

Chiral symmetry breaking in the Nambu–Jona-Lasinio model in curved spacetime with a nontrivial topology

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We discuss the phase structure (in the $1/N$ expansion) of the Nambu–Jona-Lasinio model in curved spacetime with nontrivial topology $\mathcal{M}^3 \times S^1$. The evaluation of the effective potential of the composite field $\bar{\psi}\psi$ is presented in the linear curvature approximation (topology is treated exactly) and in the leading order of the $1/N$ expansion. The combined influence of topology and curvature to the phase transitions is investigated. It is shown, in particular, that at zero curvature and for small radius of the torus there is a second order phase transition from the chiral symmetric to the chiral nonsymmetric phase. When the curvature grows and (or) the radius of S^1 decreases, then the phase transition is in general of first order. The dynamical fermionic mass is also calculated in a number of different situations.

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I. INTRODUCTION

The study of composite fermionic fields, and effects related to them, in the very early Universe has attracted much attention in the physical community. In order to be able to do such kind of research, one must first develop an effective action formalism for composite fields in a general curved spacetime (for an introduction to this subject, see [1]). However, even in flat space only very few models are known which can be treated analytically, when studying the composite bound states. The Nambu–Jona-Lasinio (NJL) model [2] (for a recent discussion, see [3]) belongs to such an interesting class. This model is usually discussed in frames of the $1/N$ expansion (see, for example, [4]), which is a very useful scheme to study nonperturbative effects. The NJL model may be considered as an effective field theory for QCD in some region, and it is also connected with the theory of superconductivity (related models with quite interesting properties were studied some time ago [5]). One can explicitly discuss the dynamical chiral symmetry breaking for the NJL model in the $1/N$ expansion, where a vacuum condensate $\langle \bar{\psi}\psi \rangle \neq 0$ appears and a dynamical fermionic mass is generated.

Recently [6, 7], a detailed study of the NJL model in curved spacetime has been started. The one-loop effec-

tive potential in the $1/N$ expansion and in the linear curvature approximation has been calculated and the existence of a curvature-induced first-order phase transition from a chiral symmetric to a chiral nonsymmetric phase has been shown [7] (for a review on curvature-induced phase transitions for elementary fields in grand unified theories (GUT's), see, for example, [1]). However, one might expect that the very early Universe had a nontrivial topology and (or) that it was very hot. This renders it interesting to investigate the phenomenon of dynamical chiral symmetry breaking in curved spacetime with nontrivial topology.

The present work is specifically devoted to the study of the NJL model in a curved spacetime of the form $\mathcal{M}^3 \times S^1$, where S^1 is the one-dimensional sphere and \mathcal{M}^3 a three-dimensional, arbitrarily curved manifold of trivial topology. Here, we will only consider the standard choice of boundary conditions (periodic and antiperiodic) for the fermion on the one-dimensional sphere S^1 [8]. However, more general choices of the boundary conditions are, of course, possible [9], in particular those in which the fermion ψ acquires an extra phase $\exp(2\pi\varphi)$ with arbitrary φ (see the second reference in [9]) each time it goes along S^1 . The $\varphi = 0$ case corresponds to periodic boundary conditions and $\varphi = 1/2$ corresponds to antiperiodic boundary conditions.

In the next section we calculate the effective potential for the NJL model in the spacetime $\mathcal{M}^3 \times S^1$, in the $1/N$ expansion and for the linear curvature approximation. In Sec. III, the phase structure of the effective potential is discussed and the dynamically generated fermionic mass is calculated. The existence of topology- and curvature-induced phase transitions from the chiral symmetric phase to the nonsymmetric one is established for different regions of the parameters of the theory.

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II. EFFECTIVE POTENTIAL FOR THE NJL MODEL

Let us start the calculation of the effective potential for the NJL model in the $1/N$ expansion. The classical action of the theory in curved spacetime is given by

$$S = \int d^4x \sqrt{g} \left\{ \bar{\psi} i \gamma^\mu(x) \nabla_\mu \psi + \frac{\lambda}{2N} \left[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right] \right\}, \quad (1)$$

where N is the number of fermions and the rest of the notation is standard [6, 7]. It is more useful in actual calculations to work with the following equivalent action, in which the auxiliary fields σ and π are introduced:

$$S = \int d^4x \sqrt{g} \left[\bar{\psi} i \gamma^\mu(x) \nabla_\mu \psi - \frac{N}{2\lambda} (\sigma^2 + \pi^2) - \bar{\psi} (\sigma + i \gamma_5 \pi) \psi \right]. \quad (2)$$

Already in flat spacetime it is known that the global chiral symmetry—which is the classical symmetry of the theory (1) or (2)—is spontaneously broken when the coupling constant λ exceeds some critical value λ_c . The contribution of the external gravitational field to this effect has been discussed in Ref. [7]. Our purpose here will be to determine the influence of combined effects, namely, of external gravity and nontrivial topology, simultaneously, on the dynamical chiral symmetry breaking and restoration. Notice that such a study cannot be done for the most simple low-dimensional analogue of the NJL model, namely, the $D = 2$ Gross-Neveu model [10], or for the—a little more complicated— $D = 2$ Schwinger [11] or $D = 2$ Thirring [12] models. In all those cases one can study only the isolated effects of either nontrivial topology (nonzero temperature [13]) or external gravity [14] (see also [1]) on the chiral symmetry-breaking pattern.

Let us now proceed with the explicit calculations. First of all, we introduce the semiclassical effective action S_{eff} as

$$Z[0, 0] = \int D\sigma D\pi \exp(iNS_{\text{eff}}), \quad (3)$$

where Z is defined, in the usual way, to be the generating functional

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} D\sigma D\pi \exp(iS + i\bar{\eta}\psi + i\bar{\psi}\eta). \quad (4)$$

This S_{eff} is trivially solved by integrating over the variables ψ and $\bar{\psi}$, what amounts to a simple Gaussian integration, namely,

$$S_{\text{eff}} = - \int d^4x \sqrt{g} \left[\frac{1}{2\lambda} (\sigma^2 + \pi^2) + i \ln \det [i \gamma^\mu(x) \nabla_\mu - (\sigma + i \gamma_5 \pi)] \right]. \quad (5)$$

Moreover, S_{eff} is the leading term in the large- N expansion of the effective action Γ .

The effective potential $V(\sigma, \pi)$ is defined by $V(\sigma, \pi) = -\Gamma(\sigma, \pi)/(N \times \text{volume})$, with constant configurations for the fields. Thus, to leading order in the $1/N$ expansion, we have

$$V(\sigma, \pi) = \frac{1}{2\lambda} (\sigma^2 + \pi^2) + i \text{Sp} \ln \langle x | [i \gamma^\mu(x) \nabla_\mu - (\sigma + i \gamma_5 \pi)] | x \rangle. \quad (6)$$

The second term is directly related to the Green function given by the equation

$$[i \gamma^\mu(x) \nabla_\mu - s] S(x, y) = \delta^4(x, y),$$

where δ refers to the scalar, coordinate-independent Dirac δ functional for the given manifold. This fact can be seen by directly applying the operator expression

$$\ln \left(\frac{A - s}{A} \right) = - \int_0^s dr \frac{1}{A - r}$$

to (6).

To calculate the second term of the left-hand side (LHS) of Eq. (6), we make use of Schwinger's proper-time method. First, we write

$$V(\sigma, \pi) = \frac{1}{2\lambda} (\sigma^2 + \pi^2) - i \text{Sp} \ln \langle x | (i \gamma^\mu \nabla_\mu - s)^{-1} | x \rangle, \quad (7)$$

and using the known expression for the propagator of a free Dirac field in a weakly varying gravitational background (which has been obtained in terms of the Riemann normal coordinate expansion [15]), we will then write the effective potential with accuracy up to linear curvature terms. Notice also that a summation over the two inequivalent spin structures which are admitted by the spacetime $\mathcal{M}^3 \times S^1$ will be performed [8]. Of course, one can consider also the corrections corresponding to periodic and antiperiodic (nonzero temperature) boundary conditions independently. We will make a few comments about this point later.

Thus, the effective potential is found to be

$$V(\sigma, 0) = \frac{1}{2\lambda} \sigma^2 - i \text{tr} \int_0^\sigma ds \frac{1}{L} \sum_{n=-\infty}^{n=\infty} \sum_{p=0,1} \int \frac{d^3k}{(2\pi)^3} \left[(\gamma^a k_a + s) \frac{1}{k^2 - s^2} - \frac{1}{12} R (\gamma^a k_a + s) \frac{1}{(k^2 - s^2)^2} + \frac{2}{3} R_{\mu\nu} k^\mu k^\nu (\gamma^a k_a + s) \frac{1}{(k^2 - s^2)^3} - \frac{1}{2} \gamma^a \mathcal{T}^{cd} R_{cda\mu} k^\mu \frac{1}{(k^2 - s^2)^2} \right], \quad (8)$$

where $\mathcal{T}^{cd} = \frac{1}{4}[\gamma^c, \gamma^d]$ and where one should integrate over k^0, k^1, k^2 and sum over the coordinate k^3 , which is given by $k^3 = (2n + \delta_{p,1})\pi/L$. In expression (8), tr only refers to the spinor indices. We have set $\pi = 0$, since there is a rotational symmetry in the fields σ and π , so that it is enough to discuss the $\sigma \neq 0$ case for the effective potential only.

Integration over s is immediate. To perform the momentum integration, one first makes the Wick rotation ($k^0 = ik^4$) and puts then a cutoff to regularize the resulting expressions. In our case, we simply restrict

$$(k^4)^2 + (k^1)^2 + (k^2)^2 \leq \Lambda^2,$$

so that our cutoff is different, when compared with the cutoff for the case of trivial topology [7].

From now on, we shall call V_1 the contribution to the effective potential which comes from the logarithm of the determinant of the operator which appears in Eq. (5). After carrying out the integrations over s and the momenta, we are led to the following expression for the contribution to V_1 coming from the $p = 0$ case, namely, purely periodic boundary conditions [this corresponds to the contribution to $V(\sigma, 0)$ obtained by taking only the $p = 0$ term in Eq. (8)]. Of course, we want to study V_1 , which is given by the sum over the two values of p , but—as we discuss below— V_1 may be written down immediately once the $p = 0$ contribution has been worked out. We obtain

$$\begin{aligned} V_1^{p=0} = & -\frac{1}{L} \sum_{n=-\infty}^{n=\infty} \left\{ \frac{1}{3\pi^2} \Lambda^3 \ln \left(1 + \frac{\sigma^2}{\Lambda^2 + \frac{4\pi^2 n^2}{L^2}} \right) \right. \\ & + \frac{2}{3\pi^2} \left[\sigma^2 \Lambda - \left(\frac{4\pi^2 n^2}{L^2} + \sigma^2 \right)^{3/2} \arctan \left(\frac{\Lambda}{\sqrt{\frac{4\pi^2 n^2}{L^2} + \sigma^2}} \right) + \left(\frac{2\pi n}{L} \right)^3 \arctan \left(\frac{\Lambda L}{2\pi n} \right) \right] \\ & + \frac{R}{3(2\pi)^2} \left[\frac{2\pi n}{L} \arctan \left(\frac{\Lambda L}{2\pi n} \right) - \sqrt{\frac{4\pi^2 n^2}{L^2} + \sigma^2} \arctan \left(\frac{\Lambda}{\sqrt{\frac{4\pi^2 n^2}{L^2} + \sigma^2}} \right) \right] \\ & \left. - \frac{R}{2(3\pi)^2} \left[\frac{\left(\frac{2\pi n}{L} \right)^2 \Lambda}{\Lambda^2 + \frac{4\pi^2 n^2}{L^2}} - \frac{\left(\frac{4\pi^2 n^2}{L^2} + \sigma^2 \right) \Lambda}{\frac{4\pi^2 n^2}{L^2} + \sigma^2 + \Lambda^2} \right] \right\}. \end{aligned} \quad (9)$$

To simplify this expression we use standard techniques drawn from complex analysis, such as the expression

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2\pi n}{L}i\right) = \frac{L}{2\pi i} \int_{-i\infty}^{i\infty} dp \frac{1}{2} [f(p) + f(-p)] + \frac{L}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp \frac{f(p) + f(-p)}{\exp(Lp) - 1}. \quad (10)$$

One has just to identify the function f for each term in Eq. (9) and then perform the integrals in (10). Another remark is in order here: When computing the term

$$-\frac{2}{L} \frac{1}{3\pi^2} \sum_{n=-\infty}^{\infty} \left[\sigma^2 \Lambda - \left(\frac{4\pi^2 n^2}{L^2} + \sigma^2 \right)^{3/2} \arctan \left(\frac{\Lambda}{\sqrt{\frac{4\pi^2 n^2}{L^2} + \sigma^2}} \right) + \left(\frac{2\pi n}{L} \right)^3 \arctan \left(\frac{\Lambda L}{2\pi n} \right) \right],$$

it is better to rewrite it as

$$\begin{aligned} -\lim_{\sigma' \rightarrow 0} \frac{2}{L} \frac{1}{3\pi^2} \sum_{n=-\infty}^{\infty} \left[\sigma^2 \Lambda - \left(\frac{4\pi^2 n^2}{L^2} + \sigma^2 \right)^{3/2} \arctan \left(\frac{\Lambda}{\sqrt{\frac{4\pi^2 n^2}{L^2} + \sigma^2}} \right) \right. \\ \left. - \sigma'^2 \Lambda + \left(\frac{4\pi^2 n^2}{L^2} + \sigma'^2 \right)^{3/2} \arctan \left(\frac{\Lambda}{\sqrt{\frac{4\pi^2 n^2}{L^2} + \sigma'^2}} \right) \right], \end{aligned}$$

since now the expression within square brackets satisfies the properties which justify the use of Eq. (10).

As we said before, once $V_1^{p=0}$ has been computed, one may write immediately $V_1^{p=1}$ and V_1 , which is given (by definition) by $V_1 = V_1^{p=0} + V_1^{p=1}$, because

$$\frac{1}{L} \sum_{n=-\infty}^{\infty} F\left(\frac{2n+1}{L}\right) = \frac{1}{L} \sum_{n=-\infty}^{\infty} F\left(\frac{n}{L}\right) - \frac{1}{L} \sum_{n=-\infty}^{\infty} F\left(\frac{2n}{L}\right) \quad (11)$$

and, then,

$$\frac{1}{L} \sum_{n=-\infty}^{\infty} \left[F\left(\frac{2n+1}{L}\right) + F\left(\frac{2n}{L}\right) \right] = \frac{1}{L} \sum_{n=-\infty}^{\infty} F\left(\frac{n}{L}\right). \quad (12)$$

Now it is apparent that the physics displayed by the model in the case of purely periodic boundary conditions and in the one where we consider both spin structures will be essentially the same, since

$$V_{1,L} = V_{1,L}^{p=0} + V_{1,L}^{p=1} = 2V_{1,2L}^{p=0}.$$

Thus we see that both cases are related by a trivial rescaling of the length and an overall factor which multiplies V_1 . That is quite an appealing result.

However, the last remark does not apply to the case of purely antiperiodic boundary conditions, where we only take $p = 1$, which gives the thermodynamics of a system in three-dimensional space with trivial topology. In fact, from the last expression we get

$$V_{1,L}^{p=1} = V_{1,L} - V_{1,L}^{p=0} = V_{1,L} - \frac{1}{2}V_{1,\frac{l}{2}}.$$

Henceforth, we shall concentrate below only on the analysis of the case in which both spin structures are taken into account, as they appear in Eq. (8).

The first term on the RHS of Eq. (10) may be computed without difficulty in all cases. The second term is, in general, rather more involved. In order to simplify this contribution as much as possible in the cases which come from Eq. (9), one should pay careful attention to the determination of the integrand along the contour of integration. After some work, the final result is found to be

$$\begin{aligned} \frac{V_1}{\Lambda^4} = & -\frac{2}{3\pi^2} \left[1 - \sqrt{1+x^2} + \frac{1}{l} \ln \left(\frac{\exp(2l\sqrt{1+x^2}) - 1}{\exp(2l) - 1} \right) \right] \\ & + \frac{1}{6\pi^2} \left[\sqrt{1+x^2} - 1 - \frac{3}{2}x^2\sqrt{1+x^2} + \frac{3}{2}x^4 \operatorname{arcsinh} \left(\frac{1}{x} \right) \right] \\ & + \frac{4}{3\pi^2} \left[\int_x^{\sqrt{1+x^2}} d\tau \frac{(\tau^2 - x^2)^{3/2}}{\exp(2l\tau) - 1} - \int_0^1 d\tau \frac{\tau^3}{\exp(2l\tau) - 1} \right] \\ & + \frac{r}{6(2\pi)^2} \left[1 - \sqrt{1+x^2} + x^2 \operatorname{arcsinh} \left(\frac{1}{x} \right) \right] \\ & + \frac{2r}{3(2\pi)^2} \left[\int_0^1 d\tau \frac{\tau}{\exp(2l\tau) - 1} - \int_x^{\sqrt{1+x^2}} d\tau \frac{\sqrt{\tau^2 - x^2}}{\exp(2l\tau) - 1} \right] \\ & - \frac{r}{2(3\pi)^2} \left(1 - \frac{1}{\sqrt{1+x^2}} \right) + \frac{r}{(3\pi)^2} \left[\frac{1}{\sqrt{1+x^2} \exp(2l\sqrt{1+x^2}) - 1} - \frac{1}{\exp(2l) - 1} \right], \end{aligned} \quad (13)$$

where $x = \sigma/\Lambda$, $l = L\Lambda$, and $r = R/\Lambda^2$.

The value of the field σ which satisfies the gap equation

$$\frac{\partial}{\partial \sigma} V(\sigma, 0) = 0$$

gives a dynamical mass to the fermions. This last equation, when written in terms of the natural variables x , l , r , and c ($c \equiv \lambda\Lambda^2$), reads

$$\begin{aligned} 0 = \frac{V'(x)}{\Lambda^4} = & \frac{x^2}{2c} + \frac{5}{6\pi^2} \frac{x}{\sqrt{1+x^2}} - \frac{4}{3\pi^2} \frac{x}{\sqrt{1+x^2}} \frac{1}{1 - \exp(-2l\sqrt{1+x^2})} \\ & + \frac{x}{\pi^2} \left[x^2 \operatorname{arcsinh} \left(\frac{1}{x} \right) - \sqrt{1+x^2} \right] + \frac{1}{2\pi^2} \frac{x}{\sqrt{1+x^2}} \\ & + \frac{4}{3\pi^2} \left[\frac{x}{\sqrt{1+x^2}} \frac{1}{\exp(2l\sqrt{1+x^2}) - 1} - 3x \int_x^{\sqrt{1+x^2}} d\tau \frac{(\tau^2 - x^2)^{1/2}}{\exp(2l\tau) - 1} \right] \\ & + \frac{rx}{3(2\pi)^2} \left[\operatorname{arcsinh} \left(\frac{1}{x} \right) - \frac{1}{\sqrt{1+x^2}} \right] - \frac{r}{4(3\pi)^2} \frac{x}{(1+x^2)^{3/2}} \\ & - \frac{2rx}{3(2\pi)^2} \left[\frac{1}{\sqrt{1+x^2}} \frac{1}{\exp(2l\sqrt{1+x^2}) - 1} - \int_x^{\sqrt{1+x^2}} d\tau \frac{1}{\sqrt{\tau^2 - x^2}} \frac{1}{\exp(2l\tau) - 1} \right] \\ & - \frac{r}{2(3\pi)^2} \left[\frac{x}{(1+x^2)^{3/2}} \frac{1}{\exp(2l\sqrt{1+x^2}) - 1} - \frac{2l}{1+x^2} \frac{\exp(2l\sqrt{1+x^2})}{(\exp(2l\sqrt{1+x^2}) - 1)^2} \right]. \end{aligned} \quad (14)$$

It is not possible to produce an exact, analytical expression for the dynamically generated fermion mass. Therefore, we will calculate it below only in a number of limiting cases.

III. SMALL- L LIMIT

One gets convinced immediately that it is very difficult to go any further without doing some simplification. In this section we will consider the case $L\Lambda \ll 1$, and we will treat the opposite case $L\Lambda \gg 1$ in the next one.

Let us expand V/Λ^4 in powers of l . Assuming now that $l\sqrt{1+x^2} \ll 1$, we readily obtain

$$\begin{aligned} \frac{lV(x)}{\Lambda^4} &= \frac{x^2}{2g} - \frac{1}{3\pi^2} \ln(1+x^2) + \frac{2}{3\pi^2} \left(-x^2 + x^3 \operatorname{arcsec} \sqrt{1 + \frac{1}{x^2}} \right) \\ &+ \frac{r}{3(2\pi)^2} x \operatorname{arcsec} \sqrt{1 + \frac{1}{x^2}} + \frac{r}{2(3\pi)^2} \left(\frac{1}{1+x^2} - 1 \right) + O(l^2), \end{aligned} \quad (15)$$

where $g = \lambda\Lambda^2/l$. In order to study the phase space of the model, one has to compute also the derivative of the effective potential. That yields

$$\begin{aligned} \frac{lV'}{\Lambda^4} &= \frac{x}{g} - \frac{2}{3\pi^2} \frac{x}{1+x^2} + \frac{2}{\pi^2} x^2 \operatorname{arcsec} \sqrt{1 + \frac{1}{x^2}} - \frac{4}{3\pi^2} \frac{x^2}{1+x^2} \\ &+ \frac{r}{3(2\pi)^2} \operatorname{arcsec} \sqrt{1 + \frac{1}{x^2}} - \frac{r}{(3\pi)^2} \frac{x}{(1+x^2)^2} - \frac{r}{6\pi^2} \frac{x}{1+x^2}. \end{aligned} \quad (16)$$

A. Phase structure in the fixed-curvature case

We will here analyze the phase structure of the model in the case of fixed curvature. In particular, the situations of zero curvature and constant nonzero curvature will be considered.

1. Zero-curvature case

As the validity of our results is restricted by the condition $r \ll 1$, it is worthwhile studying first the case $r = 0$. Here it can be proven that it is impossible to have a first-order phase transition; in fact there is a second-order phase transition at $g_{\text{cr}} = \pi^2/2$. For $g > g_{\text{cr}}$ there is chiral symmetry breaking and the symmetry is restored for $g < g_{\text{cr}}$; in other words, for a given value of λ , the symmetry is restored when l grows beyond a critical point. In the broken phase the order parameter is

given by $x_{\text{br}} = \pi \left(\frac{2}{\pi^2} - \frac{1}{g} \right)$ or, in terms of the generated mass, $m_{\text{gen}} = \frac{2\Lambda}{\pi} - \frac{L\pi}{\lambda}$. It is straightforward to study the behavior of the value of the effective potential at x_{br} , for varying g or l , in this limit of small compactification length. One finds

$$V(\sigma_{\text{br}}) = -\Lambda^4 \frac{\pi^2}{6l} \left(\frac{2}{\pi^2} - \frac{1}{g} \right)^3.$$

This last expression has the appearance of some kind of dynamical dimensional reduction (see Refs. [16, 17]). In our case we see that the vacuum becomes less and less energetic as the compactification length shrinks to zero (see Fig. 1). In this limit we obtain

$$\lim_{L \rightarrow 0} LV(\sigma_{\text{br}}) = -\Lambda^3 \frac{4}{3\pi^4}.$$

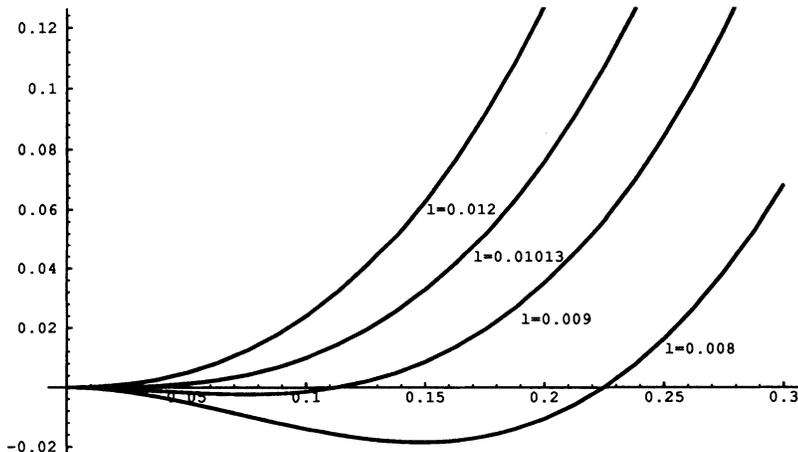


FIG. 1. Plot of the function V/Λ^4 for fixed $c = 0.05$, $r = 0$, and different values of l .

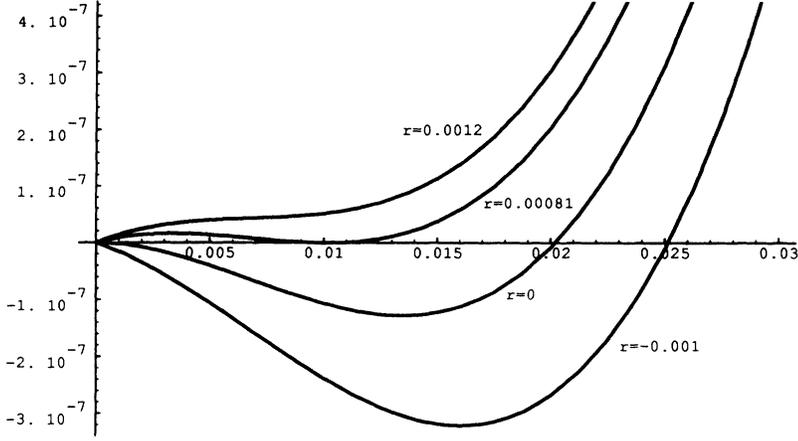


FIG. 2. Plot of V/Λ^4 for fixed $g = 5.0404$ (case $g > \pi^2/2$) and for different values of the curvature. It shows the discontinuous character of the phase transition in this case.

2. Influence of the curvature

Now we shall study the influence that the presence of curvature has on this behavior. It is immediate to notice that for negative values of the curvature, the symmetry is always broken [the slope of the effective potential is $r/(24\pi)$ at the origin; see Fig. 2]. Moreover, for fixed positive values of r (which is kept always small), as l grows there is a first-order phase transition: It has actually lost its continuous character. Now the critical value of the parameter g is

$$g_{\text{cr}} = \frac{\pi^2}{2} \left(1 + \frac{2\pi}{3} \sqrt{\frac{r}{8}} \right).$$

The approximation is consistent with the small-curvature limit. The difference between the values of the order parameter x in the broken and disordered phases at the phase transition is given by $x^2 = r/8$ (retaining again only the first correction coming from the curvature). The influence of curvature on chiral symmetry breaking in $D = 2$ fermionic models is very similar (see Refs. [1, 14]).

B. Phase structure in the case of fixed compactification length

We can now, on the other hand, study the situation when g is held fixed and the curvature takes on different values.

1. Case $g < \pi^2/2$

If $g < \pi^2/2$, there is a continuous phase transition at $r = 0$ (the symmetry is broken for negative values of r and is restored for positive curvature). As r approaches 0 from below, the order parameter tends to zero according to the expression

$$x = \frac{|r|}{24\pi\Delta},$$

or $m_{\text{gen}} = \frac{\Lambda^{-1}|R|}{24\pi\Delta}$, being $\Delta \equiv \frac{1}{g} - \frac{2}{\pi^2}$. In deriving this result we assume that $|r| \ll 1$ and that $|r| \ll \Delta$.

2. Case $g > \pi^2/2$

To finish this section, we consider the case $\Delta < 0$ or, equivalently, $g > \pi^2/2$. To be consistent with the requirement of small curvature near the critical point, we have to assume now that $|\Delta| \ll 1$. In this setting one may expect that, as r grows, there will be a phase transition from the ordered to the disordered phase at some positive value of the curvature (see Fig. 2). Retaining again the first terms of the expansions only, we obtain

$$r_{\text{cr}} = \frac{(3\pi)^2}{2} \Delta^2,$$

and the value of the order parameter at the transition is (for the ordered phase)

$$x = \frac{3\pi}{4} |\Delta|.$$

IV. LARGE- L LIMIT

We shall here consider again the phase structure in the fixed-curvature case but corresponding now to the opposite limit, when L is large.

A. Zero-curvature case

It is easy to see from Eq. (14) that, dropping exponentially vanishing contributions, one may approximate it with

$$\frac{V(x)}{\Lambda^4} = \frac{x^2}{2c} - \frac{1}{4\pi^2} \left[2 \left(\sqrt{1+x^2} - 1 \right) + x^2 \sqrt{1+x^2} - x^4 \operatorname{arcsinh} \frac{1}{x} \right], \quad (17)$$

where c is, as before, $c = \lambda\Lambda^2$.

Let us consider the gap equation $V'(x) = 0$. Differentiation of Eq. (17) yields

$$\frac{V'(x)}{\Lambda^4} = \frac{x}{\pi^2} \left(\frac{\pi^2}{c} - \sqrt{1+x^2} + x^2 \operatorname{arcsinh} \frac{1}{x} \right). \quad (18)$$

From here it is trivial to see that the symmetry is broken

when

$$\lambda > \frac{\pi^2}{\Lambda^2} .$$

Otherwise, the symmetry is respected by the vacuum.

B. Influence of the curvature

If one takes into account the presence of a background gravitational field, one can check that the terms missing from Eqs. (17) and (18) are such that now

$$\begin{aligned} \frac{V(x)}{\Lambda^4} = & \frac{x^2}{2c} - \frac{1}{4\pi^2} \left[2 \left(\sqrt{1+x^2} - 1 \right) + x^2 \sqrt{1+x^2} - x^4 \operatorname{arcsinh} \frac{1}{x} \right] \\ & + \frac{r}{6(2\pi^2)^2} \left(1 - \sqrt{1+x^2} + x^2 \operatorname{arcsinh} \frac{1}{x} \right) - \frac{r}{2(3\pi)^2} \left(1 - \frac{1}{\sqrt{1+x^2}} \right) \end{aligned} \tag{19}$$

and

$$\frac{V'(x)}{\Lambda^4} = \frac{x}{\pi^2} \left[\frac{\pi^2}{c} - \sqrt{1+x^2} + x^2 \operatorname{arcsinh} \frac{1}{x} + \frac{r}{12} \left(\operatorname{arcsinh} \frac{1}{x} - \frac{1}{\sqrt{1+x^2}} \right) - \frac{r}{18(1+x^2)^{3/2}} \right] . \tag{20}$$

1. Case $\lambda < \pi^2/\Lambda^2$

In this situation one may expect that there will be a continuous phase transition. After a short calculation, expanding Eq. (20) around the origin and keeping only the two first leading contributions to $V'(x)$, one sees that, very near the phase transition, the order parameter is found by solving

$$0 = \frac{\pi^2}{c} - 1 - \frac{5}{36}r + \frac{r}{12} \operatorname{arcsinh} \frac{1}{x} + O \left(x^2 \operatorname{arcsinh} \frac{1}{x} \right) .$$

Thus, in a first approximation,

$$x = \left[\sinh \left(\frac{12}{|r|} (\pi^2/c - 1) - \frac{5}{3} \right) \right]^{-1}$$

(remember that this is valid for negative r). The symmetry is restored for positive values of r .

2. Case $\lambda > \pi^2/\Lambda^2$

Here, a quick analysis of the previous expressions tells us that in this case there is a first-order phase transition, from the disordered to the ordered phase, as the curvature grows beyond a positive critical value. Notice that this result is very much like the one that one obtains in the small- L limit (see Fig. 3).

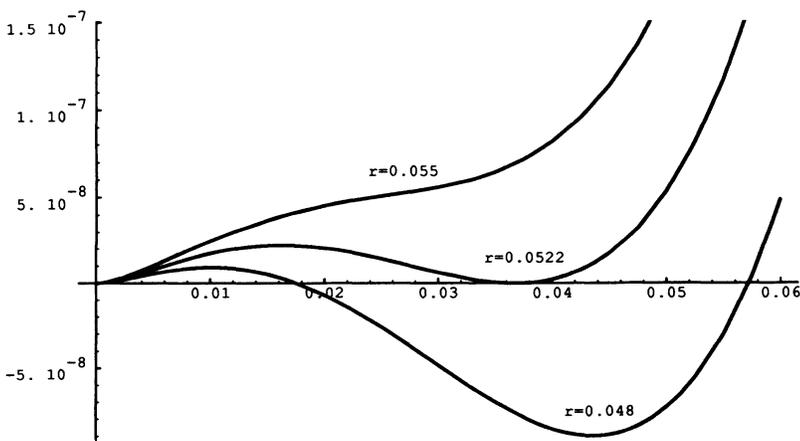


FIG. 3. Plot of the effective potential in the large- L limit, corresponding to the case $\lambda > \pi^2/\Lambda^2$ (c is held fixed at $c = \pi^2 + 0.15$) for different values of r . The discontinuous character of the phase transition is clearly seen.

V. CONCLUSIONS

Recently there has been an increased interest in the NJL model, in connection with the dynamical symmetry breaking of the electroweak interactions, using the top quark condensate as order parameter [3, 18]. Here, we have investigated the phase structure of this model in a curved spacetime with nontrivial topology—using the $1/N$ expansion and working in the linear-curvature approximation—and we have explicitly shown the possibility of curvature- and (or) topology-induced phase transitions from a chiral symmetric phase to a chiral nonsymmetric one. In our approximate analysis, we have discussed the composite field effective potential where summation over two inequivalent spin structures has been performed. Of course, one can get the results corresponding to purely periodic (or antiperiodic) boundary conditions as some particular cases of the above study.

There are different possibilities to extend our analysis.

First, one can consider different topologies, for example, $\mathcal{M}^2 \times T^2$ or the hyperbolic one $\mathcal{M}^2 \times \mathcal{H}^2/\Gamma$. Second, it would be of interest to study the same problems treating the external gravitational field exactly. In particular, one possibility is to take the de Sitter space S_4 for such a calculation. These considerations may draw some connection with renormalized quantum gravity in the $1/N$ expansion (see [19]) that could be certainly important for quantum cosmology near the Planck scale. We plan to discuss these questions in the near future.

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