to the case of the head-on collision of CP^1 solitons [1], is essentially the same as that of Ruback [7] in the special case of $p = 2$ and $m = 0$. We think this reformulation to be much more manifest.

We have implicitly assumed that in the geodesic motion $\lambda^{(p)}$ moves along a straight line in the complex plane. To verify this assumption let us promote the λ's in Eq. (12) to the role of collective coordinates and following Manton's idea of adiabatic approximation [6] let us work out the form of the effective Lagrangian [7]:

$$L_{\text{eff}} = \frac{1}{2}\int d^2x (\partial_t \phi \partial_t \phi^\star + \partial_t A_+ \partial_t A_-) \ . \tag{17}$$

For the fluctuation of the Higgs field due to Eq. (12) and the accompaning fluctuation of the gauge field, Eq. (10), we can work out the effective Lagrangian as

$$L_{\text{eff}} = \sum_{k=1}^{n} L^{(k)} \frac{d\lambda_{(k)}}{dt}\frac{d\lambda_{(k)}^\star}{dt} \ , \tag{18}$$

where

$$L^{(k)} \equiv \pi n^2 \int_0^\infty r dr \left[f^2 H_{(k)}^2 + \left(\frac{k}{r} H_{(k)} + \frac{d}{dr} H_{(k)}\right)^2 \right] \ , \tag{19}$$

where we have preserved only terms up to the second order in the time derivatives and we have to keep in mind that this Lagrangian is valid only for small λ's or when the vortices are almost coincident. Thus when we use λ's as local coordinates on the moduli space its metrics is diagonal near the point where the n vortices coincide. Particular λ's are independent from each other and they really move along straight lines as it has been anticipated above. That it is not always the case we can see in an example of Chern-Simons vortices where additional charge-flux interaction is present [11]. What we have still to assume is that the splitting configuration (16) can be extended to an asymptotic regime of largely separated vortices where the positions of vortices can be unambiguously identified with the positions of the zeros of the

Higgs field. If this assumption is correct, as we believe it is thanks to extensive numerical studies [12,13] in the case of $n = 2$, then we can conclude that after the head-on collision of n vortices the asymptotic configuration reappears but rotated by an angle of π/n with respect to the initial one, even when there is an additional vortex in the center of scattering of arbitrary topological index.

The scattering in which the n vortices collide symmetrically and simultaneously will occur very rarely even in a multivortex system of very large density. We have investigated it just as a very characteristic case which can give us an idea of what happens when n vortices scatter almost in the head-on fasion but with the help of the effective Lagrangian (18) it is possible to investigate the generic collision of n vortices without such a lot of symmetry. The only limitation is that all of the λ's have to be small. In other words our analysis is reasonable when the n vortices pass simultaneously through the coincident configuration. Some of the more complicated scattering events can be roughly approximated by a series of simultaneous collisions. The general simultaneous collision at time say $t = 0$ is described by initial conditions: $\lambda^{(k)}(0) = 0$ and $\frac{d}{dt}\lambda^{(k)}(0) \equiv u^{(k)}$, where the complex $u^{(k)}$'s are a set of initial velocities. For small t and z we obtain an asymptotics of the Higgs field

$$\phi(t,z) \sim f_0(z^n - tu^{(n)} - tu^{(n-1)}z^1 - \cdots - tu^{(1)}z^{n-1})$$

$$\equiv f_0 \prod_{s=1}^{n}[z - R_{(s)}(t)] \; . \quad (20)$$

The zeros of the Higgs field have been moved from a coincident position at $z = 0$ to the time-dependent roots $R_{(s)}(t)$ of the above polynomial. In a general case of n vortices it is difficult to establish these roots analytically but it can be easily done numerically. Thus Eq. (20) provides a basis for a general classification of multivortex simultaneous collisions.

III. STRING MODEL AND THE METRIC

Now we will show that the same results can be obtained with a help of the string model derived in [9] to describe short range interactions of vortices. Let us imagine n vortices at the positions $R(t)\exp[i\Theta(t) + ik\frac{2\pi}{n}]$, where $k = 0, ..., (n-1)$. The equation of motion of the first ($k = 0$) vortex [9] reads

$$\ddot{R}^i = 2\frac{\partial\bar{\psi}}{\partial x^i} + 2\varepsilon^{ij}\frac{\partial\bar{\varphi}}{\partial x^j} + 2\varepsilon^{ij}\dot{R}^j\frac{\partial\bar{\varphi}}{\partial t} + 2\dot{R}^i\frac{\partial\bar{\psi}}{\partial t} \quad , \quad (21)$$

where A, B take values 1,2 and denote x, y. The fields on the rhs of the equation are due to the other vortices: $\exp[-\bar{\psi} + i\bar{\varphi}] = \phi_1 \cdots \phi_{n-1}$. The ϕ_k's are taken as

$$\phi_k = \frac{f(2L_k)}{2L_k}\{[x - X_k(t)] + i[y - Y_k(t)]\} \ , \quad (22)$$

where f is a profile of the index-one vortex from Eq. (3), L_k is an actual distance between the position of the kth vortex and those of the first one, while X_k and Y_k are the Cartesian coordinates of the kth vortex. Thus we have n drifting elementary vortices. The gauge fields are neglected in Eq. (21) because at small separations of vortices their influence on the motion is relatively weak. After a little algebra in which it is convenient to use the function $F(x) = \frac{f(x)}{x}$ one obtains the equations of motion of the vortex core (zero of the Higgs field):

$$\ddot{R} - nR\dot{\Theta}^2 = -(n-1)\frac{\dot{R}^2}{R} \ , \quad (23)$$

$$R\ddot{\Theta} + 2n\dot{R}\dot{\Theta} = 0 \ . \quad (24)$$

The terms with $\frac{F'}{F}$ are neglected, they are small when $R \to 0$. It is straightforward to solve these equations. From the second one we obtain the first constant of motion: the angular momentum $J = R^{2n}\dot{\Theta}$. Substitution of this to Eq. (23) and integration yields the form of the energy: $K = R^{2(n-1)}\dot{R}^2 + \frac{J^2}{R^{4n}}$ or $K = R^{2(n-1)}[\dot{R}^2 + R^2\dot{\Theta}^2]$. The knowledge of these constants of motion enables us to find the trajectory:

$$R(\Theta) = \left\{\frac{J^2}{K}\left[1 + \tan^2\left(n\Theta - \frac{\pi}{2}\right)\right]\right\}^{\frac{1}{2n}} \ . \quad (25)$$

The difference between $\Theta_{\rm in}$ and $\Theta_{\rm out}$ is equal to $\frac{\pi}{n}$. This trajectory is expected [9] to be exact in the limit of very small R. Thus when $J \to \pm 0$ the vortex starts to be backward scattered by an angle of $\pm\frac{\pi}{n}$. Apparently there is a discontinuity, but the n vortices are indistiguishable. This reasoning can be applied to any of vortices, so the backward scatterings of any individual vortex by angles of $\pm\frac{\pi}{n}$ have to be identified. Thus the result is the same as that obtained with the moduli space approximation: the whole configuration shrinks to zero and then reappears rotated by an angle of $\frac{\pi}{n}$. We can also find how the position of the zero of the Higgs field depends on time:

$$R(t) = \left(\frac{J^2}{K} + n^2Kt^2\right)^{\frac{1}{2n}} \ , \quad (26)$$

$$\Theta(t) = \frac{\pi}{2n} + \frac{1}{n}\arctan\left(\frac{nKt}{J}\right) \ . \quad (27)$$

With the help of Eq. (26) and the definition of K one can also find the dependence of the velocity on time:

$$v^2 = \dot{R}^2 + R^2\dot{\Theta}^2 = \frac{K}{[\frac{J^2}{K} + n^2Kt^2]^{(1-\frac{1}{n})}} \ . \quad (28)$$

The velocity attains its maximal value at the minimal separation of vortices ($t = 0$). This maximal value in the limit of the head-on collision ($J \to 0$) tends to infinity, quite as in the case of two vortices [12,13,9].

Let us note that the equations of motion (23) and (24) are the Euler-Lagrange equations to the Lagrangian

$$L = R^{2(n-1)}[\dot{R}^2 + R^2\dot{\Theta}^2] \ , \quad (29)$$

so they can be interpreted as the equations of motion of the nonrelativistic point particle on a curved surface. If one identifies $\lambda^{(n)} = R^n e^{in\theta}$ the above Lagrangian will be the same as that obtained in the moduli space approximation (18).

The analysis of the case with m vortices at the center of scattering and the head-on collision of p other vortices with the help of the string model also gives the same result as in the moduli space approach. The source of this coincidence can be traced to the equivalence of the moduli space and string approximations in the case of small separations of vortices. The construction of the string model for short range interactions of vortices [9] in the case of parallel strings essentially amounts to an ansatz on the Higgs field

$$\phi(t,z) = f_0 \prod_{k=1}^{n} [z - R_{(k)}(t)] , \qquad (30)$$

where $R_{(k)}$ is an actual position of the kth vortex on the complexified plane. This ansatz can be compared with the field in Eq. (22) and with Eq. (20). The second step in the construction of the string model [9] is to find such a time evolution of $R_{(k)}$'s that this approximate ansatz satisfies the field equations at the zeros of the Higgs field. This yields an equation of motion of the vortex core like that in Eq. (21). So far as the ansatz is an analytic function in z we can expect it also to be a good approximation of the exact solution in certain neighborhoods of the zeros of the Higgs field. As the zeros are very close to one another it provides a good approximation to the exact solution in the whole region containing all of the zeros or, in other words, for small z. The effective Lagrangian (18) is based on the same approximation of the exact solution so the results of the two approaches have to be the same.

The string model can also be applied to the analysis of the general n vortex simultaneous collision. Its advantage as compared to the application of Eq. (20) is that it uses directly the positions of the zeros of the Higgs field. The price we have to pay for it is that we have to solve the equations of motion of the vortex cores (21) while in the former method they have been already solved by diagonalization of the moduli space metrics only the positions of the zeros are related in a rather involved way to λ's.

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