Low energy dynamics of $[\mathrm{U}(1)]^N$ Chern-Simons solitons

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We apply the adiabatic approximation to investigate the low energy dynamics of vortices in the parity-invariant double self-dual Higgs model with only mutual Chern-Simons interaction. When distances between solitons are large they are particles subject to the mutual interaction. The dual formulation of the model is derived to explain the sign of the statistical interaction. When vortices of different types pass through one another they behave like charged particles in a magnetic field. They can form a bound state due to the mutual magnetic trapping. Vortices of the same type exhibit no statistical interaction. Their short range interactions are analyzed. Possible quantum effects due to the finite width of vortices are discussed.

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INTRODUCTION

Experiments with high temperature superconductors seem to show no indication of parity breaking [1]. At first sight this result seems to exclude the anionic mechanism of superconductivity. But this is not the case as was shown recently [2,3]. The presence of a Chern-Simons interaction in a model does not lead inevitably to the breaking of the P and T invariance. It is just a property of the simplest models with only one Chern-Simons field. When there is an even number of Chern-Simons fields and their coupling constants are appropriately chosen the parity invariance can be restored [2,3]. One of the simplest models of this kind is the $[\mathrm{U}(1)]^2$ model of two Higgs fields each of them coupled to one of the two Chern-Simons fields [4]. The model is constructed is such a way that particles which carry one kind of charge interact with the magnetic field of the other kind. So the ordinary construction of anions as charged particles which have at the same time attached a fictitious magnetic flux is split into two parts. Particles of one kind are carriers of the charge while those of the other kind carry the magnetic flux. Thus the ordinary fractional statistics [5] is replaced by the so-called mutual statistics [2].

In this paper we investigate the Chern-Simons interactions of vortices in the relativistic self-dual $[\mathrm{U}(1)]^N$ model [4]. We apply Manton's idea of adiabatic approximation [6] to the topological solitons of the model with corrections necessary in the case of Chern-Simons vortices [7]. The topological solitons configurations satisfy the lower Bogomolny bound on energy [4] so the moduli space approximation is justified. Explicit calculations are made in the special case of the $[\mathrm{U}(1)]^2$ model but the results can be easily generalized to the case of $[\mathrm{U}(1)]^N$. At large separations the vortices of different Higgs fields exhibit the expected mutual interactions but with a sign opposite to that expected from their fluxes and charges. It is very much like the case for vortices in the standard U(1) Chern-Simons-Higgs model which exhibit ordinary fractional statistics properties [8,7]. We derive the dual formulation of the model to explain the sign of the statistical interaction.

When the vortices pass through one another their interaction is a little more complicated. The pair of vortices of different types behave like charged particles passing the flux of magnetic field similarly to vortices in the ordinary Chern-Simons-Higgs system [7,9]. Due to the fact that the spin of separated vortices is equal to zero while that of coincident vortices is nonzero there is a kind of magnetic trapping—they form a composite state. If the corrections to the standard adiabatic approximation are quantitative in nature there is a periodic solution with vortices circling in the magnetic field of the trap. On the other hand vortices of the same kind do not interact through the Chern-Simons field. If the corrections to the ordinary adiabatic approximation amount only to the renormalization of parameters in the effective Lagrangian, they would behave very much like vortices in the Abelian Higgs model. In particular the result of their head-on collision would be the right-angle scattering [10,12].

The paper is organized as follows. In Sec. I we derive the general form of the effective Lagrangian. Section II is devoted to the long-range interactions of vortices and their mutual statistics. In the next paragraph (3) we analyze what happens when various types of vortices pass through one another. Section IV is a presentation of the dual formulation of the model. In the last section we summarize and discuss the results.

I. GENERAL FORM OF THE EFFECTIVE LAGRANGIAN

The Lagrangian of the relativistic model presented in [4] when we restrict to such a choice of parameters that only mutual interactions are preserved can be written in the form

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$$L = \kappa\varepsilon^{\mu\nu\lambda}v_\mu^{(1)}\partial_\nu v_\lambda^{(2)} + D_\mu\phi^\star D^\mu\phi + D_\mu\chi^\star D^\mu\chi - V(\phi,\chi) \ , \quad (1)$$

where the covariant derivatives are defined by

$$D_\mu\phi = \partial_\mu\phi - iq_1 v_\mu^{(1)}\phi$$

and (2)

$$D_\mu\chi = \partial_\mu\chi - iq_2 v_\mu^{(2)}\chi \ ,$$

while the Higgs potential of this self-dual model is equal to

$$V(\phi,\chi) = \frac{q_1^2 q_2^2}{\kappa^2}[\mid\phi\mid^2(\mid\chi\mid^2 - c_2^2)^2 + \mid\chi\mid^2(\mid\phi\mid^2 - c_1^2)^2] \ . \quad (3)$$

$$L = -\kappa\varepsilon^{ij}v_i^{(1)}\partial_t v_j^{(2)} + \frac{\kappa}{q_1}B_{(2)}\partial_t\omega_1 + \frac{\kappa}{q_2}B_{(1)}\partial_t\omega_2 + (\partial_t|\phi|)^2 + (\partial_t|\chi|)^2 - \frac{\kappa^2 B_{(2)}^2}{4q_1^2\mid\phi\mid^2} - \frac{\kappa^2 B_{(1)}^2}{4q_2^2\mid\chi\mid^2} - \mid\mathbf{D}\phi\mid^2 - \mid\mathbf{D}\chi\mid^2 - V(\phi,\chi) \ , \quad (5)$$

where we have introduced magnetic fields corresponding to the two different gauge fields: $B_I = \varepsilon^{ij}\partial_i v_j$. ω_I's are the phases of the Higgs fields. We have separated terms containing time derivatives from those with spatial derivatives of the fields. In this form of the Lagrangian we can make replacements due to the identity

$$-\mid\mathbf{D}\phi\mid^2 = -\mid(D_1 + i\sigma_1 D_2)\phi\mid^2 - \sigma_1 q_1 B_{(1)}\mid\phi\mid^2 + i\sigma_1\varepsilon^{ij}\partial_i(\phi^\star D_j\phi - \text{ c.c. }) \quad (6)$$

and an analogous one for the field χ. The constants σ_1, σ_2 which take values from the set $\{+1,-1\}$ have the same sign as the topological indices of field configurations ϕ and χ, respectively.

In the Manton approximation we assume the fields to have the same form as in some static solution and thus to satisfy the static field equations at every instant of time. In our case the fields satisfy the self-dual equations

$$(D_1 + i\sigma_1 D_2)\phi = 0 \ \text{ and } \ (D_1 + i\sigma_2 D_2)\chi = 0, \quad (7)$$

so the first term on the right-hand side (RHS) of Eq. (6) vanishes. The assumed static solution depends in general on some finite set of parameters which will play the role of collective coordinates. To derive an effective Lagrangian we have to integrate out the spatial dependence of the fields in Eq. (5). Assuming the Coulomb gauge condition for the static solutions, which implies that the gauge potentials can be written as

$$v_i^{(I)} = \varepsilon_{ij}\partial_j U^{(I)} \quad (8)$$

with some regular functions $U^{(I)}$ we can remove first term in the first line of (5) and third term on the RHS of Eq. (6). Further simplification can be achieved with a help of the remaining static field equations

$$B_{(1)} = -2\sigma_1\frac{q_1 q_2^2}{\kappa^2}\mid\chi\mid^2(\mid\phi\mid^2 - c_1^2) \ , \quad (9)$$

$$B_{(2)} = -2\sigma_2\frac{q_2 q_1^2}{\kappa^2}\mid\phi\mid^2(\mid\chi\mid^2 - c_2^2) \ . \quad (10)$$

Neglecting terms which are constant in a given topological sector we obtain the form of the effective Lagrangian

$$L_{\text{eff}}^{(0)} = \int d^2x\left((\partial_t\mid\phi\mid)^2 + (\partial_t\mid\chi\mid)^2 + \frac{\kappa}{q_1}B_2\partial_t\omega_1 + \frac{\kappa}{q_2}B_1\partial_t\omega_2\right) \ . \quad (11)$$

Kim and Lee [7] taught us that it is important to take into account corrections to such direct application of adiabatic approximation which is oversimplified in the case of Chern-Simons vortices. Namely, even in the slow-motion approximation the fields cannot be taken just as static solutions with time dependent parameters. We also have to take into account corrections to those fields which are linear in velocities.

$$\phi \to \phi + \delta\phi, \ \ \chi \to \chi + \delta\chi, \ \ v_\mu^{(I)} \to v_\mu^{(I)} + \delta v_\mu^{(I)}, \quad (12)$$

because they give additional terms to the effective Lagrangian. It is a kind of complicated Lorenz transformation.

We can show that corrections affect only this part of

First we will work out the general form of the effective Lagrangian by a direct application of the methods of adiabatic approximation known from the papers on slow motion of vortices in self-dual Maxwell-Higgs system [10,12] and then we will discuss corrections to this oversimplified version of the Lagrangian [7].

The Lagrangian (1), when we eliminate auxiliary $v_0^{(I)}$ components of the gauge fields with the help of the Gauss law

$$q_1 v_0^{(1)} = \partial_t\omega_1 - \frac{\kappa B_2}{2q_1\mid\phi\mid^2}, \quad q_2 v_0^{(2)} = \partial_t\omega_2 - \frac{\kappa B_1}{2q_2\mid\chi\mid^2}, \quad (4)$$

can be rewritten in the form

effective Lagrangian which is quadratic in velocities. Let us take a closer look at the Lagrangian in the form of Eq. (5). The second line of the above formula is manifestly of at least second order in velocities. The third line is minus static energy density, so the static solutions are its stationary points. Thus the third line also gives only quadratic terms. The only contribution to the linear part can come from the first line. Any replacements of the form (12) will produce only extra second or higher order terms. So we have to take the first line as it stands—only with static fields with time-dependent parameters. As it has been shown the first term of the first line gives no contribution, so the linear part of the Lagrangian remains as

$$L^{(1)}_{\rm eff}=\kappa\int d^2x\left(\frac{B_{(2)}}{q_1}\partial_t\omega_1+\frac{B_{(1)}}{q_2}\partial_t\omega_2\right)\ . \tag{13}$$

Another representation of $L^{(1)}_{\rm eff}$, useful for further discussion can be obtained with use of the identity

$$\begin{aligned}\dot\omega_I&=\sigma_I\frac{d}{dt}\sum_{p_I}{\rm Arg}(\mathbf{x}-\mathbf{R}_{p_I})\\ &=\sigma_I\sum_{p_I}\varepsilon_{ij}\dot R^i_{p_I}\partial_j\ln\mid\mathbf{x}-\mathbf{R}_{p_I}\mid\end{aligned} \tag{14}$$

and certain integration by parts

$$L^{(1)}_{\rm eff}=\frac{2\pi\kappa\sigma_1}{q_1}\sum_{p_1}\dot R^i_{p_1}v^{(2)}_i(\mathbf{R}_{p_1})+\quad(1\leftrightarrow 2)\ . \tag{15}$$

$\mathbf{R}_{p_I}(t)$ are positions of vortices of type "I," $I=1,2$. From this representation one can obtain general form of the orbital part of angular momentum

$$J_{\rm orb}=\sum_I\sum_{p_I}\varepsilon_{ij}R^i_{p_I}\frac{\partial L^1_{\rm eff}}{\partial\dot R^j_{p_I}}=2\pi\kappa\left(\frac{\sigma_1}{q_1}\sum_{p_1}R^i_{p_1}\varepsilon_{ij}v^{(2)}_j(\mathbf{R}_{p_1})+\quad(1\leftrightarrow 2)\right)\ . \tag{16}$$

The form of $L^{(2)}_{\rm eff}$ can be extracted from (5) after replacement (12) and making use of static equations satisfied by "static" fields. It gives correction to the effective Lagrangian (11)

$$\delta L_{\rm eff}[\text{fields},\delta.\text{fields}] \tag{17}$$

which is a functional of "static" fields, their linear corrections and their time derivatives. It is a long expression which we will not write down. To make use of this expression we have to express "δ fields" in terms of "fields" and/or positions of vortices. We have to know corrections for a given trajectory in the parameter space. We can make replacement (12) in the field equations following from (1)

$$\begin{aligned}&D_\mu D^\mu\phi=-\frac{\partial V}{\partial\phi^\star}\ ,\\ &\kappa\varepsilon^{\mu\nu\lambda}\partial_\nu v^{(2)}_\lambda=iq_1(\phi D^\mu\phi^\star-\text{c.c.}),\end{aligned} \tag{18}$$

and an analogous one for the field χ. It is convenient to use geodesic parametrization on moduli space. By geodesic parametrization we mean such a set of parameters that during the time evolution of the field configuration their accelerations are vanishing. It is always in principle possible to construct such geodesic coordinates at least for a finite period of time. One advantage of such a set of coordinates is that we can neglect in Eqs. (18) terms which contain accelerations or higher order time derivatives of parameters. The other one also very important is that only when velocities in a given parametrization are constant there is a direct correspondence between the fact that kinetic energy is small and smallness of velocities. For example in the head-on collision of Nielsen-Olesen vortices the time derivatives of Cartesian coordinates of vortices become singular during collision [10,12], so in this case terms quadratic in such velocities are much larger than linear terms and obviously cannot be neglected. Nevertheless the kinetic energy is still small—it is still a slow motion. In fact it is possible to find geodesic coordinates good for small separations of these vortices. So only for geodesic parametrization is it reliable to preserve in Eqs. (18) only terms which are linear in velocities and neglect accelerations and higher order time derivatives.

Thus "simplified" equations read

$$\begin{aligned}0&=\partial^2_k\delta f_1+[q_1^2v^{(1)}_0v^{(1)}_0-(\partial_k\omega_1-q_1v^{(1)}_k)^2]\delta f_1+2q_1f_1(\partial_k\omega_1-q_1v^{(1)}_k)\delta v^{(1)}_k-2q_1f_1v^{(1)}_0(\dot\omega_{(1)}-q_1\delta v^{(1)}_0)\\ &\quad-\frac{\partial^2V}{\partial f_1^2}\delta f_1-\frac{\partial^2V}{\partial f_1\partial f_2}\delta f_2,\\ 0&=f_1(\dot v^{(1)}_0+\partial_k\delta v^k_{(1)})+2q_1v^{(1)}_0\dot f_1,\\ 0&=\kappa\varepsilon_{ij}\partial_i\delta v^{(2)}_j-q_1f_1^2(\dot\omega_1-q_1\delta v^{(1)}_0)+2q_1^2v^{(1)}_0f_1\delta f_1,\\ 0&=\kappa\varepsilon_{ij}(\partial_j\delta v^{(2)}_0-\dot v^{(2)}_j)+q_1^2f_1^2\delta v^{(1)}_k-2q_1f_1(\partial_k\omega_1-q_1v^{(1)}_k)\delta f_1,\end{aligned} \tag{19}$$

and an analogous set of equations for the field χ. We have introduced moduli and phases of the Higgs fields: $\phi = \frac{f_1}{\sqrt{2}} e^{i\omega_1}$, $\chi = \frac{f_2}{\sqrt{2}} e^{i\omega_2}$. An analogous set of equations was derived in [7] for self-dual Chern-Simons-Higgs model. Neglect of second order time derivatives for any kind of coordinates can lead to serious problems. For example in the papers on effective string models for Nielsen-Olesen vortices the second order derivatives on world-sheet parameters were neglected. It was shown in [16] that it was the reason why the string with rigidity was obtained, which is known to possess classical tachionic solutions [17].

With the help of Eqs. (19) $\delta L_{\rm eff}$ can be simplified to the form

$$\delta L_{\rm eff} = \sum_I \int d^2x \left\{ \tfrac{1}{2} f_I^2 [(\dot{\omega}_I - q_I v_0^I)^2 + q_I^2 (\delta v_i)^2] - f_I \delta f_I q_I v_0^{(I)} (\dot{\omega}_I - q_I v_0^{(I)}) - f_I \delta f_I (\partial_i \omega_I - q_I v_i^{(I)}) q_I \delta v_i^{(I)} \right\}. \quad (20)$$

Once we have solved Eqs. (19) we can substitute their solutions to the above functional and integrate out their spatial dependence. We will be left with a mechanical Lagrangian which should be reparametrization invariant. From that point on we will be able to use any parametrization we like. But to evaluate $\delta L_{\rm eff}$ we have first to guess geodesic coordinates and then to solve Eqs. (19). We have to try with different parametrizations and then to check whether for a given parametrization there exist solutions to Eqs. (19). If yes then as a matter of fact we have found already the trajectory. The effective Lagrangian can be evaluated for consistency check and because it is useful if we want to perform effective quantization of the theory or to investigate its effective thermodynamics [13]. The effective Lagrangian under restrictions due to the above comments reads

$$L_{\rm eff} = \int d^2x \, [(\partial_t \mid \phi \mid)^2 + (\partial_t \mid \chi \mid)^2] + \delta L_{\rm eff} + \kappa \int d^2x \left(\frac{B_2}{q_1} \dot{\omega}_1 + \frac{B_1}{q_2} \dot{\omega}_2 \right) . \quad (21)$$

To start the above procedure we have to make some "educated" guesswork. To provide an appropriate basis for it in the next two sections we investigate ordinary adiabatic approximation in detail. Because of mathematical difficulties [existence proof and/or explicit solution of Eqs. (19)] we postpone calculation of corrections to future publication.

II. LONG-RANGE INTERACTIONS OF SOLITONS

For sufficiently separated vortices we can approximate gauge invariant fields: $(\partial_k \omega_I - q_I v_k^I)$ by contributions due to particular vortices. At the core of any chosen vortex such fields due to the other vortices are very small—they vanish exponentially with distances. Since $\omega_I = \sigma_I \sum_{p_I} \mathrm{Arg}(\mathbf{x} - \mathbf{R}_{p_I})$ is an exact solution of static equations for any configuration of vortices, corrections to the simple superposition of gauge potentials are negligible. Thus values of the gauge fields in Eq. (15) can be obtained from a formula

$$v_i^{(2)}(\mathbf{R}_{p_1}) = -\frac{\sigma_2}{q_2} \sum_{q_1} \frac{\varepsilon_{ij}(R_{p_1} - R_{q_2})^j}{\mid \mathbf{R}_{p_1} - \mathbf{R}_{q_2} \mid^2} \quad , \ (1 \leftrightarrow 2) \quad (22)$$

so the linear part of effective Lagrangian is

$$L_{\rm eff}^{(1)} = 2\pi\kappa \frac{\sigma_1 \sigma_2}{q_1 q_2} \sum_{p_1=1}^{n_1} \sum_{p_2=1}^{n_2} \frac{d}{dt} \mathrm{Arg}(\mathbf{R}_{p_1}^{(1)} - \mathbf{R}_{p_2}^{(2)}) \ . \quad (23)$$

The Lagrangian contains only terms of mutual statistical interaction between vortices of different types.

To obtain the kinetic term one has to evaluate the first term of the effective Lagrangian (21) and $\delta L_{\rm eff}$. We approximate the moduli of the Higgs fields by a normalized product of the fields of isolated unit vortices:

$$\mid \phi(\mathbf{x}) \mid = c_1 \prod_{p_1=1}^{n_1} G(\mid \mathbf{x} - \mathbf{R}_{p_1}^{(1)} \mid) \ , \quad (24)$$

$$\mid \chi(\mathbf{x}) \mid = c_2 \prod_{p_2=1}^{n_2} G(\mid \mathbf{x} - \mathbf{R}_{p_2}^{(2)} \mid) \ , \quad (25)$$

where G is a profile of a unit vortex the same for the two types, which satisfies the equation

$$\nabla^2 \ln G^2 = \frac{4 q_1^2 q_2^2 c_1^2 c_2^2}{\kappa^2} (G^2 - 1) \ , \quad (26)$$

with the boundary conditions: $G(0) = 0$ and $G(\infty) = 1$. The quadratic part of the effective Lagrangian reads

$$L_{\rm eff}^{(2)} = \frac{1}{2} \bar{M} \left(c_1^2 \sum_{p_1=1}^{n_1} \mathbf{V}_{p_1}^2 + c_2^2 \sum_{p_2=1}^{n_2} \mathbf{V}_{p_2}^2 \right) + \delta L_{\rm eff} \ , \quad (27)$$

where the coefficient $\bar{M}$ equals to

$$\bar{M} = 2\pi \int r dr \left(\frac{dG(r)}{dr} \right)^2 \ . \quad (28)$$

About $\delta L_{\rm eff}$ we know only that it is quadratic in velocities. Let us restrict to the case of two vortices of the same type and choose the center-of-mass frame

$$L_{\rm eff}^{(2)} = g_{ij}(R^k) \dot{R}^i \dot{R}^j \ , \ g_{ij} = g_{ji} \ , \quad (29)$$

where R_k's are coordinates of the chosen vortex. Rotational invariance restricts this form to

$$L_{\rm eff}^{(2)} = g_1(R) \dot{R}^2 + g_2(R) R^2 \dot{\Theta}^2 \ , \quad (30)$$

where $R^1 + iR^2 = Re^{i\Theta}$. At very large R we expect the influence of one vortex on another to be very small: $g_1(R)$, $g_2(R)$ $\to$const as $R \to \infty$. This reasoning can be repeated for any pair of vortices. Finally we obtain

$$L_{\text{eff}} = \frac{1}{2}M\left(c_1^2\sum_{p_1=1}^{n_1}\mathbf{V}_{p_1}^2 + c_2^2\sum_{p_2=1}^{n_2}\mathbf{V}_{p_2}^2\right) + 2\pi\kappa\frac{\sigma_1\sigma_2}{q_1q_2}\sum_{p_1=1}^{n_1}\sum_{p_2=1}^{n_2}\frac{d}{dt}\text{Arg}(\mathbf{R}_{p_1}^{(1)} - \mathbf{R}_{p_2}^{(2)}) \ , \quad (31)$$

where $M = \bar{M} + \delta M$ is the effective mass with included corrections from δL_{eff}. Corrections can be calculated with a help of formula (20), since for fairly separated vortices "δ fields" are given by ordinary Lorenz formulas linearized in velocities. The corrected coefficient M appears to be equal 2π, what is consistent with the original field theoretical model since the energy of a static unit vortex of type "I" is equal to $2\pi c_I^2$.

We can see that when the widths of the vortices can be neglected as compared with their separations the system behaves like a set of free particles with mutual statistical interactions, at least in the slow-motion approximation. What happens if the vortices come into very close encounters of one another is a subject of the next section.

Let us remark here on the possibility of ordinary fractional statistics in the system if the short-range interactions between vortices of different types favored them to form mixed anionic $\phi - \chi$ bound states.

III. SHORT-RANGE INTERACTIONS

In this section we would like to investigate interactions of the two types of vortices when their cores overlap. To apply Manton's prescription we have to know at least an approximate static solution. Let us take as a zero order approximation the configuration of vortices sitting on top of each other and then let us find a small perturbation of the field dependent on a definite set of parameters.

Let the topological indices of the fields ϕ and χ be $\sigma_1 n_1$ and $\sigma_2 n_2$, respectively, where n's are positive integers and σ's take values $+1$ or -1. The solution corresponding to vortices sitting on top of each other takes the form

$$\phi = c_1 F(r)e^{i\sigma_1 n_1\theta} \ , \ \chi = c_2 H(r)e^{i\sigma_2 n_2\theta} \ , \quad (32)$$

$$\mathbf{v}_{(1)} = \mathbf{e}_\theta \sigma_1 V_{(1)}^\theta \ , \ \mathbf{v}_{(2)} = \mathbf{e}_\theta \sigma_2 V_{(2)}^\theta \ . \quad (33)$$

Upon substitution of the above ansatz to the static field equations (7), (9), and (10) one obtains

$$F' - \frac{n_1}{r}F + q_1 F V_\theta^{(1)} = 0 \ , \quad (34)$$

$$H' - \frac{n_2}{r}H + q_2 H V_\theta^{(2)} = 0 \ , \quad (35)$$

$$\frac{-1}{r}\frac{\partial(rV_\theta^{(1)})}{\partial r} = q_2\gamma H^2(F^2-1) \ , \quad (36)$$

$$\frac{-1}{r}\frac{\partial(rV_\theta^{(2)})}{\partial r} = q_1\gamma F^2(H^2-1), \ \ \gamma = \frac{2q_1q_2c_1^2c_2^2}{\kappa^2} \ . \quad (37)$$

This zero order solution together with a small perturbation reads

$$\phi = c_1 F(r)[1 + f(r,\theta)]e^{i\sigma_1 n_1\theta + i\alpha_1(r,\theta)} \ , \quad (38)$$

$$\chi = c_2 H(r)[1 + h(r,\theta)]e^{i\sigma_2 n_2\theta + i\alpha_2(r,\theta)} \ , \quad (39)$$

$$\mathbf{v}_{(I)} = \mathbf{e}_\theta \sigma_I V_{(I)}^\theta + \mathbf{a}_{(I)} \ , \ I = 1,2. \quad (40)$$

Upon substitution of the above form of the solution to the self-dual equations (7) and subsequent linearization one obtains the first order equations

$$q_I a_\theta^{(I)} = \frac{1}{r}\partial_\theta \alpha^{(I)} - \sigma_I \partial_r f^{(I)} \ , \quad (41)$$

$$q_I a_r^{(I)} = \partial_r \alpha^{(I)} + \sigma_I \frac{\partial_\theta f^{(I)}}{r} \ . \quad (42)$$

These equations enable us to find perturbations of the gauge fields once $f^{(I)} = (f,h)$ and $\alpha_{(I)}$'s are already known.

After substitution of Eqs. (40) to the Coulomb gauge condition, $\partial_i v_i^{(I)} = 0$, linearization and use of Eqs. (41) and (42) an equation determining the phases of the Higgs fields appears:

$$\nabla^2 \alpha_{(I)} = 0 \ . \quad (43)$$

Similarly Eqs. (34) and (35) linearized in the perturbations yield

$$\nabla^2 f = \bar{\gamma} H^2[F^2 f + (F^2-1)h] \ , \quad (44)$$

$$\nabla^2 h = \bar{\gamma} F^2[H^2 h + (H^2-1)f] \ , \quad (45)$$

where $\bar{\gamma} = 2q_1q_2\gamma$. From the above two equations one can obtain a general form of the perturbation of the moduli of the Higgs fields. Then one has to choose such a solution of Eq. (43) so as to avoid singularities of the gauge fields, see Eqs. (41) and (42). We will solve this problem explicitly in two special cases.

A. Interaction of unit ϕ vortex with unit χ vortex

In this case we have unit topological indices of both the ϕ field configuration and the χ field one

$$n_1 = 1, \ \ n_2 = 1. \quad (46)$$

Equations (34)–(37) can be simplified by a substitution

$$F(r) = H(r) \equiv G(r), \ \ \frac{V_\theta^{(1)}}{q_2} = \frac{V_\theta^{(2)}}{q_1} \equiv V_\theta(r) \ , \quad (47)$$

to the form

$$G' - \frac{G}{r} + q_1q_2 G V_\theta = 0 \ , \quad (48)$$

$$\frac{-1}{r}\frac{\partial(rV_\theta)}{\partial r} = \gamma G^2(G^2-1) \ . \quad (49)$$

From the index theorem [4] we know that in the special case of $n_1 = n_2 = 1$ there are only two splitting modes for each of the two types of the fields. We can Fourier transform f and h in θ:

$$f(r,\theta) = f(r)[\lambda_1 \cos(\sigma_1\theta) + \lambda_2 \sin(\sigma_1\theta)] \ , \tag{50}$$

$$h(r,\theta) = h(r)[\mu_1 \cos(\sigma_2\theta) + \mu_2 \sin(\sigma_2\theta)] \ . \tag{51}$$

The θ-independent terms are neglected because solitons have a definite size, while the terms higher than the first would spoil regularity of the total Higgs fields. To define the meaning of coefficients λ, μ we normalize the radial functions in such a way that

$$f(r) \sim \frac{-1}{r} \ , \ h(r) \sim \frac{-1}{r} \ , \ \text{as } r \to 0 \ . \tag{52}$$

Eventual higher order terms would have a stronger singularity which could not be matched by $G(r) \sim r$, see Eqs. (38) and (39). Thus Eqs. (51) and (52) present the general form of the perturbation compatible with the regularity of Higgs fields.

To avoid singularities in the gauge fields, Eqs. (41) and (42) we take the perturbations of phases

$$\alpha_1 = \frac{-1}{r}[\lambda_2 \cos(\sigma_1\theta) - \lambda_1 \sin(\sigma_2\theta)] \ , \tag{53}$$

$$\alpha_2 = \frac{-1}{r}[\mu_2 \cos(\sigma_2\theta) - \mu_1 \sin(\sigma_2\theta)] \ , \tag{54}$$

which satisfy Eqs. (43). Now we can see that the perturbed Higgs fields in the limit of small r are proportional to

$$\phi \simeq (z_{\sigma_1} - \lambda) \ , \ \chi \simeq (z_{\sigma_2} - \mu) \ , \tag{55}$$

where we have introduced $z_{\sigma_I} = x + i\sigma_I y$ and $\lambda = \lambda_1 + i\lambda_2$, $\mu = \mu_1 + i\mu_2$. The effect of the perturbation is a shift of the zeros of the Higgs fields to λ_{σ_1} and to μ_{σ_2}, up to linear terms. To work in the center-of-mass frame we have to choose $\mu_{\sigma_2} = -\lambda_{\sigma_1}$ together with $c_1 = c_2 = c$. The last condition can be suspended if we make certain relative rescaling of the parameters λ and μ. Without loss of generality in the evaluation of $f(r)$ and $h(r)$ we can choose $\lambda_2 = \mu_2 = 0$. Now upon substitution of Eqs. (50) and (51) to Eqs. (44) and (45) one obtains

$$\triangle_1 f(r) = \bar{\gamma} G^2[G^2 f(r) - (G^2 - 1)h(r)] \ , \tag{56}$$

$$\triangle_1 h(r) = \bar{\gamma} G^2[G^2 h(r) - (G^2 - 1)f(r)] \ , \tag{57}$$

$$\triangle_k \equiv \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2} \right) \ .$$

We can make the replacements $f(r) = w(r) + u(r)$ and $h(r) = w(r) - u(r)$, where the newly introduced functions satisfy the equations

$$\triangle_1 w(r) = \bar{\gamma} G^2 w(r) \ , \tag{58}$$

$$\triangle_1 u(r) = \bar{\gamma} G^2 (2G^2 - 1) u(r) \ . \tag{59}$$

The only solution compatible with normalization (52) is $u(r) = 0$ and $f(r) = h(r) = w(r)$. $w(r)$ can be approximated for large r by the modified Bessel function $K_1(\rho) \sim \sqrt{\frac{\pi}{2\rho}} e^{-\rho}, \rho \equiv \sqrt{\bar{\gamma}} r$. Going from infinity to zero this approximate solution is replaced by a linear combination $\delta_{+1} r + \delta_{-1} \frac{1}{r}$. In the case of $\delta_{-1} \neq 0$, which we think to be quite general, it is possible to rescale the whole function $w(r)$ in such a way that we obtain asymptotics of Eq. (52). The perturbations of the Higgs fields are square integrable.

Now the effective Lagrangian (21) reads

$$L_{\text{eff}} = \tfrac{1}{2} M_{\text{eff}} (\dot{R}^2 + R^2 \dot{\Theta}^2) + \delta L_{\text{eff}} + B_{\text{eff}} R^2 \dot{\Theta} \ . \tag{60}$$

R and Θ are the polar coordinates of the zero of the ϕ field: $\lambda \equiv R e^{i\Theta}$, while the introduced coefficients are an effective reduced mass of the two vortices

$$M_{\text{eff}} = 4\pi c^2 \int_0^\infty (r dr) G^2(r) w^2(r) \ , \tag{61}$$

and an effective "uniform external magnetic field"

$$B_{\text{eff}} = \frac{8\pi q_1 q_2 c^4}{\kappa \sigma_1 \sigma_2} \int_0^\infty dr G^2(r)[-w(r)] \ . \tag{62}$$

Let us first discuss the linear part of the Lagrangian (60), which becomes exact for very small R. It is a term which describes, as it stands, coupling of a charged particle to a uniform external magnetic field perpendicular to the plane. The total angular momentum in the effective description is (for $\dot{\Theta} = 0$, $R \to 0$)

$$\begin{aligned} J(R) &\stackrel{def}{=} -\frac{2\pi\kappa\sigma_1\sigma_2}{q_1 q_2} + \frac{\partial L_{\text{eff}}}{\partial \dot{\Theta}} \\ &= -\frac{2\pi\kappa\sigma_1\sigma_2}{q_1 q_2} + B_{\text{eff}} R^2 + O(R^3) \ , \end{aligned} \tag{63}$$

where we have shifted the scale so that for $R \to 0$, J tends to the value of spin characteristic for coincident static unit ϕ and χ vortices. With this choice of scale we obtain from (31) that for large R: $J \to 0$. This result is consistent with what we know from field-theoretical considerations. Separate ϕ or χ mutually interacting vortices carry no spin. Spin is nonzero only when their cores overlap. Equation (63) gives leading terms in expansion of $J(R)$ around $R = 0$.

It is interesting that Eq. (63) can be inverted in a remarkable way:

$$B_{\text{eff}} = \lim_{R \to 0} \frac{1}{R} \left(\frac{J(R) - J(0)}{R} \right) = \lim_{R \to 0} \frac{1}{R} \frac{dJ(R)}{dR} \ . \tag{64}$$

A natural thing is to ask whether such a formula can be generalized to an arbitrary value of R. Let us look at $L_{\text{eff}}^{(1)}$ in the form of Eq. (15):

$$L_{\text{eff}}^{(1)} = \frac{2\pi\kappa\sigma_1}{q_1} \sum_{p_1} \dot{R}_{p_1}^i v_i^{(2)}(\mathbf{R}_{p_1}) + \ (1 \leftrightarrow 2) \ . \tag{65}$$

What we see is an interaction term which couples point particle currents to fields $v_i^{(I)}$ defined on the moduli space. Due to this coupling vortex at $\mathbf{R}_{p_1}$ feels magnetic field

$$B_{\rm eff}^{(1)}(\mathbf{R}_{p_1}) = \frac{2\pi\kappa\sigma_1}{q_1}\varepsilon_{ij}\partial_i v_j^{(2)}(\mathbf{R}_{p_1}),\ \ \partial_i = \frac{\partial}{\partial R_{p_1}^i} \ . \quad (66)$$

Our pair of vortices feels double this field: $B_{\rm eff} = B_{\rm eff}^{(1)} + B_{\rm eff}^{(2)}$. From the formula (16) we obtain angular momentum, which in our case reads

$$J_{\rm orb}(\mathbf{R}) = 2\pi\kappa\frac{\sigma_1}{q_1}R^i\varepsilon_{ij}v_j^{(2)}(\mathbf{R}) - 2\pi\kappa\frac{\sigma_2}{q_2}R^i\varepsilon_{ij}v_j^{(1)}(-\mathbf{R}) \ , \quad (67)$$

Due to the rotational symmetry, in polar coordinates

$$J_{\rm orb}(R) = 2\pi\kappa\frac{\sigma_1}{q_1}Rv_\theta^{(2)}(R) + 2\pi\kappa\frac{\sigma_2}{q_2}Rv_\theta^{(1)}(R) \ . \quad (68)$$

Now we can see that

$$\frac{1}{R}\frac{dJ}{dR} = \frac{1}{R}\frac{dJ_{\rm orb}}{dR} = 2\pi\kappa\frac{\sigma_1}{q_1}\left(\frac{v_\theta^{(2)}}{R} + \frac{dv_\theta^{(2)}}{dR}\right) + (\,1\leftrightarrow 2\,)$$
$$= B_{\rm eff}^{(1)}(R) + B_{\rm eff}^{(2)}(R) = B_{\rm eff}(R) \ . \quad (69)$$

Thus Eq. (64) can indeed be generalized for any value of R:

$$B_{\rm eff}(R) = \frac{1}{R}\frac{dJ(R)}{dR} \ , \quad (70)$$

where $B_{\rm eff}(R)$ is a magnetic field felt by a reduced particle in our problem, while $J(R)$ is a total field-theoretical spin, which can be obtained numerically. It is important that this formula is based on global properties of the field configurations and not on local distortions of the fields, so $B_{\rm eff}(R)$ can be calculated numerically with great accuracy.

Just from the knowledge of spin dependence on R we can obtain qualitative understanding of interactions between ϕ and χ vortices. Vortices boosted against each other for a head-on collision will avoid direct collision. Their initial total angular momentum is zero. For very small separations spin itself would have to be close to $(-\frac{2\pi\kappa\sigma_1\sigma_2}{q_1q_2})$ so the vortices must acquire angular momentum of the opposite value. For $\frac{\sigma_1\sigma_2}{q_1q_2}$ positive they will be turned by $B_{\rm eff}(R)$ to the right while for the negative value of the coefficient they will turn left. Since $B_{\rm eff}(R)$ is short ranged, for sufficiently small initial velocity it becomes impossible to reach the center of mass for any value of impact parameter. Analogous argument shows that if a ϕ vortex initially sits on top of a χ vortex there is a velocity small enough below which they cannot escape to infinity. So it is a kind of magnetic trap for vortices. They trap each other and form a composite. From Eq. (31) is clear that such composites as a whole behave like anions. Their internal reduced dynamics is that of a particle in an external magnetic field. Upon quantization it is probable to obtain resonances classified by the Landau levels. The trapped vortices do not form a real bound state—their energy is the same as that of isolated vortices.

Now let us take a look at the quadratic part of the effective Lagrangian. If we assume that $\delta L_{\rm eff}$ only renormalizes $M_{\rm eff}$ then $L_{\rm eff}$ is a Lagrangian for a planar motion of charged particle in external uniform magnetic field perpendicular to the plane. Solutions to the equations of motion are circular trajectories

$$R(t)e^{i\Theta(t)} = A + Be^{-i\Omega t} \ , \ \ \Omega = \frac{2B_{\rm eff}}{M_{\rm eff}} \ , \quad (71)$$

where A, B are complex constants. This solution is valid only for small R. The trapped vortices rotate around circles (not necessarily around the center of mass if $A \neq 0$). We can also obtain qualitative understanding of scattering. Let us assume that this approximation is valid up to say R_0, for larger R let the vortices move along straight lines. In the head-on collision vortices cross the circle $R = R_0$, move along an arc of a circle and escape to infinity. For larger velocities, if the adiabatic approximation still works, scattering pattern evolves to a forward scattering. We can check whether $M_{\rm eff}$ is indeed a constant for small R by performing numerical simulation of the above-described head-on collision. For example, if for higher velocities right-angle scattering is obtained, $M_{\rm eff}$ must behave like R^2 for small R. From analogous simulations for Nielsen-Olesen vortices we can expect that the adiabatic approximation can still work well even up to $\frac{1}{3}$ of the light velocity.

The eventual dependence of $M_{\rm eff}$ on R or its constant renormalized value can be obtained from a measurement of the frequency of the purely orbital motion of the trapped $\phi - \chi$ pair:

$$\frac{1}{2}M_{\rm eff}(R) = \frac{B_{\rm eff}(R)}{\Omega(R)} \ , \quad (72)$$

where it is assumed that $B_{\rm eff}(R)$ is already known from (70) and $\Omega(R)$ has to be measured in numerical simulation. An alternative analytical approach is to take Θ as a geodesic coordinate and try to find such a value of $\Omega = \dot{\Theta}$ for which Eqs. (19) possess a unique solution.

B. Interactions of unit vortices of a given type

In this subsection we would like to investigate short-range interactions of say ϕ vortices when there are no χ vortices or their influence can be neglected because they are very distant. We put $\chi = c_2$ and $n_1 = n$, $\sigma_1 = +1$ in this paragraph. The configuration of n vortices splitting from their coincident position is of the form

$$\phi(r,\theta) = cF(r)[1 + f(r,\theta)]e^{in\theta + i\alpha(r,\theta)},$$
$$\mathbf{V}_{(1)} \equiv \mathbf{V} = \mathbf{e}_\theta V^\theta(r) + \mathbf{a}(r,\theta), \quad (73)$$

where the functions $F(r)$ and $V^\theta(r)$ satisfy equations

$$F' - \frac{n}{r}F + q_1FV_\theta = 0 \ , \quad (74)$$

$$\frac{-1}{r}\frac{\partial(rV_\theta)}{\partial r} = q_2\gamma c_2^2(F^2-1) \ , \tag{75}$$

Following similar steps as in Sec. III A we obtain equations satisfied by the perturbations

$$q_1 a_i = \partial_i\alpha + \varepsilon_{ij}\partial_j f \ , \tag{76}$$

$$\nabla^2\alpha = 0 \ , \tag{77}$$

$$\nabla^2 f = \bar{\gamma}F^4 f \ , \tag{78}$$

and solve them by an ansatz

$$f(r,\theta) = g(r)[\lambda_1\cos(n\theta) + \lambda_2\sin(n\theta)] \ , \tag{79}$$

$$\alpha(r,\theta) = \frac{-1}{r^n}[\lambda_2\cos(n\theta) - \lambda_1\sin(n\theta)] \ , \tag{80}$$

where $g(r)$ is a solution of equation

$$\triangle_n g(r) = \bar{\gamma}F^4(r)g(r) \ . \tag{81}$$

$g(r)$ is normalized so that $g(r) \sim \frac{-1}{r}$ for small r. We have taken the nth term in the Fourier transform because it corresponds to a uniform splitting of the n vortices from a coincident position

$$\phi \sim (z^n - \lambda) \ , \ \lambda = \lambda_1 + i\lambda_2 \ . \tag{82}$$

The positions of the n vortices are nth order roots of λ, which we denote by $Re^{i\Theta + i\frac{2\pi}{n}k}$, $k = 0, ..., (n-1)$. In the case of only ϕ vortices the effective Lagrangian reduces to

$$\begin{aligned} L_{\text{eff}} &= \int d^2x \ |\partial_t\phi|^2 \ + \delta L_{\text{eff}} \\ &= \pi c_1^2 \int r dr \ F^2 g^2(\dot{\lambda}_1^2 + \dot{\lambda}_2^2) \ + \ \delta L_{\text{eff}} \ . \end{aligned} \tag{83}$$

When we take into account that $\lambda = R^n e^{in\Theta}$ we can rewrite the Lagrangian as

$$\begin{aligned} L_{\text{eff}} &= \tfrac{1}{2}M_{\text{eff}}(\dot{\lambda}\dot{\lambda}^\star) + \delta L_{\text{eff}} \\ &= \tfrac{1}{2}M_{\text{eff}}R^{2(n-1)}(\dot{R}^2 + R^2\dot{\Theta}^2) + \delta L_{\text{eff}} \ . \end{aligned} \tag{84}$$

If we neglected δL_{eff} in (83) it would follow that in the head-on collision λ turns to $-\lambda$ and it is clear from (82) that it means scattering by an angle of $\frac{\pi}{n}$. The configuration of n vortices shrinks to a coincident position and then reappears but rotated by an angle of $\frac{\pi}{n}$ with respect to the initial one.

Let us analyze corrections due to δL_{eff}. First we make a hypothesis that λ is a geodesic coordinate or $\ddot{\lambda} = 0$ for small values of λ. The "fields" are regular as functions of λ in $\lambda = 0$. By regularity we mean finiteness and single valuedness. The "δ fields" are defined as linear in $\dot{\lambda}$ and we take them as a series in the powers of λ (components λ_1, λ_2). If we substitute such "fields" and "δ fields" to the field equations (18) we will find that because "fields" are regular the "δ fields" can also be taken self-consistently as regular in $\lambda = 0$. In the limit $\lambda \to 0$ we will obtain equations which are linear in $\dot{\lambda}_1$,$\dot{\lambda}_2$ and in the values of "δ fields" at $\lambda = 0$. If there is a solution to these equations we can use it to calculate δL_{eff} and because "δ fields" are regular at $\lambda = 0$ correction to the effective Lagrangian amounts only to renormalization of M_{eff}. If there is no solution it will mean that the initial assumption $\ddot{\lambda} = 0$ was wrong and we have to look for other candidates for geodesic coordinates.

The following argument can restrict the set of acceptable candidates. Let us take the head-on collision of two ϕ vortices. Initially they are coming to the center of mass along the x axis. Such a configuration is invariant under successive charge conjugation and reflection with respect to the y axis. Since our theory is CP-invariant the time evolution has to preserve this symmetry of the initial configuration, so if the zeros of the Higgs field pass through the center of mass there is possible only forward scattering or right-angle scattering. For the right-angle scattering λ_i's are good geodesic coordinates but the forward scattering is well described by the Cartesian coordinates of vortices (zeros of the Higgs field). If $\ddot{\lambda} = 0$ leads to contradiction that means that we will have to try with R, Θ.

Since these calculations are a fairly nontrivial problem we will only conclude that only forward or $\frac{\pi}{n}$ scattering are possible in the head-on collision of vortices of the same type. Knowledge of the exact form of L_{eff} could be useful to effective quantization of the model and to investigations of its thermodynamics [13].

IV. DUAL FORMULATION

In this section we derive dual formulation for mutually interacting vortices, following similar steps as in [7] for ordinary Chern-Simons-Higgs system. There are two reasons for it. We would like to show that the dual transformation can be in a natural way generalized to the systems with more complicated Chern-Simons terms. Second and more important—it explains why the sign of statistical interaction is inverse to what could be expected from calculation of naive Aharonov-Bohm phase. We will use Lagrangian (1) with extra couplings to external currents and external field

$$L = \kappa\varepsilon^{\mu\nu\lambda}v_\mu^{(1)}\partial_\nu v_\lambda^{(2)} - V(f_1,f_2) + \sum_{I=1}^{2}[\ \tfrac{1}{2}(\partial_\mu f_I)^2 + \tfrac{1}{2}f_I^2(\partial_\mu\omega_I - q_I v_\mu^{(I)} - e_I A_\mu^{\text{ext}})^2 + v_\mu^{(I)}J_{(I)}^\mu\ ] \ , \tag{85}$$

where $A_\mu^{\rm ext}$ is an external field and $J_{(I)}^\mu$ are external currents. We have rewritten Higgs fields as

$$\phi = \frac{1}{\sqrt{2}} f_1 e^{i\omega_1} \quad , \quad \chi = \frac{1}{\sqrt{2}} f_2 e^{i\omega_2} \quad . \tag{86}$$

The partition function is

$$Z = \int \prod_I [df_I][d\omega_I][dv_I^\mu] \prod_x f_I(x) \ \exp\ i \int d^3x\ L. \tag{87}$$

Phases of the Higgs fields can be split into multivalued and regular parts

$$\omega_I(x) = \bar{\omega}_I(x) + \eta_I(x) \ , \tag{88}$$

where $\bar{\omega}$'s are given by

$$\bar{\omega}_I(t,\mathbf{r}) = \sum_{p_I} \sigma_{p_I} \mathrm{Arg}[\mathbf{r} - \mathbf{R}_{p_I}(t)] \ . \tag{89}$$

$\mathbf{R}_{p_I}$ are positions of unit vortices ($\sigma_{p_I} = 1$) and unit antivortices ($\sigma_{p_I} = -1$) of the type "I." It is convenient to construct out of $\bar{\omega}$ vortex currents

$$\begin{aligned} K_I^\mu(x) &= \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda \bar{\omega}_I \\ &= \sum_{p_I} \sigma_{p_I} \int d\tau \frac{dR_{p_I}^\mu(\tau)}{d\tau} \delta[x - R_{p_I}(\tau)] \end{aligned} \tag{90}$$

in a covariant fashion. By definition K satisfies the conservation law: $\partial_\mu K^\mu = 0$. Integration over $\bar{\omega}$ can be replaced by integration over vortex world lines

$$[d\omega] = [d\bar{\omega}][d\eta] = [dR_{p_I}^\mu][d\eta] \ . \tag{91}$$

The Jacobian $\prod_I \prod_x f_I(x)$ can be removed by introducing pair of auxiliary fields C_I^μ:

$$\begin{aligned} &\prod_I \prod_x f_I(x) \ \exp\ i \int d^3x \sum_I f_I^2 (\partial_\mu \omega_I - q_I v_\mu^I - e_I A_\mu^{\rm ext})^2 \\ &\quad = \int \prod_I [dC_I^\mu] \ \exp\ i \int d^3x \sum_I \left[-\frac{1}{2f_I^2} C_\mu^I C_I^\mu + C_I^\mu (\partial_\mu \bar{\omega}_I + \partial_\mu \eta_I - q_I v_\mu^I - e_I A_\mu^{\rm ext}) \right]. \end{aligned} \tag{92}$$

Integration over η's will introduce $\prod_I \delta(\partial_\mu C_I^\mu)$ to the path integral measure. These δ functions can be removed by introducing a pair of dual gauge fields H_I^μ:

$$\int \prod_I [dC_I^\mu] \delta(\partial_\mu C_I^\mu) = \int \prod_I [dC_I^\mu][dH_I^\mu] \delta\left(C_I^\mu - \frac{1}{2\pi q_I} \varepsilon^{\mu\nu\lambda} \partial_\nu H_\lambda^I \right) \tag{93}$$

and integrating over auxiliary fields $C_\mu^{(I)}$. Now the vortex currents can be introduced by the identity

$$\int d^3x \ \frac{1}{2\pi q_I} \varepsilon^{\mu\nu\lambda} \partial_\mu \bar{\omega}_I \partial_\nu H_\lambda^I = \frac{1}{q_I} \int d^3x \ K_I^\mu H_\mu^I \ , \tag{94}$$

where integration by parts has been done and we have made use of the definition of K_I^μ , see (90). The present intermediate form of the Lagrangian reads

$$\begin{aligned} L &= \kappa \varepsilon^{\mu\nu\lambda} v_\mu^{(1)} \partial_\nu v_\lambda^{(2)} + \sum_I \frac{1}{2\pi} v_\mu^I \varepsilon^{\mu\nu\lambda} \partial_\nu H_\lambda^I \\ &\quad + \sum_I \left(-\frac{1}{16\pi^2 f_I^2 q_I^2} H_{\mu\nu}^I H_I^{\mu\nu} + \frac{1}{q_I} H_\mu^I K_I^\mu + \frac{1}{2} (\partial_\mu f_I)^2 - \frac{e_I}{4\pi q_I} \varepsilon^{\mu\nu\lambda} H_\mu^I F_{\nu\lambda}^{\rm ext} + v_\mu^I J_I^\mu \right) - V(f_1, f_2) \end{aligned} \tag{95}$$

and the integration measure

$$\prod_I [df_I][dR_{p_I}^\mu][dH_I^\mu][dv_I^\mu] \ . \tag{96}$$

We would like to remove the gauge fields v_I^μ from the Lagrangian. Let us take a look at their classical field equations following from (95):

$$\kappa \varepsilon^{\mu\nu\lambda} \partial_\nu v_\lambda^{(2)} + \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu H_\lambda^{(1)} + J_{(1)}^\mu = 0 \ , \tag{97}$$

$$\kappa \varepsilon^{\mu\nu\lambda} \partial_\nu v_\lambda^{(1)} + \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu H_\lambda^{(2)} + J_{(2)}^\mu = 0. \tag{98}$$

Motivated by these equations we write

$$v_\mu^{(1)} = -\frac{1}{2\pi\kappa} H_\mu^{(2)} + G_\mu^{(1)}, \quad v_\mu^{(2)} = -\frac{1}{2\pi\kappa} H_\mu^{(1)} + G_\mu^{(2)}, \tag{99}$$

where $G_\mu^{(I)}$ are extra fields due to the presence of external currents and containing quantum fluctuations around classical solution. Thus finally the partition function reads

$$Z = \int \prod_I [df_I][dR^\mu_{p_I}][dH^\mu_I][dG^\mu_I] \exp\left(i \int d^3x \, L_D\right) , \tag{100}$$

where the dual Lagrangian is

$$L_D = -\frac{1}{4\pi^2\kappa}\varepsilon^{\mu\nu\lambda} H_\mu^{(1)}\partial_\nu H_\lambda^{(2)} + \sum_I \left(-\frac{1}{16\pi^2 f_I^2 q_I^2} H^I_{\mu\nu}H_I^{\mu\nu} + \frac{1}{q_I} H^I_\mu K^\mu_I\right) + \sum_I [\tfrac{1}{2}(\partial_\mu f_I)^2] - [V(f_1,f_2)]$$
$$+\kappa\varepsilon^{\mu\nu\lambda} G_\mu^{(1)}\partial_\nu G_\lambda^{(2)} \sum_I \left(-\frac{e_I}{4\pi q_I}\varepsilon^{\mu\nu\lambda} H^I_\mu F^{\rm ext}_{\nu\lambda} + G^I_\mu J^\mu_I\right) - \frac{1}{2\pi\kappa} H_\mu^{(2)} J^\mu_{(1)} - \frac{1}{2\pi\kappa} H_\mu^{(1)} J^\mu_{(2)} . \tag{101}$$

When there is no external current the fields G_μ^I decouple and can be integrated out giving contribution to the normalization factor. If in addition $F^{\rm ext}_{\mu\nu} = 0$ we are left only with the first line of the above dual Lagrangian.

Vortex current couple to the dual fields H^μ. Since the Chern-Simons term for the dual fields has an opposite sign to that for the original gauge fields v_I^μ it explains why the statistical interaction term in the effective Lagrangian (12) has an opposite sign to that expected from the values of vortex fluxes and charges. The dual Aharonov-Bohm interaction between vortices is mediated by the dual fields and it gives rise to the correct value of the statistical interaction.

V. CONCLUSIONS

We have made an analysis of interactions of self-dual Chern-Simons vortices in the limit of very large and very small separations. We have shown the existence of mutual statistical interaction between vortices of different types in $[\mathrm{U}(1)]^2$ model but the results can be easily generalized to the general $[\mathrm{U}(1)]^N$ theory. The sign of the statistical interaction is inverse to expectations based on ordinary Aharonov-Bohm effect. That is why we have derived dual formulation of the system in which it is clear that vortices interact via dual gauge field with the sign of the mutual Chern-Simons term inverse to that in the original formulation.

We have not attempted calculating corrections to the standard adiabatic approximation but a possible method how it could be done was discussed. If the corrections are only quantitative in nature the qualitative picture of short-range interactions obtained in ordinary adiabatic approximation remains unchanged. In the head-on collision of vortices of the same type we should observe right-angle scattering. For vortices of different types at large separations dual Aharonov-Bohm effect is observed but when their cores overlap they behave like charged particles crossing magnetic flux.

The analysis of the short-range interactions of vortices in Chern-Simons-Higgs systems presented in both this paper and in [7,9] shows the possibility of periodic solutions very much like bound states of vortices. The semiclassical quantization of these solutions [11] can give rise to some discrete spectra of energy. The spectra can be expected as an additional quantum effect to the statistical interaction due to the finite width of vortices. The special effects of the short-range interactions could be expected to vanish in the limit of vanishing thickness of vortices but we know from the studies of the string limit for vortices in the Abelian Higgs model [14] that even when classical vortices become very thin the quantum fluctuations cause that they preserve nonzero effective thickness. So it is possible that a classical vortex of finite width is a better zero order approximation to the full quantum theory [15].

The other topic worth of detailed study is the possibility of existence of ordinary fractional statistics in the apparently only mutually interacting system. The pair of ϕ and χ vortices can form a composite thanks to the magnetic trapping. If the potential of the model $V(\phi,\chi)$ were slightly deformed in such a way that it would prefer energetically overlapping of the ϕ vortices and χ vortices but it would discourage vortices of the same type to overlap, than we would expect vortices of different types to form true stable bound states. If we wanted bound states to be composed of exactly one vortex of each type we would have to make a repulsion of the species of the same kind to be stronger than attraction of vortices of different types. Such a multivortex system in a sufficiently low temperature would be a gas of such anionic bound states. In a higher temperature the average kinetic energy of the anions could be large enough to split them into particular mutually interacting vortices. Thus we can construct a system with two phases: an anionic one and a phase with mutual statistics. The composed anions might have an interesting internal structure. If the corrections to the Higgs potential do not change to much the interactions patterns at small separations the two vortices will feel both the charge-flux interaction and an oscillatory-type interaction due to the attractive properties of the Higgs potential. The phase diagram of the system could be even more complicated if the Higgs potential itself depended on temperature. We think all these topics to be worthy of further investigation.

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