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Instantons in the Schwinger model

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The known calculations of the fermion condensate $\langle \bar{\psi}\psi \rangle$ and the correlator $\langle \bar{\psi}\psi(x)\ \bar{\psi}\psi(0) \rangle$ have been interpreted in terms of *localized* instanton solutions presenting the minima of path integrals *with quantum corrections being taken into account.* Their size is of the order of the massive photon Compton wavelength μ^{-1}. At high temperature, these instantons become quasistatic and present the two-dimensional analog of the "walls" found recently in four-dimensional gauge theories. In spite of the static nature of these solutions, they should not be interpreted as "thermal solitons" living in Minkowski space: the mass of these would-be solitons does not display itself in the physical correlators. At small but nonzero fermion mass, the high-T partition function of two-dimensional QED is saturated by the rarified gas of instantons and antiinstantons with density $\propto m \exp\{-S^{\rm inst}\} = m \exp\{-\pi T/\mu\}$ to be confronted with the dense strongly correlated instanton-antiinstanton liquid saturating the partition function at $T=0$.

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I. MOTIVATION

The appearance of instantons (topologically nontrivial Euclidean configurations minimizing the action) is a very common feature of many different field theories [1]. Of particular interest are the instantons in QCD [2] which are so beautiful that a serious hope existed [3] and still exists [4] that, working with instantons, one can perceive the essential features of the QCD vacuum state.

However, soon after the discovery of instantons, it was understood that the *naive* instanton calculations meet problems. The problem is that the quasiclassical approximation on which these calculations are based just does not work in QCD—everything depends on the instantons of large sizes where quantum corrections are extremely important. It is impossible to calculate analytically these quantum corrections in QCD for $\rho \sim \Lambda_{\rm QCD}^{-1}$ (though for smaller ρ where the quantum effects are still under control and can be considered as a perturbation, it is possible [5]).

Generally, the situation is much better, however, at high temperatures $T \gg \Lambda_{\rm QCD}$. The high-T instanton action $S^I = 8\pi^2/g^2(T)$ is large and the quasiclassical approximation works: the amplitude of quantum fluctuations is small compared to the amplitude of the classical field. As the temperature goes up, the instantons cool down. That allows one to perform some explicit instanton calculations in high-T QCD which are under control, e.g., the fermion condensate in high-T QCD with *one* quark flavor can be found [6].

The *pure* Yang-Mills theory involves, besides instantons, also planar topologically nontrivial Euclidean field configurations. They appear due to nontrivial $\pi_1[\mathcal{G}] = Z_N$ where the true gauge group $\mathcal{G}$ for the pure Yang-Mills theory is $SU(N)/Z_N$ rather than just $SU(N)$ (gluon fields belong to the adjoint representation and are not transformed under the action of the elements of the center—see Ref. [7] for a detailed discussion). The action of these configurations involves the large area factor $\sim L^2$ where L is the size of the box, but, if the size of the box is finite (as is always so in practical numerical calculations in QCD), these topologically nontrivial sectors (known also as 't Hooft fluxes [8]) do contribute in the partition function.

At low temperatures, the quasiclassical approximation does not work, and little can be said about the properties of these configurations. But at high temperatures, it works and the characteristic field configurations in the path integral present the classical wall-like solutions with the width of order of the Debye screening length $\sim (gT)^{-1}$ and the surface action density[1]

$$\frac{S^{SU(N)}}{\mathcal{A}} = \frac{4\pi^2(N-1)T^2}{3\sqrt{3N}g}, \tag{1.1}$$

and quantum fluctuations are relatively small.

These high-temperature wall-like solutions turn out to be static (for the simple reason that, when the size of the Euclidean cylinder $\beta = 1/T$ on which the theory is considered is small, higher Fourier harmonics are not excited). Originally, they were interpreted as real Minkowski space domain walls separating distinct Z_N high-T states. However, there are serious reasons to believe that these solutions have relevance only for the Euclidean path integral and *cannot* be interpreted as real physical objects in the Minkowski space [7,10]. That means that the common assertion about spontaneous breaking of Z_N symmetry in high-T Yang-Mills theory is misleading—there is no symmetry breaking in the physical meaning of this word as the physical domain walls separating the distinct phases do not appear. Note that the Euclidean quasiclassical wall-like solutions exist also in high-T QED. Their surface action density can be found by the same token as in the non-Abelian case [7,10].

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[1]The result (1.1) has been derived in Ref. [9]. The authors of that work tried to calculate also the next-to-leading term $\propto gT^2$ in the action density, but it was shown in Refs. [7,10] that this calculation is not infrared stable.

We want to emphasize that the narrow wall-like solutions appear only as the solutions for the *effective* action (after adding the logarithm of the one-loop determinant). At the pure classical level, walls do not appear—the nontrivial 't Hooft fluxes still exist, but they are delocalized [8,11].

II. SCHWINGER MODEL

Our main remark here is that the full scope of questions (of the relevance of the instanton vacuum picture, of applicability of the quasiclassical approximation, and of the physical meaning of the high-T wall-like solutions) can be effectively studied in the Schwinger model (SM) (two-dimensional massless QED). The SM is exactly soluble so that practically every reasonable question can be given an exact and exhaustive answer and, on the other hand, resembles QCD in all gross features. The action of the model reads

$$S=\int d^2x[-\tfrac{1}{4}F_{\mu\nu}^2+i\bar{\psi}\mathcal{D}_\mu\gamma_\mu\psi] , \tag{2.1}$$

where $\mathcal{D}_\mu=\partial_\mu-igA_\mu$ and g is the coupling constant having the dimension of mass.

The SM involves confinement—as in QCD, the spectrum involves only "mesons" with the mass

$$\mu=\frac{g}{\sqrt{\pi}} \tag{2.2}$$

but not free fermions and photons. The axial vector current is anomalous:

$$\partial_\mu\bar{\psi}\gamma_\mu\gamma^5\psi=\frac{1}{2\pi}\epsilon_{\alpha\beta}F_{\alpha\beta} . \tag{2.3}$$

And, what is most important for us here, it involves topologically nontrivial Euclidean gauge field configurations with integer topological charge

$$\nu=\frac{g}{2\pi}\int E(x)d^2x , \tag{2.4}$$

where $E=F_{01}$ (ν is the two-dimensional analog of the Pontryagin class), and hence the instantons, configurations with $\nu=\pm1$ realizing the minimum of the action.

What is the explicit form of these configurations? If we proceed along the same lines as people usually do in QCD and look for the configuration which minimizes the pure *bosonic* Euclidean action

$$S_B=\frac{1}{2}\int d^2xE^2 \tag{2.5}$$

with the constraint $\nu=1$, we are led to the *constant field strength* solution

$$E=\frac{2\pi}{g\mathcal{A}} ,$$

where $\mathcal{A}$ is the total area of the Euclidean manifold on which the theory is defined. Introducing the compact manifold is necessary in this approach to provide for the infrared regularization of the theory.

Different choices for this manifold are possible. In Ref. [12], the Euclidean functional integral on the two-dimensional sphere in different topological sectors has been calculated. But, bearing in mind the parallels with four-dimensional theories and also the generalization of the results for the finite temperature case, it is more instructive to consider the theory on the two-dimensional torus with the spatial size L and the imaginary time size β [13]. Here we always assume that $L\gg\mu^{-1}$. When also $\beta\gg\mu^{-1}$, the boundary effects become irrelevant for physical quantities.[2] When $\beta\lesssim\mu^{-1}$, they are relevant and should be interpreted as the effects due to finite temperature $T=\beta^{-1}$.

The constant field strength solution presents a fiber bundle on the torus. Bearing in mind the subsequent discussion of the high-T case and the parallels with four-dimensional gauge theories we are going to draw, it is convenient to choose the gauge $A_1=0$. The solution then takes the form

$$A_0(x,\tau)=-\frac{2\pi}{gL\beta}x . \tag{2.6}$$

It satisfies the twisted boundary conditions (BC's)

$$A_0(x,\beta)=A_0(x,0) ,$$
$$A_0(L,\tau)=A_0(0,\tau)-\frac{i}{g}\Omega^\dagger\partial_0\Omega , \tag{2.7}$$

where

$$\Omega(\tau)=\exp(-2\pi i\tau/\beta) \tag{2.8}$$

is the gauge transformation matrix. In finite temperature applications, the corresponding B.C.'s for fermion fields in imaginary time directions are antiperiodic, and the B.C.'s in the spatial direction involve an extra gauge transformation

$$\psi(L,\tau)=\Omega(\tau)\psi(0,\tau) .$$

The transition matrix (2.8) satisfies the self-consistency condition $\Omega(\beta)=\Omega(0)$ which *is* the reason for the topological charge (2.4) to be quantized. The solution (2.6) is the direct analog of the four-dimensional 't Hooft toron solutions [11].

The configurations with $\nu=\pm1$ are responsible for the formation of the fermion condensate in the Schwinger model

$$\langle\bar{\psi}\psi\rangle_{\text{vac}}=-\frac{\mu}{2\pi}e^\gamma , \tag{2.9}$$

where γ is the Euler constant. There are many ways to get this result. The easiest way is to employ the bosonization technique [17,15]. But bosonization rules are specific for two dimensions, and it is more instructive to extract the condensate from the Euclidean path integral in the sectors with $\nu=\pm1$. This has been done in Refs. [12,13]. In particular, Sachs and Wipf [13] considered the SM on a two-dimensional torus and calculated the functional integral over the quantum fluctuations around the constant strength solution (2.6). The amplitude of these quantum fluctuations turns out to be large [the characteristic field configurations contributing to the path integral are rather far in Hilbert space from the classical bosonic solution (2.6), and the quasiclassical picture does not work]. The

[2]On the large torus, the boundary effects are always exponentially suppressed [14–16]. On the large two-dimensional sphere with $R\gg\mu^{-1}$, they are suppressed only as a power due to finite curvature of the manifold [15].

calculation is still possible, however, due to the fact that the functional integral is *exactly* Gaussian.

The result (2.9) is obtained in the limit when both dimensions of the torus β and L are large compared to μ^{-1}, the finite boundary effects bringing about exponentially small corrections.

III. PATH INTEGRALS AND THEIR SADDLE POINTS

In this section, we shall be concerned only with the large torus $\beta \sim L \gg \mu^{-1}$. Finite β and L should be thought of then not as quantities of physical relevance but rather as tools of infrared regularization to be disposed of at the end of the calculations. In our discussion, we rely heavily on the results of Ref. [13]. As not everybody may have access to the journal where the paper by Sachs and Wipf was published, it makes sense to outline here in some detail the method they used and the results they obtained.

Any field in the topological sector $\nu=1$ can be decomposed as

$$A_\mu = A_\mu^{\text{cl.i.}} + A_\mu^{(0)} - \epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\chi \ , \tag{3.1}$$

where

$$A_\mu^{(0)} = \left(\frac{2\pi}{g\beta}h_0, \frac{2\pi}{gL}h_1 \right) , \quad (0 \le h_{0,1} \le 1) \tag{3.2}$$

is the constant part of the potential, $A_\mu^{\text{cl.i.}}$ is the classical instanton solution (2.6), $\partial_\mu\chi$ is the gauge part, and the part $-\epsilon_{\mu\nu}\partial_\nu\phi$ carries nontrivial dynamic information. To calculate the partition function, one has first to do the integral over fermion fields i.e., evaluate the fermion determinant $\det\|i\mathcal{D}-m\|$ in an external field (3.1) (m is the small fermion mass, $m \ll g$), and to integrate over gauge fields afterwards. It is convenient to find first the determinant in the "reference field" $\phi=0$. The spectrum and eigenfunctions of the Dirac operator for the field $A_\mu^{\text{cl.i.}} + A_\mu^{(0)}$ can be determined explicitly (as the determinant does not depend on the gauge part $\partial_\mu\chi$, we set it to zero here and in the following). Actually, this problem is exactly the same as for electron moving along our two-dimensional torus in the magnetic field $B(x) \equiv E(x)$ which is normal to the surface and has the unit flux (2.4). In the strictly massless case, the index theorem dictates the presence of one zero eigenvalue in the spectrum and the determinant vanishes. For small but nonzero m, the determinant $\det\|i\mathcal{D}-m\|$ does not vanish but, when $mg\beta L \ll 1$, the only source of the mass dependence in the determinant is the former zero mode which gives rise to a common small factor m (see Ref. [16] and also Sec. V of this paper for more detailed discussion).

When $\nu=1$, the massless zero mode is left handed. It depends in a nontrivial way on the constant part of the potential A_μ^0, and, if $\phi \neq 0$, also on ϕ:

$$\psi_L(\phi) = \exp\{-g\phi(x)\}\psi_L(\phi=0) \ . \tag{3.3}$$

To find the determinant at nonzero ϕ, one has to make use of the property (3.3) and also of the fact that the Dirac operator on an arbitrary two-dimensional Abelian background is related to that on a reference background with $\phi=0$ as

$$\mathcal{D} = e^{\gamma^5\phi}\mathcal{D}_0 e^{\gamma^5\phi} \ . \tag{3.4}$$

Then one has to consider the one-parametric family of operators

$$\mathcal{D}_\alpha = e^{\gamma^5\phi\alpha}\mathcal{D}_0 e^{\gamma^5\phi\alpha} \tag{3.5}$$

which interpolates between $\mathcal{D}_0$ and $\mathcal{D}$ as α changes from 0 to 1. The derivative of the determinant over α can be evaluated using ζ-function regularization and heat kernel technique, and the integration of that derivative over α gives the final result

$$\det\|i\mathcal{D}-m\| = -m \int d^2x e^{-2g\phi(x)} \Phi_{x,\tau}(h_0,h_1) \times \exp\left\{ \frac{\mu^2}{2} \int \phi\Delta\phi d^2y \right\} , \tag{3.6}$$

where Φ is some nontrivial function of the constant harmonics of the potential. The only thing we need to know about it here is that $\int d^2h \Phi_{x,\tau}(h_0,h_1)$ is a constant which does not depend on x and τ but only on the geometry of the box. Thus, the partition function in the sector $\nu=1$ can be written as

$$Z_1 \propto m \int \Pi d\phi \int d^2x e^{-2g\phi(x)} \exp\left\{ -\frac{1}{2} \int \phi(\Delta^2-\mu^2\Delta)\phi d^2y \right\} \overset{\text{def}}{=} \int \prod d\phi \exp\{-S^{\text{eff}}[\phi]\} \ . \tag{3.7}$$

Derivation is tricky but the result is rather simple and its interpretation is straightforward. $\phi\Delta^2\phi/2=(\Delta\phi)^2/2 = E^2/2$ is the classical part of the action density, the term $\propto \mu^2\phi\Delta\phi$ in the effective action is the local part of the fermion determinant and gives mass to one photon, and the factor $m\int d^2x e^{-2g\phi(x)}$ comes from the fermion zero mode: m is the eigenvaluc and the integral $\int d^2x e^{-2g\phi(x)}$ is the normalization factor of the zero mode (3.3). The functional integral (3.7) is Gaussian and has been explicitly calculated in [13]. To get a clearer understanding of the dynamics, it is desirable, however, to not only know the bulk answer but also what field configurations $\phi(x)$ are mainly responsible for it and, to this end, to try to find the saddle points of the integral (3.7). Unfortunately, for the integral (3.7) as it stands, it is not so easy. The equations of motion determining the minimum $\phi_0(x)$ of the effective action $S^{\text{eff}}[\phi]$ are

$$(\Delta^2-\mu^2\Delta)\phi_0(x) = -2g\frac{e^{-2g\phi_0(x)}}{\int d^2y e^{-2g\phi_0(y)}} \ . \tag{3.8}$$

The problem here is not only that Eq. (3.8) is highly nonlinear and complicated, but also that it is ill defined on the infinite plane. As we shall see soon, the fermion mode (3.3) is only quasinormalizable—the integral $\int d^2x e^{-2g\phi(x)}$ diverges logarithmically at large $|x|$. To find the solution, one should recall that the theory has been defined in the first place on a large but finite torus, take the proper account of finite size effects, etc. As, for large β and L, characteristic fields in the integral (3.7) deviate essentially from $\phi_0(x)$ due to significant quantum fluctuations, as finite size effects which affect $\phi_0(x)$ do *not* affect physical quantities, and also as the solution $\phi_0(x)$ can anyway be found only numerically, it is not obvious that this complicated problem is worth the effort which must be spared for its solution.

We shall see later that the situation is much better at high temperatures. Speaking of the large β, L case considered in this section, we may do the following trick. Let us substitute in Eq. (3.7)

$$\int d^2x \exp\{-2g\phi(x)\} \rightarrow \mathcal{A} \exp\{-2g\phi(x_0)\} ,$$

where $\mathcal{A} = \beta L$ is the total area of our Euclidean manifold. We arrive thus at the new functional integral which has, however, the same value as the former due to translational invariance. This new integral also has a direct physical meaning. The fermion condensate $\langle \bar{\psi}_R \psi_L(x_0) \rangle$ is determined by [13]

$$\langle \bar{\psi}_R \psi_L(x_0) \rangle \propto \int \Pi d\phi e^{-2g\phi(x_0)} \times \exp\left\{ -\frac{1}{2} \int \phi(\Delta^2 - \mu^2 \Delta)\phi d^2x \right\} \tag{3.9}$$

without a troublesome $\int d^2x_0$ integration. We shall *define* the effective instanton (e.i.) as the stationary point of Eq. (3.9) or else of the modified path integral for Z_1 which involves compared to Eq. (3.9) the overall factor $m\mathcal{A}$. It satisfies the equation

$$(\Delta^2 - \mu^2 \Delta)\phi^{\text{e.i.}}(x) = -2g\delta(x - x_0) . \tag{3.10}$$

Thus, $\phi^{\text{e.i.}}$ is just the Green's function of the operator $O = \Delta^2 - \mu^2 \Delta$. It has the form[3]

$$\phi^{\text{e.i.}}(x) = \frac{1}{g}[K_0(\mu|x - x_0|) + \ln(\mu|x - x_0|)] + \text{const} . \tag{3.11}$$

[The possible linear in x part of $\phi^{\text{e.i}}(x)$ can be absorbed in the constant component of the gauge field in the decomposition (3.1).] The solution (3.11) is regular at zero. At large $|x - x_0|$, only the logarithmic piece survives, which means that the fermion zero mode $\sim \exp\{-g\phi(x)\} \sim |x - x_0|^{-1}$ is only quasinormalizable as was mentioned above. The electric field

$$E^{\text{e.i.}}(x) = \Delta\phi^{\text{e.i.}}(x) = \frac{g}{\pi} K_0(\mu|x - x_0|) \tag{3.12}$$

falls off exponentially at large distances. The topological charge (2.4) of this solution is equal to 1 as it should be.

We see that, in contrast to the delocalized classical instanton (2.6), the effective instanton (3.11) *is* localized—quantum effects changed the properties of the solution drastically. The parameter x_0 may be thought of as the collective coordinate of the center of the instanton.

Let us forget for the moment about a not yet fixed constant in the right-hand side in Eq. (3.11) and find the action of this instanton. Setting const=0, we get,

$$S^{\text{e.i.}} = g\phi^{\text{e.i.}}(0) = \ln 2 - \gamma . \tag{3.13}$$

Then the *exact* result for the partition function (3.7) in the sectors with $\nu = \pm 1$ can be presented as

$$Z_1 = Z_{-1} = m\mathcal{A} \frac{\mu}{2\pi} \exp\{-S^{\text{e.i.}}\} Z_0 . \tag{3.14}$$

The factor m comes from the fermion zero mode, the factor $\mathcal{A}$ arises due to integration over collective coordinates d^2x_0, and the factor μ appears for dimensional reasons. The hard (in this approach) part of the problem is to determine correctly the numerical factor $(2\pi)^{-1}$.

To do it, one should proceed more accurately. First of all, the exact proportionality coefficient in Eq. (3.7) should be found—more exactly, the *ratio* of this coefficient to the corresponding coefficient in the functional integral for the partition function in the topologically trivial sector $\nu = 0$. This ratio depends on the particular method of the infrared regularization and thereby on the size of our torus. Second, Eq. (3.10) as it stands has no solution at all on a compact manifold. The correct procedure is to project out the zero modes of the operator $\Delta^2 - \mu^2\Delta$ by substituting $\delta(x) \rightarrow \delta(x) - 1/\mathcal{A}$ and imposing the constraint

$$\int \phi(x) d^2x = 0 . \tag{3.15}$$

Thereby the constant in the right-hand side of Eq. (3.11) is fixed. This constant (and hence the instanton action) also depends on the size of the torus.[4] For large L, β, this dependence cancels exactly the similar dependence of the normalization constant, and the finite result (3.14) is obtained [13].

We cannot suggest in this respect anything new compared to the calculation by Sachs and Wipf, but the final result (3.14) looks so suspiciously simple that one is tempted to guess that an easier way to derive it may exist. Differentiating Eq. (3.14) over the fermion mass and adding the equal contribution from the sector $\nu = -1$, we

[3]The notion of effective (or "induced") instanton in the SM was introduced long ago [18], but seems to be very well forgotten since that time.

[4]To be quite precise, the fixing of this constant requires also taking into account the modification of the solution to Eq. (3.10) due to finite size effects which are essential when $|x - x_0|$ is comparable with the size of the box. The solution has the simple form (3.11) only at the vicinity of the center of the instanton, $|x - x_0| \ll \beta, L$.

reproduce the result (2.9),

$$\langle \bar\psi\psi \rangle = -\frac{1}{\mathcal{A}Z_0}\frac{\partial}{\partial m}[Z_1+Z_{-1}] = -\frac{\mu}{2\pi}e^{\gamma} . \quad (3.16)$$

There is however, a simple *indirect* way to fix the coefficient in Eq. (2.9). To this end, one should consider the correlator $\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle$ in the topologically trivial sector $\nu=0$ [19]. It can be calculated rather straightforwardly using the *exact* expression for the fermion Green's function in *any* gauge field background

$$S_\phi(x,y) = \exp\{-g\gamma^5\phi(x)\}S_0(x-y)\exp\{-g\gamma^5\phi(y)\} , \quad (3.17)$$

where $S_0(x-y)$ is the free fermion Green's function.[5] We have

$$\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\nu=0} = -Z_0^{-1}\int\prod d\phi \exp\left\{-\frac{1}{2}\int\phi(\Delta^2-\mu^2\Delta)\phi d^2y\right\}\mathrm{Tr}\{S_\phi(x,y)S_\phi(y,x)\}$$
$$= \frac{1}{2\pi^2x^2}Z_0^{-1}\int\prod d\phi \exp\left\{-\frac{1}{2}\int\phi(\Delta^2-\mu^2\Delta)\phi d^2y\right\}\exp\{2g[\phi(x)-\phi(0)]\} , \quad (3.18)$$

where we substituted $\cosh\{\ \}\to\exp\{\ \}$ as odd powers of $\phi(x)-\phi(0)$ give zero after integration.

The integral is Gaussian. Its stationary point is the solution to the equation

$$(\Delta^2-\mu^2\Delta)\phi^{\text{stat}}(y) = 2g[\delta(y-x)-\delta(y)] , \quad (3.19)$$

which is

$$\phi^{\text{stat}}(y) = \phi^{\text{e.i.}}(y) - \phi^{\text{e.i.}}(y-x) \quad (3.20)$$

with $\phi^{\text{e.i.}}(y)$ being taken from Eq. (3.11). $\phi^{\text{stat}}(y)$ presents an *instanton-antiinstanton configuration.* Note that the free constant in the right-hand side of Eq. (3.11) cancels out completely in the difference.[6]

The calculation is standard. Introduce a new integration variable

$$\varphi(y) = \phi(y) - \phi^{\text{stat}}(y) . \quad (3.21)$$

We have

$$\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\nu=0}$$
$$= Z_0^{-1}\frac{1}{2\pi^2x^2}\exp\{g[\phi^{\text{stat}}(x)-\phi^{\text{stat}}(0)]\}$$
$$\times\int\prod d\varphi \exp\left\{-\frac{1}{2}\int(\Delta^2-\mu^2\Delta)\varphi d^2y\right\} . \quad (3.22)$$

The integral $\int\prod d\varphi\exp\{\dots\}$ exactly cancels out the identical functional integral for Z_0 and the result is

$$\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\nu=0} = \frac{\mu^2}{8\pi^2}e^{2\gamma}\exp\{-2K_0(\mu x)\} . \quad (3.23)$$

At large x, this correlator tends to a constant. One may be tempted to extract the square root of this constant and call it the fermion condensate, but that would not be quite correct. Only the full correlator (the contributions of all topological sectors being summed over) enjoys the cluster decomposition property. One can show [12] that, besides the sector $\nu=0$, only the configurations with topological charge $\nu=\pm2$ contribute to the large x asymptotics of the correlator in the small mass, large volume limit. The following relation is valid:

$$\lim_{x\to\infty}\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\nu=2} = \lim_{x\to\infty}\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\nu=-2}$$
$$= \frac{1}{2}\lim_{x\to\infty}\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\nu=0} . \quad (3.24)$$

The result (3.24) follows from Ward identities which dictate the particular form of θ dependence of the partition function. It holds both in the SM [19] and in the QCD [16]. In our language, it can be interpreted quite naturally. Large x asymptotics of the correlator acquire equal contributions from instanton-antiinstanton and antiinstanton-instanton configurations taken into account in (3.23), and also from two-instanton ($\nu=2$) and two-antiinstanton ($\nu=-2$) configurations. Adding together all contributions and extracting the square root, we arrive at the result (2.9). What is attractive in this derivation is that the effects due to finite β, L play absolutely no role and can be safely forgotten not only in the final stage but right from the beginning.

There is an important point which we want to emphasize here. In the zero-temperature SM as well as in

[5]The result (3.17) as well as the results (3.3) and (3.4) are specific for the SM. Unfortunately, no similar simple formula is known for four-dimensional theories.

[6]The *antiinstanton-instanton* configuration $-\phi^{\text{stat}}(y)$ plays exactly the same role (we could substitute $\cosh\{\dots\}\to\exp\{-\dots\}$ with equal ease). A quite precise way would be to present $\cosh\{\dots\}$ as the sum of two exponentials and split the path integral (3.18) in two equal parts. The stationary point of one of them [which describes the correlator $\langle \bar\psi_L\psi_R(x)\,\bar\psi_R\psi_L(0)\rangle$] is $\phi^{\text{stat}}(y)$ while the stationary point of the other corresponding to $\langle \bar\psi_R\psi_L(x)\,\bar\psi_L\psi_R(0)\rangle$ is $-\phi^{\text{stat}}(y)$.

zero-temperature QCD, the quasiclassical picture does not really work— the quantum fluctuations are large and the characteristic field configurations in the path integral *have nothing to do* with either classical (2.6) or effective (3.11) instanton solutions. We have shown in Ref. [20] that all essential properties of the characteristic vacuum fields in the SM are well reproduced in the model of a vortex-antivortex liquid. The basic ingredient of this model was a vortex configuration

$$\phi^{\text{vort}}(y)=\frac{1}{2g}\ln[(y-x_0)^2+\rho^2] . \tag{3.25}$$

It carries unit topological charge (2.4) but does not minimize either classical nor effective action. The full field in our model presented a stochastic superposition of vortices and antivortices with certain correlation properties. All results of Ref. [20] (in particular, the results for the fermion condensate, Wilson loop average, and topological susceptibility) can be, however, rederived using the solution (3.11) rather than (3.25) as a basic ingredient. The form of the instanton is anyway distorted by quantum fluctuations and, for modeling purposes, the configurations (3.11) and (3.25) are equivalent. The really essential feature of characteristic vacuum fields is fluctuating *local density* $E(x)$ of the topological charge (2.4). Both vortex (3.25) and effective instanton (3.11) configurations model such local fluctuations well. We think also that all spectacular results of the instanton-antiinstanton liquid model in QCD [4] can be reproduced if we choose as a basic ingredient not the BPST instanton but any other localized field configuration carrying topological charge. At large temperatures, however, the situation simplifies a lot and it is very instructive to analyze the SM in the $T \gg \mu$ region.

IV. HIGH TEMPERATURES

Let us repeat all derivations of the previous section for the case when the imaginary time extension of our torus $\beta=1/T$ is small compared to the massive photon Compton wavelength μ^{-1} (and the spatial size is still large $L \gg \mu^{-1}$). The partition function in the sector $\nu=1$ is given again by Eq. (3.7), only the integral $\int d^2x$ extends now over the narrow torus. If $\beta\mu \ll 1$, one can assume the field $\phi(x,\tau)$ to be static (the contribution of higher Fourier harmonics $\propto \exp\{2\pi i k\tau/\beta\}$ in the path integral is suppressed), and we can substitute $\int d^2x \to \beta \int dx$, $\Delta \to \partial^2/\partial x^2$ (from now on x will always be assumed to be spatial). To start with, let us do the same trick as earlier and substitute

$$\int dx \exp\{-2g\phi(x)\} \to L\exp\{-2g\phi(x_0)\}$$

in the path integral for the partition function. Then the effective instanton solution satisfies the equation

$$\left[\frac{\partial^4}{\partial x^4}-\mu^2\frac{\partial^2}{\partial x^2}\right]\phi^{\text{e.i.}}_{\text{high }T}(x)=-2gT\delta(x-x_0) . \tag{4.1}$$

The solution has the form

$$\phi^{\text{e.i.}}_{\text{high }T}(x)=\frac{\pi T}{g}\left[\frac{1}{\mu}\exp\{-\mu|x-x_0|\}+|x-x_0|\right] +\text{const} . \tag{4.2}$$

The corresponding gauge field is

$$\begin{aligned} A_0^{\text{e.i.}}(x)|_{\text{high }T} &= -\frac{\partial}{\partial x}\phi^{\text{e.i.}}_{\text{high }T}(x)-\frac{\pi T}{g} \\ &= \frac{\pi T}{g}\text{sgn}(x-x_0)[\exp\{-\mu|x-x_0|\}-1] \\ &\quad -\frac{\pi T}{g} . \end{aligned} \tag{4.3}$$

[The gauge $A_1=0$ is chosen; the term $-\pi T/g$ comes from the properly fixed constant term $A_\mu^{(0)}$ in the decomposition (3.1) [13].] When going from $x=-\infty$ to $x=\infty$, A_0 changes from zero to $-2\pi T/g$ which corresponds to unit net topological charge

$$\nu=\frac{g\beta}{2\pi}\int E(x)dx=-\frac{g}{2\pi T}[A_0(\infty)-A_0(-\infty)]=1 .$$

But the electric field

$$E^{\text{e.i.}}_{\text{high }T}=-\frac{\partial}{\partial x}A_0^{\text{e.i}}\bigg|_{\text{high }T}=T\sqrt{\pi}\exp\{-\mu|x-x_0|\} \tag{4.4}$$

is localized at $x \sim x_0$ in contrast to the classical solution (2.6).

As we have already noticed, the notion of the instanton has much more physical content at high T where quantum fluctuations are relatively small. To see this, let us make a simple estimate [10]. Let us expand the integrand in the path integral around the classical solution (4.2)–(4.4) and express the fluctuating part in terms of $a_0=-\partial/\partial x[\phi(x)-\phi^{\text{e.i.}}_{\text{high }T}(x)]$,

$$Z \propto \int \prod da_0 \exp\left\{-\frac{\beta}{2}\int dx[(\partial_x a_0)^2+\mu^2 a_0^2]\right\} . \tag{4.5}$$

Let us expand $a_0(x)$ in the Fourier series

$$a_0(x)=\sum_{n=-\infty}^{\infty} c_n \exp\{2\pi i n x/L\},\quad c_{-n}=c_n^* ,$$

and rewrite the functional integral as the integral over $\prod_n dc_n$. It is easy to see that

$$\langle |c_n|^2\rangle_{\text{char}} \sim \frac{1}{\beta L[\mu^2+4\pi^2n^2/L^2]} .$$

Assuming stochastic phases α_n for $c_{n>0}$, we may estimate

$$\begin{aligned} a_0^{\text{fluct}}(x) &\sim \sqrt{LT}\sum_{n=1}^{\infty}\frac{\cos(\alpha_n)}{\sqrt{4\pi^2n^2+\mu^2L^2}} \\ &\sim \sqrt{LT}\left[\sum_{n=1}^{\infty}\frac{1}{4\pi^2n^2+\mu^2L^2}\right]^{1/2} \\ &\sim \sqrt{T/\mu} , \end{aligned} \tag{4.6}$$

which is much less than the amplitude of the classical solution (4.3) $\sim T/\mu$. Characteristic momenta in Eq. (4.6) $k^{\text{char}} \sim n^{\text{char}}/L$ are of order μ.

If setting const=0 in the right-hand side of Eq. (4.2), the action of the instanton is

$$S^{\text{e.i.}}_{\text{high } T}=g\phi^{\text{e.i.}}_{\text{high } T}(0)=\frac{\pi T}{\mu}\gg 1 \ . \tag{4.7}$$

We want to emphasize that, in contrast to the zero-temperature case where the effective instanton (3.11) was the stationary point of the path integral (3.9) for the fermion condensate but *not* the stationary point of the original path integral for the partition function (3.7) and differed essentially from the latter, at high T the situation is much better. The stationary point of the high-T path integral for the partition function is determined from the equation

$$\left[\frac{\partial^4}{\partial x^4}-\mu^2\frac{\partial^2}{\partial x^2}\right]\phi_0^{\text{high } T}(x)=-2gT\frac{e^{-2g\phi_0^{\text{high } T}(x)}}{\int dy\, e^{-2g\phi_0^{\text{high } T}(y)}} \ . \tag{4.8}$$

Though Eq. (4.8) differs from (4.1) and their solutions also differ, this difference is small. Let us solve Eq. (4.8) by iteration. Let us choose as the zero approximation $\phi^{(0)}$ our old solution (4.2). Substitute it in the right-hand side of Eq. (4.8). Note that the zero mode $\exp\{-g\phi^{(0)}(x)\}$ is now normalizable and is localized at the distance $|x-x_0|\sim(\mu T)^{-1/2}$ from the instanton center which is *much less* than the size of the instanton $|x-x_0|\sim\mu^{-1}$ [this is the region where the electric field (4.4) is localized]. As a result, the source distribution in the right-hand side of Eq. (4.8) is comparatively very narrow and is simulated by $\delta(x-x_0)$ quite well. And that means that the corrections to the zero approximation (4.2) are going to be small.

Let us see it. The first iteration for the electric field $E^{(1)}(x)=\partial^2/\partial x^2\phi^{(1)}(x)$ satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2}-\mu^2\right]E^{(1)}(x)=2gT\left[\delta(x)-\sqrt{T\mu}\,e^{-\pi T\mu x^2}\right]$$
$$\sim-\frac{g}{2\pi\mu}\delta''(x) \tag{4.9}$$

[we put $x_0=0$ for simplicity and neglected higher order terms in the expansion of $\phi^{(0)}(x)$ in x which are small in the relevant region of x]. The solution to (4.9) is

$$E^{(1)}(x)=-\frac{g}{4\pi}\left[e^{-\mu|x|}-\frac{2}{\mu}\delta(x)\right] , \tag{4.10}$$

which is much less not only than the amplitude of the zero-approximation solution (4.4) but also than the characteristic amplitude of quantum fluctuations $E^{\text{fluct}}(x)\sim\mu a_0^{\text{fluct}}(x)\sim\sqrt{T\mu}$.[7] The correction (4.10) is of the same order as the corrections brought about by nonstatic modes in $\phi(x,\tau)$ and can be safely neglected.

The exact result for the partition function in the sectors $\nu=\pm1$ at high temperatures can be written as

$$Z_1=Z_{-1}=mL\exp\{-S^{\text{e.i.}}_{\text{high } T}\}Z_0 \ . \tag{4.11}$$

The factor m comes from the fermion zero mode and the factor L from the integral over the collective coordinate x_0 describing the spatial instanton position. The numerical coefficient is just 1. From this, one can easily derive

$$\langle\bar\psi\psi\rangle_{T\gg\mu}=-\frac{1}{\beta L Z_0}\frac{\partial}{\partial m}[Z_1+Z_{-1}]$$
$$=-2T\exp\{-\pi T/\mu\} \ . \tag{4.12}$$

Again, the results (4.11), (4.12) look suspiciously simple and again we cannot suggest a simple direct way to derive them (other than to use the whole machinery of the quantization in the box as in Ref. [13]). And again, the simplest way to fix the exact coefficient in $\langle\bar\psi\psi\rangle_{T\gg\mu}$ we know of is to study the correlator $\langle\bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{T\gg\mu}$ in the topologically trivial sector at large spatial x. The stationary point of the corresponding path integral is the instanton-antiinstanton configuration

$$\phi^{\text{stat}}_{\text{high } T}(y)=\frac{\pi T}{g}\left[\frac{1}{\mu}\exp\{-\mu|y|\}+|y|-\frac{1}{\mu}\exp\{-\mu|y-x|\}-|y-x|\right] . \tag{4.13}$$

The integral over fluctuations cancels out with the same integral in Z_0, and we get

$$\langle\bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{T\gg\mu}=\langle\bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{\text{free}}\times\exp\{g[\phi^{\text{stat}}_{\text{high } T}(x)-\phi^{\text{stat}}_{\text{high } T}(0)]\} , \tag{4.14}$$

where the free correlator on the cylinder (one can forget about finite L in this method of derivation) is

$$\langle\bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{T\gg\mu}=2T^2\left[\sum_{n=0}^{\infty}\exp\{-\pi T(2n+1)x\}\right]^2$$
$$=\frac{T^2}{2\sinh^2(\pi T x)} \ . \tag{4.15}$$

It falls off exponentially at large distances, the exponent being given by the lowest fermion Matsubara frequency $\omega_{\min}=\pi T$.

The factor $\exp\{\dots\}$ in the right-hand side of Eq. (4.14), however, rises exponentially at large x with the same exponent so that

$$\lim_{x\to\infty}\langle\bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{T\gg\mu}=2T^2\exp\{-2\pi T/\mu\} \ . \tag{4.16}$$

Adding the equal contribution from the sectors $\nu=\pm2$ [cf. Eq. (3.24)] and taking the square root, we arrive at the result (4.12).

Let us look now at the solution (4.3) more intently. As

[7] One should not be worried by the $\delta(x)$ term in the right-hand side of Eq. (4.10) because the coefficient is small and the contribution of this term to the flux (2.4) is of order $\mu/T\ll1$. This contribution is duly canceled out by the contribution from the first term so that the corrected solution $E^{(0)}+E^{(1)}$ also belongs to the instanton topological class.

has been explained in detail in [7], it may be thought of as the configuration interpolating between two adjacent minima of the high-T effective potential $V^{\text{eff}}(A_0)$ in the constant A_0 background. The form of the potential is

$$V^{\text{eff}}(A_0)=\frac{\mu^2}{2}\left[\left[A_0+\frac{\pi T}{g}\right]_{\text{mod}2\pi T/g}-\frac{\pi T}{g}\right]^2, \qquad (4.17)$$

and it has minima exactly at $A_0^{\min}=2\pi nT/g$.

Thus, the solution (4.3) is closely analogous to the "walls" which appear in the Euclidean path integral in four-dimensional gauge theories [9,7,10]—the planar static field configurations which interpolate between different minima of effective potential $V^{\text{eff}}(A_0)$. In Ref. [9], they were interpreted as real walls in Minkowski space separating different thermal Z_N phases. We argued in Refs. [7,10] that such domain walls do not actually exist in Minkowski space and there is only *one* physical phase both in QED and pure Yang-Mills theory at high temperature.

One of the arguments comes from SM analysis—the static solutions (4.3) can (or cannot) be interpreted as real Minkowski space "solitons" with the mass

$$M^{\text{sol}}=TS^{\text{e.i.}}_{\text{high } T}=\frac{\pi T^2}{\mu} \qquad (4.18)$$

exactly by the same token as the planar four-dimensional Euclidean static solitons can (or cannot) be interpreted as real walls.

Here we want to present some additional arguments why they cannot. If the solitons with the mass (4.18) really exist in some reasonable sense, this new mass scale should display itself somehow in the physical correlators, in particular in the correlator $\langle\bar\psi\psi(x)\bar\psi\psi(0)\rangle$. The operator $\bar\psi\psi$ is in some sense "intimately connected" with the solution (4.2–4.4)—its expectation value is proportional to $\exp\{-S^{\text{e.i.}}_{\text{high } T}\}$. If trying to interpret $\bar\psi\psi$ as a creation operator for this soliton, one could expect that the correlator (4.14) would fall off $\sim\exp\{-M^{\text{sol}}x\}$ at large distances. But it does not.

The correlator (4.14) has three asymptotic regions.

(1) At very small $x\ll T^{-1}$, it is not affected by the boundaries of the box and behaves as $\sim 1/2\pi^2x^2$.

(2) At $T^{-1}\ll x\ll\mu^{-1}$, it is affected by the boundary but not yet by the fluctuating gauge fields, and is given by the expression (4.15).

(3) At $x>\mu^{-1}$, it starts to be affected by the gauge field dynamics and rapidly levels off at a constant (4.16), the preasymptotic terms being of order $\exp\{-\mu x\}$. The scale $\propto T^2/\mu$ is absent.

The full correlator, with the contributions from all topological sectors being summed over

$$\langle\bar\psi\psi(x)\bar\psi\psi(0)\rangle_T =\langle\bar\psi\psi\rangle_T^2\cosh\left\{\int dk e^{ikx}\frac{1}{\sqrt{k^2+\mu^2}}\coth\frac{\sqrt{k^2+\mu^2}}{2T}\right\}, \qquad (4.19)$$

exhibits the same qualitative behavior. The derivation of the result (4.19) is given in the Appendix.

Strictly speaking, the absence of the scale M^{sol} in the spatial correlators (4.14), (4.19) still leaves room for doubt—it does not imply directly the absence of *a collective excitation* with mass M^{sol} in the spectrum.[8] To answer unambigously this question, one should study the retarded correlator $R_T(x,t)=\theta(t)\langle[\bar\psi\psi(x,t),\bar\psi\psi(0)]_-\rangle_T$ in Minkowski space at large real times t. (At finite T, Lorentz invariance is lost, and large real t behavior and large spatial x behavior are not necessarily related to each other.) If such a stable soliton exists, it should display itself in a δ-function singularity in the spectral density

$$\rho(E,p=0)=\text{Im}\int dxdt\, e^{-iEt}R_T(x,t) \qquad (4.20)$$

at $E=M^{\text{sol}}$. This spectral density has been evaluated recently [21] using a bosonization technique [22]. [As written in Eq. (A2c), the operator $\bar\psi\psi$ is dual to the nonlinear operator $\langle\bar\psi\psi\rangle_0\cos(\sqrt{4\pi}\phi)$ depending on the free scalar field ϕ with mass μ.] The result is

$$\rho(E,0)=2\pi\langle\bar\psi\psi\rangle_T^2\sinh\frac{\beta E}{2}\times\sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!}\prod_{i=1}^{2k}\int d^2q_i\delta(q_i^2-\mu^2)\times\sum_{\pm q_i^0}\frac{1}{\sinh(\beta|q_i^0|/2)}\delta^2\left[P-\sum_{i=1}^{2k}q_i\right], \qquad (4.21)$$

$P\equiv(E,0)$. Unfortunately, this infinite sum over multiple integrals, each term corresponding to the square of a particular matrix element $\langle n_\phi|\cos(\sqrt{4\pi}\phi)|m_\phi\rangle$, $n_\phi+m_\phi=2k$, integrated over initial and final phase space with proper thermal weights, is difficult to analyze. Only when the function (4.21) is directly plotted versus energy at different temperatures shall we acquire the final and comprehensive understanding of the problem. But we find it most improbable that the sum (4.21) would exhibit a singularity at one given energy $E=M^{\text{sol}}$.

V. PARTITION FUNCTION AND CHARACTERISTIC FIELD CONFIGURATIONS AT HIGH T

As we have seen, instantons display themselves in the path integrals for $\langle\bar\psi\psi\rangle$ and for $\langle\bar\psi\psi(x)\bar\psi\psi(0)\rangle$. But it is important to note that the path integral for the partition function itself is not affected by instantons in the strictly massless Schwinger model. Really, at $m=0$, only the topologically trivial sector $\nu=0$ contributes (all $Z_{\nu\neq 0}$ involve the factor $m^{|\nu|}$ due to $|\nu|$ fermion zero modes and vanish in the massless limit). The path integral for Z_0 has a trivial form (4.5) and no instantons are seen there.

[8] I am indebted to L. McLerran who brought my attention to this fact.

To understand better what actually happens and why instantons reappear in the path integral for the correlator $\langle \bar\psi\psi(x)\,\bar\psi\psi(0)\rangle_{T\gg\mu}$ in the same topologically trivial sector, let us estimate the contribution of the instanton-antiinstanton pair (4.13) in the path integral (4.5). The configuration $A_0^{\rm IA}(y)$ is depicted in Fig. 1. Obviously, the corresponding contribution to Z_0 is

$$Z_0^{\rm IA}\propto\exp\left\{-\frac{\beta x}{2}\mu^2\left[\frac{2\pi T}{g}\right]^2\right\}=\exp\{-2\pi T x\}\ , \qquad (5.1)$$

where x is the separation between instanton and antiinstanton (we assumed $x\gg\mu^{-1}$).

Now, the suppression (5.1) can be interpreted as being due to *quasizero* modes in the fermion determinant. Really, an individual instanton (4.2) (the field configuration with $\nu=1$) involves an exact fermion zero mode

$$\psi_L(y)=\exp\{-g\phi(y)\}\propto\exp\{-\pi T|y|\} \qquad (5.2)$$

(where we changed the notations $x\to y$ and put $x_0=0$). For instanton-antiinstanton (IA) configuration (4.13), the former zero mode (5.2) and its counterpart for the antiinstanton located at $y=x$

$$\psi_R(y)\propto\exp\{-\pi T|y-x|\} \qquad (5.2')$$

are no longer exact eigenfunctions of the Dirac operator. But, if instanton and antiinstanton are well separated, they are *almost* the solutions. The true solutions and their eigenvalues can be found by solving the secular equation taking the functions (5.2) and (5.2′) as basis and regarding the effects due to finite IA separation as perturbation. As a result, two quasizero modes with $\lambda\propto\exp\{-\pi T x\}$ appear. Their product in $\det\|i\mathcal{D}\|$ brings about the suppression (5.1).

It is clear now why the IA configuration displayed itself in the correlator $\langle\bar\psi\psi(x)\,\bar\psi\psi(0)\rangle$—the fermion operators "absorbed" the zero modes form the fermion determinant, and the answer is finite in the limit $x\to\infty$ [see Eq. (4.16)]. The finite result (4.12) for the fermion condensate $\langle\bar\psi\psi\rangle_{T\gg\mu}$ being determined by the path integral in the sectors $\nu=\pm1$ is obtained for exactly the same reason.

However, the instantons *reappear* even in the path integral for the partition function in the high-T SM if we allow for a small but nonzero fermion mass. It is clear why—the determinant factor is now $\det\|i\mathcal{D}-m\|$ rather than just $\det\|i\mathcal{D}\|$, and the contribution of the quasizero modes remains finite $\propto m^2$ even at large IA separation.

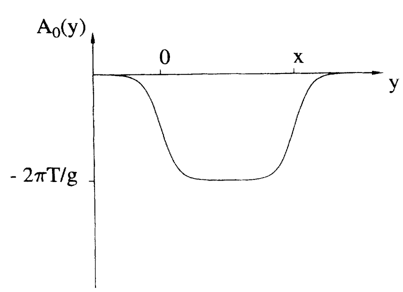

FIG. 1. Instanton-antiinstanton configuration.

If one chooses, one may speak of the *confinement* of instantons in the strictly massless case and their *liberation* for any small nonzero mass. The suppression (5.1) can be interpreted as behind due to the linearly rising "potential" between an instanton and antiinstanton. At $m\neq0$, the potential levels off at a constant value at large separations. Freely moving instantons bring about a large entropy factor, and the contribution of the IA configuration to Z_0 is

$$Z_0^{\rm IA}=[mL\exp\{-\pi T/\mu\}]^2 Z_{m=0}\ . \qquad (5.3)$$

If the spatial box is large enough,

$$\kappa=2mL\exp\{-\pi T/\mu\}=m\,|\langle\bar\psi\psi\rangle_T|\beta L\gg1\ , \qquad (5.4)$$

the contribution of the IA configuration to Z_0 *dominates* over the purely perturbative contribution $Z_{m=0}$ given by Eq. (4.5).

Let us find now the contribution due to N instantons and N antiinstantons (in the sector $\nu=0$, their number should be the same). It is

$$Z_0^{N(\rm IA)}=\frac{1}{(N!)^2}[mL\exp\{-\pi T/\mu\}]^{2N}Z_{m=0}\ . \qquad (5.5)$$

The factor $(N!)^{-2}$ appeared due to indistinguishability of instantons (and, separately, antiinstantons). Summing over all N, we get

$$Z_0=\sum_{N=0}^{\infty}Z_0^{N(\rm IA)}=Z_{m=0}I_0(\kappa) \qquad (5.6)$$

(I_0 is the exponentially rising modified Bessel function). The sum (5.6) is saturated at $N_{\rm char}\sim\kappa$, i.e., when the condition (5.4) is satisfied, characteristic field configuration involve *many* (of order κ) instanton-antiinstanton pairs.

The partition function in the topologically nontrivial sectors $\nu\neq0$ can be calculated in the same way. Assume for definiteness $\nu>0$. Then the contribution of $N+\nu$ instantons and N antiinstantons in the partition function is

$$Z_\nu(N_I=N+\nu,N_A=N)=\frac{1}{N!(N+\nu)!}\left[\frac{\kappa}{2}\right]^{2N+\nu}Z_{m=0}\ . \qquad (5.7)$$

Summing over N, we get

$$Z_\nu=Z_{m=0}I_\nu(\kappa)\ . \qquad (5.8)$$

The full partition function is very simple (and could, of course, be written right from the beginning from the most general premises):

$$Z=\sum_{\nu=-\infty}^{\infty}Z_\nu=Z_{m=0}e^{\kappa}=Z_{m=0}\exp\{-m\langle\bar\psi\psi\rangle_T\beta L\}\ . \qquad (5.9)$$

The sum (5.9) is saturated at $\nu_{\rm char}\sim\sqrt{\kappa}$. That can be rather easily understood. When no constraint is put on the net topological charge, we have two independent Poisson distributions for instantons and antiinstantons with central values of order κ. The dispersion of these distributions and the average mean square deviation

$\langle (N_+ - N_-)^2 \rangle^{1/2}$ are of order $\kappa^{1/2}$.

Thus, we are arriving at the physical picture of a rarified IA gas. It is important to understand that it arises only when the condition (5.4) is fulfilled. In the opposite limit $\kappa \ll 1$, which is realized by keeping the spatial box length large but fixed and tending m to zero, there are no extra IA pairs and the partition function in the sector ν is saturated by ν instanton configurations. In this limit, $Z_\nu \propto m^{|\nu|}$ and the only sources of mass dependence of the partition function are ν exact zero modes of the massless Dirac operator (see [16] for more details).

The results (5.6), (5.8) are very general; they hold not only in the high-T but also in the low-T SM, and also in QCD [16]. They can be rigorously derived by studying the θ dependence of the partition function [16]. It is very instructive to see, however, how these results could be rederived here in another way in the case where the characteristic field configurations in the path integral are known exactly and explicitly.

The last remark concerns topological susceptibility. It is defined as

$$\chi = \left[\frac{g}{2\pi} \right]^2 \int d^2x \langle E(x)E(0) \rangle = \frac{1}{V} \langle \nu^2 \rangle , \tag{5.10}$$

where $V = L\beta$ is the total Euclidean volume. As has been mentioned, $\nu_{\text{char}} \sim \sqrt{\kappa}$. The exact result is

$$\langle \nu^2 \rangle = V\chi = m \,|\langle \bar{\psi}\psi \rangle|\, V . \tag{5.11}$$

It can be derived either from Ward identities [23] or directly from the explicit result (5.8) and holds universally at any temperature. But the *mechanism* providing the suppression $\propto m$ in χ is much different at high temperatures compared to that at low temperatures. At $T \gg \mu$, the characteristic field configurations present a noninteracting instanton-antiinstanton gas. The density of the instantons is low—it involves the fermion mass factor m (in addition to the factor $\exp\{-S_{\text{high }T}\} = \exp\{-\pi T/\mu\}$). This factor exhibits itself in the topological susceptibility. On the other hand, at low temperatures, the characteristic vacuum fields present a dense strongly interacting instanton-antiinstanton liquid [20] (with large quantum fluctuations distorting the shape of individual instantons). The suppression in χ appears in that case due to strong correlations in this liquid providing effective screening of the topological charge.

It is worthwhile to mention that the whole analysis can be transferred without essential change to QCD with *one* nearly massless quark. The fermion condensate survives in this theory even at very large temperatures [6] and, if the quark mass is small but nonzero and the spatial volume is large enough, characteristic field configurations include of order

$$\kappa^{\text{QCD}} = m \,|\langle \bar{\psi}\psi \rangle_T^{\text{QCD}}|\, \beta V^{(3)} \tag{5.12}$$

instantons and about the same number of antiinstantons which do not interact with each other (versus the dense strongly correlated instanton-antiinstanton liquid at $T=0$).

Note added. After this work was completed, I became aware of Ref. [24] which also discusses instantons in the Schwinger model. The authors of [24] considered the model at zero temperature $\beta = \infty$ but on a small spatial circle $L \ll \mu^{-1}$. Certainly, it is equivalent to the high temperature $\beta \ll \mu^{-1}$, large spatial volume case up to a rotation by 90°. The main difference between their work and ours is that they restricted themselves to large masses. The explicit results for the instanton profile and action have been written down only in the case $m \gg T$ ($mL \gg 1$ in their approach). Then fermion zero modes are not relevant and the analysis is ideologically much simpler—it is just a quantum mechanical problem with a local Hamiltonian. But the large mass case is, in our opinion, not so interesting due to lack of analogies with QCD.

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APPENDIX: CORRELATOR $\langle \bar{\psi}\psi(x)\,\bar{\psi}\psi(0) \rangle$ IN BOSONIZATION APPROACH

Let us derive the result (4.19) for the full fermion correlator $\langle \bar{\psi}\psi(x)\,\bar{\psi}\psi(0) \rangle$ at finite temperature. The simplest way to do it is to use the bosonization technique [22]. The Lagrangian of the SM is dual to the free massive scalar field Lagrangian

$$\mathcal{L}_{\text{bos}} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 , \tag{A1}$$

and bilinear fermion combinations can be written in terms of the scalar field ϕ as

$$\bar{\psi}\gamma_\mu\psi \equiv \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi , \tag{A2a}$$

$$\bar{\psi}\gamma^5\gamma_\mu\psi \equiv \frac{1}{\sqrt{\pi}}\partial^\mu\phi , \tag{A2b}$$

$$\bar{\psi}\psi \equiv -\frac{\mu}{2\pi}e^{\gamma}\mathcal{N}_\mu \cos\sqrt{4\pi}\phi , \tag{A2c}$$

where $\mathcal{N}_\mu$ means normal ordering with respect to creation and annihilation operators of the free boson field with the mass μ, and the numerical constant in Eq. (A2c) is nothing but the zero-temperature fermion condensate (2.9). Then all correlators of the fermion bilinears calculated with the original SM action (2.1) coincide with the correlators of the corresponding bosonized expressions calculated with the Lagrangian (A1).

The typical graphs contributing to the correlator of two cosine function in the right-hand side of Eq. (A2c) are depicted in Fig. 2. Note that the tadpole loops which describe the pairing of ϕ operators at one and the same point and which did not contribute at $T=0$ due to the normal ordering prescription $\mathcal{N}_\mu$ *do* appear at nonzero temperature [15]—an annihilation operator gives zero

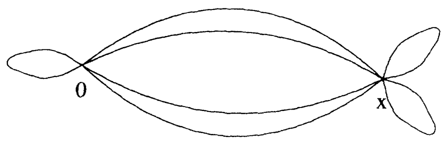

FIG. 2. A typical bosonized graph contributing to $\langle \bar{\psi}\psi(x)\,\bar{\psi}\psi(0)\rangle_T$.

when acting on the vacuum state but does not give zero when acting on a state from the thermal heat bath involving excited states.

Working out combinatorics and employing the Gaussian nature of the path integral, it is not difficult to see that the tadpole contributions factorize and the result is

$$\langle \bar{\psi}\psi(x)\,\bar{\psi}\psi(0)\rangle_T = \langle \bar{\psi}\psi\rangle_T^2 \cosh\{4\pi D_T(0,x)\}\ , \qquad \text{(A3)}$$

where $D_T(0,x)$ is the thermal Green's function of the scalar and $\langle \bar{\psi}\psi\rangle_T$ has only tadpole contributions,

$$\langle \bar{\psi}\psi\rangle_T = -\frac{e^\gamma}{2\pi}\mu \exp\{-2\pi[D_T(0)-D_{T=0}(0)]\}\ . \qquad \text{(A4)}$$

The Green's function of the Klein-Gordon operator $\Delta+\mu^2$ on a cylinder is

$$D_T(0,x) = \frac{T}{2\pi}\sum_{n=-\infty}^{\infty}\int \frac{dk e^{ikx}}{(2\pi nT)^2+k^2+\mu^2}$$

$$= \frac{1}{4\pi}\int dk e^{ikx}\frac{1}{\sqrt{k^2+\mu^2}}\coth\frac{\sqrt{k^2+\mu^2}}{2T}\ , \qquad \text{(A5)}$$

and we arrive at the result (4.19). At $T \gg \mu$, the expression (A4) for $\langle \bar{\psi}\psi\rangle_T$ takes the simple form (4.12).

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