Fractional spin and Galilean symmetry in a Chern-Simons matter system

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We study the dynamics of N nonrelativistic point particles interacting with Chern-Simons gauge field in $2+1$ dimensions. The model is canonically quantized in the Hamiltonian formalism without any gauge fixing. While the first quantization of the particles is retained, the fields are second quantized. The generators of the Galilean transformations are defined and shown to satisfy the requisite algebra. Fractional spin is computed, which is related to the number of particles.

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I. INTRODUCTION

Gauge field theories in $2+1$ dimensions with a Chern-Simons (CS) three-form have been a hot topic of investigation for their applications to quantum field theory and condensed matter physics [1]. For a CS term coupled to either complex scalars [2–5] or Dirac fermions [6] suitable (anyon) operators displaying fractional spin and statistics have been found. Poincaré invariance of these theories has also been exhibited [4–6]. It is quite interesting to observe that, in this context, it is possible to construct models, which are Galilean invariant rather than Poincaré invariant [7,8]. This is feasible because the CS term does not have an elementary photon associated with it so that the Bargmann superselection rule on the mass can be accommodated. Purely Galilean-invariant models are important, since these are useful to study problems, which are difficult when analyzed within the full formalism of special relativity.

It may be recalled that the geometrical idea of fractional spin in $2+1$ dimensions is rooted in the occurrence of multiply connected spaces having the fundamental group as the braid group (see for example Forte [9]). This gets manifested as weight factors (which are nothing but one-dimensional unitary representations of the associated braid group) corresponding to inequivalent classes of paths, when the system consisting of a *fixed* number of identical particles (say N) is quantized in the path integral formalism. The natural Lagrangian is, therefore, $L_0=\sum_{\alpha=1}^{N}\frac{1}{2}m\dot{x}_\alpha^2$, which is quantized in a multiply connected configuration space. As the phase acquired by a particle in traversing a closed loop around another particle has a universal form $e^{i\alpha}$ (α being a characteristic of the species of particles), it can be mimicked in the manner of Aharanov and Bohm (AB) by introducing a fictitious gauge field A_μ, and the corresponding Lagrangian has the form

$$L=\sum_{\alpha=1}^{N}\frac{1}{2}m\dot{x}_\alpha^2-\int d^2x\, j^\mu(x)A_\mu(x)+\theta\int d^2x\,\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\ , \tag{1}$$

where

$$j^0=e\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{x}_\alpha) \tag{2a}$$

and

$$j^i=e\sum_{\alpha=1}^{N}\dot{x}_\alpha^i\delta(\mathbf{x}-\mathbf{x}_\alpha) \tag{2b}$$

are the particle and the current density, respectively. The fully antisymmetric tensor $\epsilon^{\mu\nu\lambda}$ is defined such that $\epsilon^{012}=\epsilon_{012}=1$. The particles, labeled by α, have the same mass m and charge e.

The analogy with the AB example is illuminated by looking at the time component of the equation of motion for the gauge field [see Eq. (8a) below]. Heuristically speaking, the gauge field simulates the effects of multiply connected space.

The canonical quantization (second) of the model [Eq. (1)] has been discussed in [7], where L_0 is replaced by its corresponding Schrödinger term. Though the first and second quantized formalisms are equivalent as far as Galilean-invariant models are concerned, the second quantized version of Eq. (1) is problematic to analyze in the path integral scheme (as discussed in the preceding paragraph), since there is no obvious way of incorporating the fact that the system has a fixed N number of particles. This is in contrast with the canonical second quantized scheme, where one has to just restrict to the N-particle sector. It may be recalled that there is no particle production in a Galilean-invariant field theory. Also these N-particle states are constructed by superposing, in terms of first quantized N-particle wave functions, the states obtained by N-fold actions of the creation operators on the vacuum. Thus, once we restrict ourself to the N-particle sector, while quantizing canonically, we recover the first quantized N-particle wave function. Consequently it is desirable to start with the first quantized version [as given in Eq. (1)] itself. Though mentioned, the first quantized version was neither pursued nor its Galile-

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an invariance was explicitly demonstrated in [8]. There it was shown that one cannot solve the quantum-mechanical problem of the N-particle Schrödinger equation. It is worthwhile to mention here that the Lagrangian given by Eq. (1) does not have manifest invariance (not even up to total time derivative) under Galileo boosts. This can be easily seen by considering the interaction term, i.e., the second term in Eq. (1).

The object of this paper is to systematically analyze the canonical quantization of Eq. (1), using the gauge-independent formulation of Dirac's constraint Hamiltonian analysis [10]. This has been exploited by one of us [4,6] to discuss the quantization of relativistic field theories with a CS term. There are definite advantages of this approach over the conventional [7] gauge fixed analysis. Ambiguities related to gauge fixing conditions [11] are avoided, just as the proof of the equivalence of (physical) results in different gauges can be dispensed with. Moreover, the method of quantizing (1) by eliminating the gauge degrees of freedom cannot be generalized to the non-Abelian case. Our approach, on the contrary is completely general and can be applied to models with non-Abelian CS term [12]. The model [Eq. (1)] we are considering in this paper provides in fact the simplest example, in a "quantum-mechanical" context, where the power of this gauge-independent analysis can be exhibited.

We find that the fields and particle variables transform canonically, so that there are no anomalies. In [7], however a rotational anomaly was reported—which could be shifted away by a redefinition of the angular-momentum operator. Such redefinitions are not allowed in our analysis, where the angular momentum is defined unambiguously using Noether's theorem. Furthermore, in [7] A_0 was found to transform as a covariant vector under Galileo boosts, whereas we find it to be a Galilean scalar, as expected. We also show that the full Galilean algebra (including boosts) is reproduced acting on the physical state, though the action defined through [Eq. (1)] does not seem to have the desired manifest invariance under Galileo boost. Finally the computation of the angular-momentum operator reveals the occurrence of "fractional spin." This spin is related to the number of particles and agrees with the result quoted by Boyanovsky, Newman, and Rovelli [13]. The issue of statistics, in the context of second quantized field theory, is naturally, not covered, as the matter sector of our model has been first quantized, disallowing particle creation or annihilation from the vacuum.

The plan of this paper is as follows: Section II describes the model introduced in Eq. (1); an analysis of constraints leading to a gauge-independent quantization is done in Sec. III; Sec. IV shows that the Galilean algebra is satisfied by the various symmetry transformations; fractional spin is computed in Sec. V, while the conclusions are given in Sec. VI.

II. THE MODEL

The Lagrangian L in Eq. (1) can be rewritten in the standard form as

$$L=\frac{1}{2}m\sum_{\alpha}\dot{\mathbf{x}}_{\alpha}^2+e\sum_{\alpha=1}^{N}[\dot{\mathbf{x}}_{\alpha}\cdot\mathbf{A}(\mathbf{x}_{\alpha})-A_0(\mathbf{x}_{\alpha})] +\theta\int d^2x\,\epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda}\ . \tag{3}$$

The canonically conjugate momenta for $\mathbf{x}_{\alpha}$ is given by

$$\mathbf{p}_{\alpha}=m\dot{\mathbf{x}}_{\alpha}+e\,\mathbf{A}(\mathbf{x}_{\alpha}) \tag{4}$$

and the Euler-Lagrange equation for $\mathbf{x}_{\alpha}$ is

$$\frac{d\mathbf{p}_{\alpha}}{dt}=e\left[\dot{\mathbf{x}}_{\alpha}\cdot\frac{\partial\,\mathbf{A}(x_{\alpha})}{\partial\mathbf{x}_{\alpha}}-\frac{\partial A_0(\mathbf{x}_{\alpha})}{\partial\mathbf{x}_{\alpha}}\right], \tag{5}$$

which on further simplification yields

$$m\ddot{x}_{\alpha}^{i}=e(E^{i}+\epsilon^{ij}\dot{x}_{\alpha}^{j}B) \equiv e(F^{i0}+\epsilon^{ij}v_{\alpha}^{j}F_{12})\ , \tag{6}$$

where

$$F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}\ .$$

The Euler-Lagrange equation for the gauge field is

$$2\theta\epsilon^{\mu\nu\lambda}(\partial_{\nu}A_{\lambda})=j^{\mu}\ . \tag{7}$$

In component notation this becomes

$$2\theta F_{12}=2\theta B=j^{0}\ , \tag{8a}$$

$$-2\theta\epsilon^{ij}E^{j}=j^{i}\ . \tag{8b}$$

Equation (8a), as shall be shown subsequently, is the Gauss constraint.

Previous Hamiltonian analysis of similar models was done by eliminating the gauge field by using equations of motion [7]. Naturally it is mandatory to choose a gauge in order to do this elimination. Since the structure of the gauge field is very different in different gauges, it becomes quite nontrivial to establish the equivalence of (physical) results in various gauges. We bypass these difficulties in our gauge-independent formalism. Besides this, the very method of tackling the quantization of [Eq. (1)] by eliminating the gauge fields, using the classical equation of motion is conceptually problematic. This is because the constraints associated with the gauge field are lost and remain unaccounted. This may lead to an inequivalent theory at the quantum level. In the functional integral language the (classical) elimination of gauge fields just corresponds to doing the "naive" (i.e., ignoring the presence of constraints) Gaussian integral over the gauge field leading to an effective theory of the original model [Eq. (1)]. Effective theories of models (particularly those afflicted with constraints, as in this) are known to differ field theoretically from the original model. A dramatic example being the large N limit of the CP^{N-1} model [14]. In this model it is possible to eliminate the gauge fields completely and obtain an effective theory comprising the complex fields z and z^* only. On the contrary it is possible to directly compute the one-loop partition function. It yields a remarkable result—the generation of Maxwell term, which never appears in the effective theory [14]. In our analysis we dispense with any effective theory and work with the original model [Eq. (1)] retaining all the dynamical degrees of freedom.

III. CONSTRAINED HAMILTONIAN ANALYSIS

The various canonically conjugate momenta of the system, given by Eq. (1) are

$$p^i_\alpha=\frac{\partial L}{\partial \dot{x}^i_\alpha}=m\dot{x}^i_\alpha+eA^i(x_\alpha)\ ,\qquad \pi_0=\frac{\partial \mathcal{L}}{\partial \dot{A}_0}=0\ ,\qquad \pi_i=\frac{\partial \mathcal{L}}{\partial \dot{A}^i}=\theta\epsilon_{ij}A^j\ . \tag{9}$$

Here $\mathcal{L}$ is the CS term in Eq. (1). Thus, according to Dirac's classification [9], the primary constraints of the system are

$$\pi_0\approx 0\ ,\qquad \chi_i=\pi_i-\theta\epsilon_{ij}A^j\approx 0\ . \tag{10}$$

To find the secondary constraint, the canonical Hamiltonian H_c is first computed by a Legendre transformation

$$H_c=\sum_{\alpha=1}^{N}\mathbf{p}_\alpha\cdot\dot{\mathbf{x}}_\alpha+\int d^2x(\pi_1\dot{A}^1+\pi_2\dot{A}^2)-\frac{1}{2}m\sum_{\alpha=1}^{N}\dot{\mathbf{x}}^2_\alpha-e\sum_{\alpha=1}^{N}[\dot{\mathbf{x}}_\alpha\cdot\mathbf{A}(\mathbf{x}_\alpha)-A_0(\mathbf{x}_\alpha)]-\theta\int d^2x\,\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\ , \tag{11}$$

which on simplification yields

$$H_c=\sum_{\alpha=1}^{N}\left[\frac{1}{2m}[\mathbf{p}_\alpha-e\,\mathbf{A}(x_\alpha)]^2+eA_0(x_\alpha)\right]-2\theta\int d^2x\,A_0\epsilon^{ij}\partial_iA_j \tag{12}$$

The primary Hamiltonian is

$$H_p=H_c+\int d^2x(u_0\pi_0+u_i\chi_i)\ , \tag{13}$$

where u_0 and u_i are arbitrary multipliers. Time conserving the primary constraints with H_p and using the basic Poisson brackets

$$\{x^i_\alpha,p^j_\beta\}=\delta^{ij}\delta_{\alpha\beta}\ ,\qquad \{A^i(\mathbf{x}),\pi_j(\mathbf{y})\}=\delta^i_j\delta(\mathbf{x}-\mathbf{y})\ ,\qquad \{A^0(\mathbf{x}),\pi_0(\mathbf{y})\}=\delta(\mathbf{x}-\mathbf{y}) \tag{14}$$

leads to the secondary constraints

$$\xi=2\theta B-e\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{x}_\alpha)\approx 0\ . \tag{15}$$

The physical content of this constraint, which is the generator of gauge transformation, is that the support of the CS magnetic field B is only (weakly) at the site of the particles and that too with the same strength (say) B_0. Thus the profile of $B(x)$ can be written as

$$B(\mathbf{x})\approx B_0\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{x}_\alpha)\ , \tag{16}$$

which gives a (weak) relation between B_0, θ, and e as

$$2\theta B_0\approx e\ . \tag{17}$$

We observe that Eq. (15) is identical to Eq. (8a), on substitution of j^0 from Eq. (2a). Since χ_i [Eq. (10)] and ξ have nonvanishing Poisson brackets, these constraints are second class. One may be surprised to observe that the constraint ξ in Eq. (15) is second class, since it is the analog of the Gauss operator, which is known to be first class. This happens, since we have not yet extracted the maximal set of first class constraints of the theory [15]. It is simple to verify that the combination

$$\chi=\partial^i\chi_i+\xi \tag{18}$$

is first class. We can now classify the constraints. The two constraints π_0 [Eq. (10)] and χ [Eq. (18)] are first class, while χ_i [Eq. (10)] are second class. It is possible to eliminate the second class constraints completely from the theory by using Dirac brackets when χ becomes equal to ξ [Eq. (18)]. Thus Gauss constraint ultimately emerges to be a first class one, as it ought to be.

The physical state $|\psi\rangle$ of our model are defined to be those states that are annihilated by the constraint ξ [Eq. (15)],

$$\xi|\psi\rangle=\left[2\theta B-e\sum_\alpha\delta(\mathbf{x}-\mathbf{x}_\alpha)\right]|\psi\rangle\equiv 0\ . \tag{19}$$

The Dirac bracket (DB) between any two variables is defined to be

$$\{U(x),V(y)\}_{\rm DB}=\{U(x),V(y)\}_{\rm PB}-\int d^2z\,d^2z'\{U(x),\chi_i(z)\}_{\rm PB}\chi^{-1}_{ij}(z,z')\{\chi_j(z'),V(y)\}_{\rm PB}\ , \tag{20}$$

where

$$\chi^{-1}_{ij}(z,z')=\frac{1}{2\theta}\epsilon_{ij}\delta(z-z') \tag{21}$$

is the inverse of the matrix of the Poisson bracket (PB) $\{\chi_i(z),\chi_j(z')\}$. The DB, which differ from their corresponding PB are given explicitly as

$$\{A_i(x),A_j(y)\}_{\rm DB}=\frac{1}{\theta^2}\{\pi_i(x),\pi_j(y)\}_{\rm DB}=\frac{1}{2\theta}\epsilon_{ij}\delta(x-y)\ ,\qquad \{A_i(x),\pi_j(y)\}_{\rm DB}=\frac{1}{2}\delta^i_j\delta(x-y)\ , \tag{22}$$

which are compatible with setting the second class constraints χ_i strongly zero.

The total Hamiltonian can be written as

$$H_T = H_0 + \int d^2x \, u_0(x)\pi_0(x) + \int d^2x \, v(x)\xi(x) \ , \tag{23}$$

where u_0 and v are arbitrary parameters reflecting the gauge invariances of the theory. There are now two methods of proceeding further with the quantization program. The conventional method is to choose two gauge conditions to convert the first-class constraints π_0 and $\xi \approx 0$ to second class. A new set of Dirac brackets are computed, compatible with putting π_0 and ξ strongly zero. The freedom in the Hamiltonian is thus eliminated. Alternatively, we may fix the multipliers to obtain the Heisenberg equation

$$\{ A_\mu, H_T \} = \partial_0 A_\mu \ , \tag{24a}$$

$$\{ \mathbf{x}_\alpha, H_T \} = \partial_0 \mathbf{x}_\alpha \ . \tag{24b}$$

They are reproduced with the choice

$$u_0 = \partial_0 A_0 \ , \tag{25a}$$

$$v = 0 \ . \tag{25b}$$

Exactly as happens in usual gauge theories (e.g., in Maxwell's theory), the variable A_0 is not of any physical significance and its time derivative u_0 is also completely arbitrary. This variable is, therefore, of no interest and may, without loss of generality, be taken to be an arbitrary constant c, which implies that

$$u_0 = 0 \ , \quad v = 0 \ . \tag{26}$$

Henceforth we put $A^0 = 0$. Thus, both A^0 and π_0 disappear from the subsequent analysis, as these are not the true dynamical degrees of freedom. It has to be mentioned that putting $A^0 = 0$ is not a gauge-fixing condition [10,16].

However, it is interesting to note that, A^0 being a Galilean scalar, $A^0 = 0$ preserves manifest Galilean covariance here in contrast to the Maxwell case, where this choice destroys manifest Lorentz covariance. This is because A^μ transforms contravariantly under Galileo boosts. To see this, it may be noted that the transformation properties of A^i's can be identified by looking at Eq. (4), where the first term represents the mechanical momentum. In the absence of the CS gauge field, when the situation corresponds to a free-particle case, the canonically conjugate momentum for $\mathbf{x}_\alpha$ is just this mechanical momentum. Now it is well known that the nonrelativistic expressions for energy and momentum for a free particle can be obtained from the corresponding relativistic one in the limit $c \to \infty$, provided the transformations are taken to be contravariant in nature. In other words, only the contravariant components of the energy-momentum 3 $(=1+2)$ vectors can be identified with energy and momentum of a nonrelativistic particle in the Galilean limit. Thus it is naturally expected that the term involving A^i in (4) should also transform contravariantly when the CS gauge fields are present. This in turn implies that the component A^0 also transform contravariantly under Galileo boosts. Unlike the relativistic case, here we cannot construct a corresponding covariant object, since there does not exist any such metric in Galilean space time. Thus both A^μ and A_μ transform contravariantly under Galileo boosts, but A^i and A_i are really the contra and covariant counterparts, as far as spatial rotations are concerned, because the two-dimensional space has a natural Euclidean metric. This has to be contrasted with Hagen's [7] case, where the CS gauge fields were found to transform as a covariant vector.

IV. GALILEAN GENERATORS AND THEIR ALGEBRA

In this section, we shall use Noether's theorem to identify various generators of symmetry transformations. These generators will be shown to satisfy the Galilean algebra.

Let us first consider the generator of translation. In the absence of CS term, i.e., when the theory is analogous to the coupling of particles with a background gauge field, this generator is given by $\mathbf{p}_\alpha$ [Eq. (4)] [17]. The contribution from the CS term can be easily computed by Noether's theorem, so that the full translational generator is

$$P^i = \sum_\alpha p^i_\alpha + \theta \int d^2x \, \epsilon_{jk} (\partial^i A^j) A^k \ . \tag{27}$$

Again one has the freedom of adding a linear combination of first-class constraints to the P^i. Exactly as happens in the case of the Hamiltonian the multipliers are found to vanish, so that $\{ x^i_\alpha, P^j \}_{\rm DB} = \delta^{ij}$ and $\{ A^j(x), P^i \}_{\rm DB} = \partial^i A^j$ are satisfied.

Proceeding analogously, the generator of rotation J, and Galilean boosts G^i are found to be (see the Appendix):

$$J = \sum_{\alpha=1}^{N} \epsilon^{ij} x^i_\alpha p^j_\alpha + \theta \int d^2x [\epsilon^{jk}\epsilon^{li} x_j A_i (\partial_k A_l) - A^i A^i] \ , \tag{28a}$$

$$G^i = tP^i - m \sum_\alpha x^i_\alpha \ , \tag{28b}$$

yielding the following transformations for the physical (gauge independent) degrees of freedom of the theory:

$$\{ x^i_\alpha, J \}_{\rm DB} = \epsilon^{ij} p^j_\alpha \ , \tag{29a}$$

$$\{ x^i_\alpha, G^j \}_{\rm DB} = t \delta^{ii} \ , \tag{29b}$$

which are the normal transformation properties without any anomalies. Note that we have omitted the corresponding algebra for the gauge fields, since these are not gauge invariant. Indeed under gauge transformations generated by the Gauss law constraint $\xi(x)$,

$$\delta A_i(y) = \int d^2x \, \alpha(x) \{ \xi(x), A_i(y) \}_{\rm DB} = \partial_i \alpha(y) \tag{30}$$

the A_i's are modified in the usual way with $\alpha(x)$ being the gauge parameter. Our results may be compared with previous analysis [7], where an additional rotational anomaly was found in both radiation and axial gauges. Since this anomaly was found to differ in two gauges, we feel that it is an artifact of gauge. Though this anomaly in angular momentum could be defined away in [7] by appropriate shifting, the same cannot be done in our case, as the expression of angular momentum here has been derived unambiguously through Noether's theorem (see the Appendix; where we have also shown the gauge invariance of this operator on the physical states).

We also observe at this stage that $\{A^0, G^i\}_{\rm DB}=0$, corroborating the fact that A^0 is a Galilean scalar. Indeed it should be of the form $t(\partial A^0/\partial x^i)$, but this vanishes as we have $A^0=0$. This may be contrasted with the result of [7], where a noncanonical term in the transformation law for A^0 was obtained.

Using the various generators H_T, P^i, G^i, J, and the Dirac brackets, we find after an extensive computation

$$\{P^i,P^j\}_{\rm DB}=\{P^i,H^T\}_{\rm DB}=\{J,H_T\}_{\rm DB}=\{G^i,G^j\}_{\rm DB}=0 \ , \tag{31a}$$

$$\{J,P^i\}_{\rm DB}=\epsilon^{ij}P^j \ , \tag{31b}$$

$$\{J,G^i\}_{\rm DB}=\epsilon^{ij}G^j \ , \tag{31c}$$

$$\{P^i,G^j\}_{\rm DB}=\delta^{ij}mN \ , \tag{31d}$$

$$\{G^i,H\}_{\rm DB}=-\sum_\alpha m\dot{x}^i_\alpha \ . \tag{31e}$$

We note here that, but for the last relation [Eq. (31e)], all other relations satisfy the Galilean algebra. But this is not serious, as it can be easily seen that on the constraint surface [Eqs. (15) and (17)] the momentum $\mathbf{P}$ is equal to $\sum_\alpha m\dot{\mathbf{x}}_\alpha$. Thus even this relation also conforms to the Galilean algebra, when acting on the physical states $|\psi\rangle$ [see Eq. (19)]

$$\{G^i,H\}|\psi\rangle=-P^i|\psi\rangle \ , \tag{32}$$

which proves the validity of full Galilean algebra.

One difference between the Galilean algebra with that of Poincaré is that in the former case $\{G^i,G^j\}_{\rm DB}=0$ but in the later case $\{G^i,G^j\}_{\rm DB}\sim J$. Thus, redefining J by making a constant shift will not affect the Galilean algebra unlike in the Poincaré case. However, the shifted J will be different from the canonical J obtained through Noether's theorem, as we have emphasized above.

Quantization can now be done by converting the DB's into commutators using $i\{A,B\}_{\rm DB}\to[A,B]$. Moreover, operator symmetrization is implied whenever products of operators occur.

V. ANGULAR MOMENTUM

This section is devoted to the revelation of fractional spin by the computation of angular momentum [Eq. (28a)] acting on the physical state $|\psi\rangle$:

$$J|\psi\rangle=\left[\sum_\alpha \epsilon^{ij}x^i_\alpha p^j_\alpha+\theta\int d^2x\,(\epsilon^{jk}x_j\epsilon^{li}A_i\partial_k A_l-A^iA^i)\right]|\psi\rangle \ . \tag{33}$$

The first term $J_c=\sum_\alpha \epsilon^{ij}x^i_\alpha p^j_\alpha$ is just the canonical contribution coming from the particle sector. As $\mathbf{p}^j_\alpha=-i(\partial/\partial x^j_\alpha)$ is the generator of the translation for the particle sector, the canonical angular momentum J_c can be written in polar coordinate as $\sum_\alpha -i(\partial/\partial\phi_\alpha)$ [17]. The state $|\psi\rangle$ in (polar) coordinate basis (r_α,ϕ_α) can be written as

$$\begin{aligned}\psi(t;r_\alpha,\phi_\alpha)&\equiv\psi(t;r_1,\phi_1,\ldots,r_N,\phi_N)\\&=\langle r_1,\phi_1,\ldots,r_N,\phi_N|\psi(t)\rangle \ .\end{aligned} \tag{34}$$

Single valuedness of wave functions demands $\psi(t;r_\alpha,\phi_\alpha)$ should be periodic in $\phi_\alpha(0\le\phi_\alpha<2\pi)$ for all ϕ_α. Thus, Fourier expanding, one gets

$$\psi(t;r_\alpha,\phi_\alpha)\equiv\sum_{m_1,\ldots,m_N=-\infty}^{+\infty}\exp[i(m_1\phi_1+\cdots+m_N\phi_N)]f(m_1,\ldots,m_N;r_1,\ldots,r_N) \tag{35}$$

where m_α's are integers and $f(\{m_\alpha,r_\alpha\})$ is some function. Thus the eigenfunctions $\exp[i\sum_\alpha m_\alpha\phi_\alpha)]$ will have integer $(\sum_\alpha m_\alpha)$ eigenvalues corresponding to J_c. We now focus on the second θ-dependent contribution (denoted as J_θ)

$$J_\theta|\psi\rangle=\theta\int d^2x\,[\epsilon^{jk}\epsilon^{li}x_j A_i\partial_k A_l-A^iA^i]|\psi\rangle \ . \tag{36}$$

Using the identity

$$\epsilon^{jk}\epsilon^{li}=\delta^{jl}\delta^{ki}-\delta^{ji}\delta^{kl} \ , \tag{37}$$

Eq. (36) simplifies to

$$J_\theta|\psi\rangle=-2\theta\int d^2x\,\epsilon^{ij}x_i A_jB|\psi\rangle \ . \tag{38}$$

Using the defining equation for $|\psi\rangle$ [see Eq. (19)], we find

$$\left[A_i(x)-\frac{1}{2\theta}\epsilon_{ij}\partial^x_j\int d^2y\,D(\mathbf{x}-\mathbf{y})j_0(y)\right]|\psi\rangle=0 \tag{39}$$

where

$$\nabla^2D(\mathbf{x})=-\delta(\mathbf{x}) \tag{40}$$

is the scalar Green's function having the explicit form

$$D(\mathbf{x}) = -\frac{1}{4\pi}\ln|\mathbf{x}|^2 + \text{const} . \tag{41}$$

Using (39), we may express B and A_j in (38) in terms of j_0 to obtain

$$J_\theta|\psi\rangle = \frac{1}{2\theta}\int d^2x\, d^2y\, j^0(\mathbf{x}) x_i[\partial_i D(\mathbf{x}-\mathbf{y})] j^0(\mathbf{y})|\psi\rangle . \tag{42}$$

Substituting $D(\mathbf{x})$ from (41), we find

$$J_\theta|\psi\rangle = \frac{e^2N^2}{8\pi\theta}|\psi\rangle . \tag{43}$$

This additional piece of the angular momentum is clearly seen to be independent of the choice of origin of coordinates. It may, therefore, be regarded as the fractional spin of the system. The above equation connects this spin with the particle number. This result agrees with that of [13], where the nonrelativistic (Schrödinger) matter fields were second quantized, where also no gauge-fixing condition was used. We, on the contrary, stuck to the first quantized version of the matter sector. Fractional spin in an effective theory corresponding to a model similar to (1), obtained by eliminating the gauge degrees of freedom, was also reported in [7].

VI. CONCLUSIONS

We have showed, in a gauge-independent analysis, the occurrence of fractional angular momentum in a system of N nonrelativistic (first quantized) point particles with CS interactions. Previous attempts [8,13] to discuss this theory employed the second quantized (for the matter sector) version, converting the Schrödinger problem to a nonrelativistic field theory. Though these approaches are equivalent, we do not follow the second quantized version here. Another difference with the conventional approach [7,8] is that the gauge-field variables have not been directly eliminated in favor of matter variables by using the equations of motion. Although the gauge fields themselves are not physically meaningful (and hence are not referred directly) the actual observables (which must be gauge invariant) are constructed from these fields. Examples of these observables are the various space-time generators like the Hamiltonian, momentum, angular momentum, etc. In fact the gauge independence of the angular momentum (which involves the gauge fields) has been explicitly demonstrated in the Appendix. A similar demonstration can be done for the other observables too. We may also mention that the conventional [7,8] elimination of the gauge variables in favor of the matter ones is not conceptually clean as it utilizes the equations of motion, which are strictly valid only at the classical level. One can, and in this model one does, miss the constraints involving the phase space variables. Indeed the (second class) constraint $\pi_i = \theta\epsilon_{ij}A^j$ [Eq. (9)] remains unaccounted. Furthermore, in the non-Abelian case this elimination is quite involved and cannot be done in general. It has also led to the violation of Poincaré algebra [7]. All this reveals that, even within the usual second quantized approach, the analysis is not well understood. We also mention that the results in different gauges differ and the equivalence of (physical) results needs to be established [7]. Since our approach is gauge independent, all such problems are bypassed. The fractional spin obtained here is, therefore, a physical finding and not a consequence of gauge fixing. Our result also agrees with that of [13].

The present example may be regarded as a model, though lacking Poincaré invariance, having Galilean invariance. This led us to define, using Noether's theorem, the generators of the Galilean transformations. Contrary to earlier assertions [7] we prove that the fields and particle variables transform normally under these transformations, so that there is no anomaly. The complete Galilean algebra was reproduced thereby proving the Galilean invariance of the model. It is important to stress at this stage the Galilean algebra closes only on the physical states. In other words the complete Galilean invariance is restored only when we restrict to the physical subspace—the kernel of the Gauss law operator, i.e., those states, which are annihilated by this constraint. It is not there in the entire Hilbert space. So the demonstration of the Galilean invariance of the model (specially under boost) is, we feel, an important result of our paper.

Here we would like to make the observation that though it is possible to couple nonrelativistic particles to the photonless gauge fields (like CS gauge fields) maintaining the complete Galilean invariance, it is not possible to do the same with Maxwell's gauge field. This is because, in that case, the matter sector will have invariance under Galileo boost and the gauge-field sector will have the invariance under Lorentz boost. CS term on the contrary is invariant under any general coordinate transformation.

The present gauge independent investigation supplements the earlier analysis done by the first author [4,6] to display fractional spin and statistics in relativistic field theories involving the CS kinetic action. It may be mentioned that, although fractional spin in relativistic CS field theories have been obtained from various view points [1–6,18], a corresponding exhaustive analysis for the point particle case was lacking. In this context, our investigations have filled a gap, as well as revealed the general viability of the gauge independent approach [4,6]. Extension of our analysis for the relativistic N particle problem with CS interaction is under progress.

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APPENDIX

Here we first show how the rotation generator J [Eq. (28a)] is found, and then prove its gauge invariance. In the absence of the CS term, the translational generator is given by $\mathbf{p}_\alpha$ [Eq. (4)] (see Ref. [16]). Correspondingly, the rotation generator J_c is given by

$$J_c = \sum_{\alpha=1}^{N} \epsilon^{ij} x^i_\alpha p^j_\alpha \ , \tag{A1}$$

which is also the result obtained by an application of Noether's theorem.

When the CS term is added, its contribution to the rotation generator is obtained by considering an infinitesimal rotation in the XY plane as $\delta x^i = \epsilon^i_j x^j$ and the corresponding variation of the CS part of the action S_{CS}, which can be written as

$$\delta S_{CS} = \tfrac{1}{2} \int d^3x \ \epsilon^{ij} \partial^\alpha \mathcal{M}_{\alpha ij} \tag{A2}$$

whereby the additional contribution due to this CS term to the angular momentum J_θ is given by

$$J_\theta = \int d^2x \ \mathcal{M}_{012} \ . \tag{A3}$$

Thus, getting

$$J_\theta = \int d^2x \left[\epsilon^{ij} x_i T_{0j} + \frac{\delta \mathcal{L}_{CS}}{\delta(\partial A^i)} \Sigma^{ij}_{12} A_j \right] , \tag{A4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor obtained by Noether's theorem,

$$T_{\mu\nu} = \frac{\delta \mathcal{L}_{CS}}{\delta(\partial^\mu A_\lambda)} \partial_\nu A_\lambda - \mathcal{L}_{CS} g_{\mu\nu}$$

$$= \theta \epsilon_{\rho\mu\lambda} A^\rho \partial_\nu A^\lambda - \theta(\epsilon_{\rho\lambda\sigma} A^\rho \partial^\lambda A^\sigma) g_{\mu\nu} \tag{A5}$$

and

$$\Sigma^{ij}_{kl} = (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \ . \tag{A6}$$

Substituting (A5) and (A6) in (A4) and after some algebra, one finds

$$J_\theta = \theta \int d^2x \, [\epsilon^{jk} \epsilon^{li} x_j A_i \partial_k A_l - A^i A^i] \ . \tag{A7}$$

Adding (A1) and (A7) yields the full rotation generator J

$$J = J_c + J_\theta \tag{A8}$$

which reproduces (28a).

In order to check the gauge invariance of J, we have to show that its Dirac bracket with the Gauss operator ξ [Eq. (15)] vanishes weakly. Then from Eq. (A8) we have,

$$\{\xi, J\}_{DB} = \{2\theta B, J_\theta\}_{DB} - e \left\{ \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{x}_\alpha), J_c \right\}_{DB} , \tag{A9}$$

where we have written only the nontrivial bracket.

An explicit computation shows

$$\{2\theta B, J_\theta\}_{DB} = 2\theta \epsilon^{ij} x^i \partial^j B \tag{A10}$$

and

$$\left\{ \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{x}_\alpha), J_c \right\}_{DB} = \epsilon^{ij} x^i \partial^j \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{x}_\alpha) \ . \tag{A11}$$

Combining these results with Eq. (A9), we obtain

$$\{\xi, J\}_{DB} = \epsilon^{ij} x^i \partial^j \xi \ , \tag{A12}$$

which vanishes acting on the physical states $|\psi\rangle$

$$\{\xi, J\}_{DB} |\psi\rangle = 0 \ , \tag{A13}$$

which shows the gauge invariance of J on the physical states. The expression for the Galilean boost G_i (28b) has occurred earlier [7,8] and is well known.

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