Currents, charges, and canonical structure of pseudodual chiral models

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We discuss the pseudodual chiral model to illustrate a class of two-dimensional theories which have an infinite number of conservation laws but allow particle production, at variance with naive expectations. We describe the symmetries of the pseudodual model, both local and nonlocal, as transmutations of the symmetries of the usual chiral model. We refine the conventional algorithm to more efficiently produce the nonlocal symmetries of the model, and we discuss the complete local current algebra for the pseudodual theory. We also exhibit the canonical transformation which connects the usual chiral model to its fully equivalent dual, further distinguishing the pseudodual theory.

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I. INTRODUCTION

Many integrable models in two dimensions have the limiting feature of *no particle production.* There is a variant of the σ model for which this is not so, however, the so-called *pseudodual chiral model* of Zakharov and Mikhailov [1], for which all interactions are distilled into a very simple, constant torsion term in the Lagrangian. The essential quantum features of the model were first identified by Nappi [2], who calculated the nonvanishing $2 \to 3$ production amplitude for the model, and who also demonstrated that the model was inequivalent to the usual chiral model in its behavior under the renormalization group: The pseudodual model is not asymptotically free. Although these were lowest order perturbative calculations carried out for a massless theory, and are subject to the well-known interpretation problems inherent to a field theory of massless scalar particles in two dimensions, it is nonetheless clear that the physics of the pseudodual model is very different from that of the usual chiral model, and therefore a full comparison between the two theories is warranted.

The models were previously compared within the framework of covariant path integral quantization by Fridling and Jevicki, and similarly by Fradkin and Tseytlin [3]. However, the focus of those earlier comparisons was to exhibit dualized σ models, with torsion, which were completely equivalent to the usual σ model. Indeed, it was shown that a model fully equivalent but dual to the usual chiral model could be constructed, provided both nontrivial torsion and metric interactions were included in the Lagrangian.

In this paper, we focus on the differences between the pseudodual model and the usual chiral model without enforcing equivalence. We investigate the pseudodual model at the classical level and within the framework of canonical quantization, with emphasis on the symmetry structure of the theory. We consider both local and nonlocal symmetries, and compare with corresponding structures in the usual chiral model. We exhibit a canonical transformation which connects the usual chiral model with its fully equivalent dual version, further clarifying the inequivalence of the pseudodual theory. We provide a technically refined algorithm for constructing the nonlocal currents of the pseudodual theory, an algorithm which is particularly well suited to models with topological currents for which the usual recursive algorithm temporarily stalls at the lowest steps in the recursion before finally producing genuine nonlocals at the third step and beyond. We also consider in some detail the current algebra for the full set of local currents in the pseudodual theory, thereby providing an extension of several recent studies [4]. Other related, more recent investigations can be found in [5].

II. PSEUDODUAL CHIRAL MODEL

The familiar two-dimensional chiral model (CM) for matrix-valued fields g is defined by

$$\mathcal{L}_1 = \text{Tr}\ \partial_\mu g \partial^\mu g^{-1}, \tag{2.1}$$

whose equations of motion are conservation laws

$$\partial_\mu J^\mu = 0 \quad \Longleftrightarrow \quad \partial_\mu L^\mu = 0. \tag{2.2}$$

Here, $J_\mu \equiv g^{-1}\partial_\mu g$ and $L_\mu \equiv g\partial_\mu g^{-1}$, the right- and left-rotation Noether currents of $G_{\text{left}} \times G_{\text{right}}$, respectively. E.g., for $G = \text{O}(N)$, these currents are antisymmetric $N \times N$ matrices. We may consider this particular case in the following discussion without essential loss of generality. Note that $\mathcal{L}_1 = -\text{Tr}\ (J_\mu J^\mu) = -\text{Tr}\ (L_\mu L^\mu)$. From the pure-gauge form of the local currents, it follows that the appropriate non-Abelian field strength vanishes:

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$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \iff \varepsilon^{\mu\nu}\partial_\mu J_\nu + \varepsilon^{\mu\nu} J_\mu J_\nu = 0 \ , \tag{2.3}$$

and likewise for L_μ. In this sense, the local currents are curvature free. Curvature-free currents generally underlie nonlocal-symmetry generating algorithms, as will be reviewed in a forthcoming section.

Alternatively, the roles of current conservation and vanishing field strength may be transmuted. Consider a remarkable transformation [1,2] of the "pseudodual" type (in the language of [3]) which leads to a drastically different model for an antisymmetric matrix field ϕ. For conserved currents, one may always define

$$J_\mu = \varepsilon_{\mu\nu}\partial^\nu \phi. \tag{2.4}$$

This, of course, is conserved identically. On the other hand, the curvature-free condition above may now serve instead as the equation of motion

$$\partial^\mu\partial_\mu\phi - \tfrac{1}{2}\varepsilon_{\mu\nu}[\partial^\mu\phi, \partial^\nu\phi] = 0, \tag{2.5}$$

which follows from the Lagrangian of Zakharov and Mikhailov [1]:

$$\mathcal{L}_2 = -\tfrac{1}{4}\mathrm{Tr}\left(\partial^\mu\phi\partial_\mu\phi + \tfrac{1}{3}\phi\varepsilon_{\mu\nu}[\partial^\mu\phi, \partial^\nu\phi]\right). \tag{2.6}$$

This is the definition of the pseudodual chiral model (PCM). Having transmuted the conservation and curvature conditions obeyed by the local currents under the interchange CM↔PCM, let us also consider the transmutation of the fundamental symmetries generated by these and other currents.

Consider first the charges for the local J_μ,

$$Q = \int dx\ J_0(x). \tag{2.7}$$

For the currents of the chiral model, the time variation of Q vanishes for field configurations which extremize $\mathcal{L}_1$ by Noether's theorem, while for the currents of the pseudodual model, Q are time independent for *any* configurations with fixed boundary conditions by merely supposing the local field ϕ is temporally constant at spatial infinity. For the PCM, $Q = \phi(\infty) - \phi(-\infty)$ is just a topological "winding" of the field onto the spatial line, and thus invariant under the continuous flow of time [6].

Noether's procedure does yield results for the pseudodual model, but says nothing about the previous Q. Instead, the G_{right}-transformation invariance of $\mathcal{L}_2$, $\phi \to O^T\phi O$, yields the (on-shell conserved) Noether currents[1]

$$R_\mu = [\phi, \tilde{J}_\mu] + \tfrac{1}{3}[\phi, [J_\mu, \phi]] = [\phi, \partial_\mu\phi] + \tfrac{1}{3}\varepsilon_{\mu\nu}[\phi, [\partial^\nu\phi, \phi]], \tag{2.8}$$

where $\tilde{J}_\mu \equiv \varepsilon_{\mu\nu} J^\nu$. In contrast to the chiral model, it is these currents, and not J_μ, which generate (adjoint) right rotations in the PCM.

Moreover, as remarked by Nappi [2] in footnote 7, the action specified by the integral of $\mathcal{L}_2$ is also invariant under the nonlinear symmetries $\phi \to \phi + \xi$ whose Noether currents are [2]

$$Z_\mu = \tilde{J}_\mu + \tfrac{1}{2}[J_\mu, \phi] = \partial_\mu\phi + \tfrac{1}{2}\varepsilon_{\mu\nu}[\partial^\nu\phi, \phi]. \tag{2.9}$$

Indeed, the conservation law for these currents simply amounts to the equations of motion (2.5) for the PCM, originally introduced as a null-curvature condition for the topological J_μ currents of the model. Thus the equations of motion have been transmuted from conservation of J_μ for the chiral model to conservation of Z_μ for the pseudodual model. These Z_μ currents are not curvature free, however, but are instead J-covariant curl free:

$$\varepsilon^{\mu\nu}\partial_\mu Z_\nu + \varepsilon^{\mu\nu}[J_\mu, Z_\nu] = 0\ . \tag{2.10}$$

In some contrast to $\mathcal{L}_1$, we note further that $\mathcal{L}_2 = \frac{1}{12}\mathrm{Tr}\left[J_\mu(J^\mu + 2\tilde{Z}^\mu)\right]$.

We will demonstrate in Sec. IV below that these "new" local conserved currents Z_μ and R_μ are actually transmutations of the usual first and second nonlocal currents of the chiral model, respectively. All three sets of currents, J_μ, Z_μ, R_μ, transform in the adjoint representation of $O(N)_{\text{right}}$ (the charge of R_μ). By inspection of the transformation on the field ϕ, it would appear that the charges for the shift symmetry *commute* among themselves. This is not quite correct, however, as we shall see in the results of Sec. V below. In anticipation of those later results, it turns out that the shift charges induce a transformation of the ϕ conjugate momenta which depends on the topological charge Q: This only vanishes on states with trivial winding. We shall therefore refer to these shift charges as "pseudo-Abelian."

In Ref. [2] it was demonstrated how the PCM is radically *different* from the chiral model, as the former, unlike the latter, is not asymptotically free, and it allows particle production already at the tree (semiclassical) level. For σ models such as the above chiral ones, the suppression of particle production has been argued [7] on the basis of the nonlocal conservation laws of Lüscher and Pohlmeyer [8]. Nevertheless, we will show that the yet higher (third and beyond) transmutations of the nonlocal charges of the chiral model are, in fact, genuine nonlocal charges for the PCM, and thus the pseudodual model rather remarkably exhibits both particle production and

[1]Occasionally it may be useful to recall the identities $\varepsilon^{\kappa\lambda}\varepsilon^{\mu\nu} = g^{\kappa\nu}g^{\lambda\mu} - g^{\kappa\mu}g^{\lambda\nu}$ and $g^{\kappa\lambda}\varepsilon^{\mu\nu} + g^{\kappa\mu}\varepsilon^{\nu\lambda} + g^{\kappa\nu}\varepsilon^{\lambda\mu} = 0$.

[2]This transformation preserves the constant-at-infinity boundary conditions in the PCM, as always in spontaneous symmetry breaking and, even when the values of the constant field at $x = \pm\infty$ differ, will not change the value of the winding Q.

an infinite sequence of nonlocal charges.

Further note that, properly speaking, the left invariance $G_{\rm left}$ has degenerated: For the field ϕ, left transformations are inert, and thus right, or axial, or vector transformations are all indistinguishable. The $G_{\rm left} \times G_{\rm right}$ symmetry of the chiral model, the axial generators of which are realized nonlinearly, has thus mutated in the PCM. On the one hand it has been reduced by the loss of $G_{\rm left}$, but on the other hand it has been augmented by the nonlinearly realized pseudo-Abelian Q_Z charges.

The reader may wonder then how the conserved left currents L_μ of the chiral model are realized on the solution set of the PCM. They actually do not generate left rotations on the fields ϕ, any more than the J_μ generate right rotations. By analogy to the phenomenon detailed in Sec. IV to follow, in the PCM the left currents may be realized nonlocally. This follows by a direct construction. Upon identifying the right currents with $\varepsilon_{\mu\nu}\partial^\nu\phi$, one may write $\partial_\mu g = g\ \varepsilon_{\mu\nu}\partial^\nu\phi$. That is,

$$\partial_1 g = g\ \partial_0\phi\ , \tag{2.11}$$

which may now be integrated at a fixed time to obtain

$$g(x,t) = g_0\ P\exp\left(\int_x^\infty dy\ \partial_0\phi(y,t)\right)\ , \tag{2.12}$$

assuming the boundary conditions $g(\infty,t) = g_0$. The left currents are now realizable as explicit nonlocal functions of ϕ as obtained from using expression (2.12) for $g(x,t)$ to similarity transform the right currents:

$$L_\mu = g\partial_\mu g^{-1} = -g\ (g^{-1}\partial_\mu g)\ g^{-1} = -g\ J_\mu\ g^{-1} = -\varepsilon_{\mu\nu}\ g\ \partial^\nu\phi\ g^{-1} = -\varepsilon_{\mu\nu}\partial^\nu(g\ \phi\ g^{-1}) + g\ [\partial_\mu\phi\ ,\phi]\ g^{-1}. \tag{2.13}$$

The first term is trivially conserved while the second one resembles a similarity transform of the right currents R_μ and, in fact, to leading order, likewise generates adjoint right rotations sandwiched within the arbitrary boundary-condition g_0 similarity transformation. These nonlocal currents transform in the adjoint of $G_{\rm left}$, albeit somewhat speciously, since these transformations only serve to rotate the arbitrary boundary conditions g_0, and do not affect the dynamical fields ϕ of the action at all. They thus commute with the right rotations. As a consequence, removing g_0 from the above currents banishes $G_{\rm left}$ from the theory altogether.

As is evident from the equivalent status of left versus right in the chiral model, none of the above results hinges crucially on the difference between left and right currents. Left↔right-reflected identical results would have followed the above pseudodual transmutation upon interchange of left with right.

III. CANONICALLY EQUIVALENT DUAL σ MODEL

The above expression for $g(x,t)$, as an explicitly nonlocal function of $\partial_0\phi(y,t)$ up to boundary conditions, would seem to suggest that the two versions of the chiral model are equivalent, providing as it does an invertible, fixed-time map relating all g and ϕ field configurations. Nevertheless, the point is that this map is *not* a canonical transformation. Hence, the g and ϕ theories are *not* canonically equivalent. The quantum theories for $\mathcal{L}_1$ and $\mathcal{L}_2$ are thus inequivalent, if effects are computed in the standard way, say, in perturbation theory, which assumes canonical variables.

One direct way to see this point would be to compare Poisson brackets for various expressions in the g and ϕ theories. This is done below for the currents of the ϕ theory.

It is instructive, however, to take an indirect approach and construct a canonical transformation which maps the usual chiral σ model onto an equivalent dual σ model, with torsion, which is different from the PCM. The transformation identifies conserved, curvature-free currents differently than in (2.4). Such a construction is the canonical, fixed-time analogue of the Lagrange multiplier technique, and accompanying change of variables, employed by Fridling and Jevicki [3] in the covariant path integral formalism. We exhibit here such a transformation.

For specificity, consider the standard O(4) $\simeq$ O(3)×O(3) $\simeq$ SU(2)×SU(2) chiral σ model, parametrized in the usual way, $g = \varphi^0 + i\tau^j\varphi^j$, with coordinates φ^0, φ^j $(j = 1,2,3)$, subject to the constraint $(\varphi^0)^2 + \varphi^2 = 1$, with $\varphi^2 \equiv \sum_j (\varphi^j)^2$. Resolve the constraint and substitute $\varphi^0 = \pm\sqrt{1-\varphi^2}$, to obtain the standard form for the defining chiral Lagrangian

$$\mathcal{L}_1 = \tfrac{1}{2}\left(\delta^{ij} + \frac{\varphi^i\varphi^j}{1-\varphi^2}\right)\partial_\mu\varphi^i\partial^\mu\varphi^j . \tag{3.1}$$

We now show that this model is *canonically equivalent* to the dual σ model (DSM) defined by the Lagrangian[3]

$$\mathcal{L}_3 = \frac{1}{1+4\psi^2}\left[\tfrac{1}{2}\left(\delta^{ij} + 4\psi^i\psi^j\right)\partial_\mu\psi^i\partial^\mu\psi^j - \varepsilon^{\mu\nu}\varepsilon^{ijk}\psi^i\partial_\mu\psi^j\partial_\nu\psi^k\right] . \tag{3.2}$$

[3]Up to normalizations, this is essentially the Lagrangian of Fridling and Jevicki, as in Ref. [3], although the reader will note some significant sign differences from their Eq. (13).

Of course, this Lagrangian is different than that for the PCM, $\mathcal{L}_2$, since it contains both a nontrivial metric and torsion on the field manifold. Nonetheless, the reader is invited to compare the two to leading orders in the fields ψ and ϕ, respectively, i.e., to examine the weak ψ field limit in the above and in what follows.

As we shall verify in detail, a suitable generator for a canonical transformation relating φ and ψ is simply given by a spacelike line integral of an O(3)-invariant bilinear $\psi^i J_i^\mu[\varphi,\varpi]$, where J_i^μ is either the left or the right, conserved, curvature-free O(3) current for the φ theory. Choosing here the right current $(V+A)$, at any fixed time our generator is $F[\psi,\varphi]=\int_{-\infty}^{\infty}dx\;\psi^i J_i^1[\varphi]$, as given explicitly by

$$F[\psi,\varphi]=\int_{-\infty}^{+\infty}dx\;\psi^i\left(\sqrt{1-\varphi^2}\,\frac{\partial}{\partial x}\varphi^i-\varphi^i\frac{\partial}{\partial x}\sqrt{1-\varphi^2}+\varepsilon^{ijk}\varphi^j\frac{\partial}{\partial x}\varphi^k\right). \tag{3.3}$$

Note that left rotations on φ alone do nothing to this F. It is then consistent to assume that the new field variable ψ^i is a left-transformation singlet, just like its conjugate quantity $J_i^1[\varphi]$, and that $F[\psi,\varphi]$ is left invariant. On the other hand, the effects of right rotations (or separate axial or vector rotations) on φ must be compensated by appropriate isospin transformations of ψ analogous to the ones generated by R_μ for the PCM in the previous section.[4]

By construction, F is linear in ψ, but nonlinear in φ. Note that integrating by parts just gives back the original F, without any surface term. Also, while we have written $\pm\infty$ as the limits of x integration in the expression for F, the reader should be aware that finite limits of integration are also acceptable with appropriate boundary conditions. For example, x could be a circle, with both ψ and φ satisfying periodic boundary conditions. Finally, note that the weak φ field limit of the generating function reduces to the well-known duality transformation between free scalar and pseudoscalar fields, as generated by $F_0[\psi,\varphi]=\int dx\;\psi^i\frac{\partial}{\partial x}\varphi^i$.

Having exhibited the canonical transformation which relates the chiral model to its fully equivalent dual, we may now allow modifications in the form of F to see if it is also possible to connect the chiral model to the PCM. The results of such an investigation are negative: The nontrivial metric of the chiral model cannot be converted into the trivial metric and constant torsion of the PCM through canonical transformations which equate the curvature-free currents of the two theories. This follows from explicit calculations similar to those below. It should not be difficult, in principle, for a sufficiently motivated reader to see this, say, by allowing arbitrary invariant functions to appear in F. Nevertheless, the details can be tedious, and shortcuts are not available, at present. We forego the details here and simply state the result.

Also, in contrast to the emergent nonlinear pseudo-Abelian Z charges in the PCM model, no such local symmetry appears present in the DSM. Thus, there is no *local* analogue of the three axial φ symmetries for the theory $\mathcal{L}_3$. It is straightforward to verify this for the classical theory, and it is best left as an exercise. (At the quantum level, these facts are consistent with the properties of the transformation functional discussed below.)

Let us now verify that F generates a canonical transformation which identifies the curvature-free currents of the two models defined by $\mathcal{L}_1$ and $\mathcal{L}_3$. As a consequence, it follows that the energy-momentum tensors for the two theories are equal under the transformation. It is a textbook exercise to check that F fulfills its mission at the classical level. Classically, functionally differentiating F generates the conjugate momenta. The conjugate of ψ^i is given by

$$\begin{aligned}\pi_i=\frac{\delta F[\psi,\varphi]}{\delta\psi^i}&=\sqrt{1-\varphi^2}\;\frac{\partial}{\partial x}\varphi^i-\varphi^i\frac{\partial}{\partial x}\left(\sqrt{1-\varphi^2}\right)+\varepsilon^{ijk}\varphi^j\frac{\partial}{\partial x}\varphi^k\\ &=\left(\sqrt{1-\varphi^2}\;\delta^{ij}+\frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}}-\varepsilon^{ijk}\varphi^k\right)\frac{\partial}{\partial x}\varphi^j.\end{aligned} \tag{3.4}$$

The generator F was, in fact, chosen to yield this result. The conjugate of φ^i is obtained through

$$\begin{aligned}\varpi_i=-\frac{\delta F[\psi,\varphi]}{\delta\varphi^i}&=\left(\sqrt{1-\varphi^2}\;\delta^{ij}+\frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}}+\varepsilon^{ijk}\varphi^k\right)\frac{\partial}{\partial x}\psi^j\\ &\quad+\left(\frac{2}{\sqrt{1-\varphi^2}}\left(\varphi^i\psi^j-\psi^i\varphi^j\right)-2\varepsilon^{ijk}\psi^k\right)\frac{\partial}{\partial x}\varphi^j,\end{aligned} \tag{3.5}$$

[4] Although, as discussed below, the actual curvature-free current for ψ is a combination of isospin and topological currents, and so when comparing the effect of charges for the curvature-free currents between φ and ψ theories, there is a possibility of subtle surface term contributions reminiscent of the constants of integration g_0 that appeared in (2.12).

a "mixed" expression involving both φ and ψ. Now (3.4) may be inverted to replace $\frac{\partial}{\partial x}\varphi$ in (3.5) by π. Thus

$$\varpi_i = \left(\sqrt{1-\varphi^2}\,\delta^{ij} + \frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}} + \varepsilon^{ijk}\varphi^k\right)\frac{\partial}{\partial x}\psi^j$$
$$+2\left(\psi^k\varphi^k\delta^{il} - \psi^i\varphi^l\right)\pi_l + 2\left(\sqrt{1-\varphi^2}\,\delta^{ij} + \frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}}\right)\varepsilon^{jkl}\psi^k\pi_l\ . \tag{3.6}$$

If viewed as classical relations, we may substitute for π_i and ϖ_i, in (3.4) and (3.6), in terms of $\frac{\partial}{\partial t}\varphi^j$ and $\frac{\partial}{\partial t}\psi^j$, as follows from the Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_3$:

$$\pi_i = \frac{1}{1+4\psi^2}\left(\left(\delta^{ij}+4\psi^i\psi^j\right)\frac{\partial}{\partial t}\psi^j + 2\varepsilon^{ijk}\psi^j\frac{\partial}{\partial x}\psi^k\right), \qquad \varpi_i = \left(\delta^{ij} + \frac{\varphi^i\varphi^j}{1-\varphi^2}\right)\frac{\partial}{\partial t}\varphi^j. \tag{3.7}$$

The resulting covariant pair of first-order, nonlinear, partial differential equations for φ and ψ constitutes a Bäcklund transformation connecting the two theories. Consistency of this Bäcklund transformation is equivalent to the classical equations of motion for φ and ψ .

Note, as a consequence of (3.4), we find the mixed inner product relation

$$\varphi^i\pi_i = -\frac{\partial}{\partial x}\left(\sqrt{1-\varphi^2}\right). \tag{3.8}$$

Integrating over all x, we obtain a mixed "adiabatic" (topological) invariant, which represents a nontrivial global constraint obeyed by the correlated pair of solutions for the two theories that are connected by the canonical map. For example, assuming trivial or periodic boundary conditions on φ, we have $\int dx\ \varphi^i\pi_i = 0$.

Now, in the DSM, what is the conserved, curvature-free current? In contrast to the PCM, where it was essentially *forced* to be a topological current, here a topological current by itself will not suffice. Neither will a conserved Noether current. For instance, under the isospin transformation $\delta\psi^i = \varepsilon^{ijk}\psi^j\omega^k$, the conserved Noether current of $\mathcal{L}_3$ is defined as usual by $I_i^\mu = \delta\mathcal{L}_3/\delta(\partial_\mu\omega^i)$ so that $I_i^0 = \varepsilon^{ijk}\psi^j\pi_k$. But this is not curvature free. Instead, the conserved, curvature-free current $\mathcal{J}_i^\mu$, to be compared with J_i^μ of the CM, is a mixture of this Noether current and a topological current: $\mathcal{J}_i^\mu = 2I_i^\mu - \varepsilon^{\mu\nu}\partial_\nu\psi^i$ so that $\mathcal{J}_i^1 = \pi_i$. In covariant form,

$$\mathcal{J}_i^\mu = \frac{-1}{1+4\psi^2}\left[\left(\delta^{ij}+4\psi^i\psi^j\right)\varepsilon^{\mu\nu}\partial_\nu\psi^j + 2\varepsilon^{ijk}\psi^j\partial^\mu\psi^k\right]. \tag{3.9}$$

So, to complete verification that the canonical F-generated transformation does the job, at least classically, it remains to show that the curvature-free currents[5] are equal for the φ and ψ theories when expressed in terms of the respective fields and their conjugate variables. That is, we must show $J_i^\mu[\varphi,\varpi] = \mathcal{J}_i^\mu[\psi,\pi]$. This requires us to establish the following:

$$\mathcal{J}_i^1 \equiv \pi_i = \left(\sqrt{1-\varphi^2}\,\delta^{ij} + \frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}} - \varepsilon^{ijk}\varphi^k\right)\frac{\partial}{\partial x}\varphi^j \equiv J_i^1, \tag{3.10}$$

$$\mathcal{J}_i^0 \equiv -\frac{\partial}{\partial x}\psi^i - 2\varepsilon^{ijk}\psi^j\pi_k = -\sqrt{1-\varphi^2}\,\varpi_i - \varepsilon^{ijk}\varphi^j\varpi_k \equiv J_i^0. \tag{3.11}$$

The first of these is precisely (3.4), the form for π_i given directly by the generating functional.

The second (time component) current identity takes more effort to establish, but it also follows from (3.4) and (3.6). To see this, reduce each of the expressions involving the fields and their conjugate momenta in (3.11) to the following mixed result involving the fields and their spatial derivatives:

$$\mathcal{J}_i^0 = -\frac{\partial}{\partial x}\psi^i + 2\psi^j\left(\varphi^j\frac{\partial}{\partial x}\varphi^i - \varphi^i\frac{\partial}{\partial x}\varphi^j\right) + 2\varepsilon^{ijk}\psi^j\left(\varphi^k\frac{\partial}{\partial x}\left(\sqrt{1-\varphi^2}\right) - \sqrt{1-\varphi^2}\frac{\partial}{\partial x}\varphi^k\right) = J_i^0. \tag{3.12}$$

[5]Our normalizations here are such that "curvature-free current" means $\varepsilon_{\mu\nu}\left(\partial^\mu J_i^\nu + \varepsilon^{ijk}J_j^\mu J_k^\nu\right) = 0$.

Thus, the validity of our canonical transformation at the classical level is now established. What about the ensuing relation between the quantum theories associated with $\mathcal{L}_1$ and $\mathcal{L}_3$?

To carry out a comparison at the quantum level, we combine Schrödinger wave-functional methods with the transformation theory of Dirac, as explained, for example, within the context of Liouville theory in [9]. We find that the energy-momentum eigenfunctionals of the quantum ψ theory, $\Psi_{E,\mathbf{p}}[\psi]$, are related to those of the quantum φ theory, $\Phi_{E,\mathbf{p}}[\varphi]$, and vice versa, by exponentiating the classical generating function of the canonical transformation to obtain a transformation functional. That is, at any fixed time[6] the eigenfunctionals of the two theories are nonlinear (in φ) functional Fourier transforms of one another:

$$\Psi_{E,\mathbf{p}}[\psi] = N \int d\varphi \; e^{iF[\psi,\varphi]} \; \Phi_{E,\mathbf{p}}[\varphi]. \tag{3.13}$$

Note that we have allowed for an adjustment of the overall normalization of eigenfunctionals in the two theories by including an undetermined energy-momentum dependent factor N. In principle, it is straightforward to determine N, but we have not.

The previous classical relations between the fields and their conjugate momenta, or alternatively, between the curvature-free currents, now become nonlinear first-order functional differential equations obeyed by the transformation functional

$$\begin{aligned} \mathcal{J}_i^1 \left[\pi \equiv -i\frac{\delta}{\delta\psi}\right] \; e^{iF[\psi,\varphi]} &= J_i^1[\varphi] \; e^{iF[\psi,\varphi]}, \\ \mathcal{J}_i^0 \left[\psi, \pi \equiv -i\frac{\delta}{\delta\psi}\right] \; e^{iF[\psi,\varphi]} & \\ &= -J_i^0 \left[\varphi, \varpi \equiv -i\frac{\delta}{\delta\varphi}\right] \; e^{iF[\psi,\varphi]} \;. \end{aligned} \tag{3.14}$$

Presumably these differential equations are exact, and the transformation functional provides an exact solution of them. Imposing a cutoff on the theory, it is straightforward to check that this is indeed so simply as a consequence of the classical equations (3.10) and (3.11). [In this regard, we note that dimensional regularization is particularly convenient here, as well known for the CM, since it permits us to naively discard certain $\delta(0)$ terms which would otherwise mar the Hermiticity of F and the currents as written, and which would also confront one in comparing the energy-momentum tensors for the ψ and φ theories.] However, a nontrivial analysis is required to remove the cutoff, that is, to renormalize the transformation in (3.13). This renormalization analysis will not be given here.

Rather, here we take the expression (3.13) as is and use it to gain a better understanding of how symmetries in one *quantum* theory are related to those of the other *quantum* theory, a technique previously illustrated in the context of simple potential models [16]. From the transformation properties of the ϕ-dependent current in F, we expect that V and A transformations on the eigenfunctionals must coincide insofar as $e^{iF[\psi,\varphi]}$ projects onto the left invariants. This behavior for the quantized canonically equivalent dual σ model would correspond, in the functional framework, to the degeneration of G_{left} for the classical PCM illustrated in the previous section. Indeed, when acting on the above transformation functional, and consequently also when acting on the eigenfunctionals, the axial and vector charges of the CM (φ theory) produce equivalent effects on the ψ theory. Both the inhomogeneous, nonlinearly realized axial symmetries and the linear vector O(3) symmetry of the chiral φ theory are projected by the canonical transformation into the same linear right-isospin O(3) symmetry of the $\mathcal{L}_3$ theory:

$$\begin{aligned} \left(\int_{-\infty}^{\infty} dy\,\sqrt{1-\varphi^2(y)}\frac{\delta}{\delta\varphi^i(y)}\right) e^{iF[\psi,\varphi]} &= -\left(\int_{-\infty}^{\infty} dy\,\varepsilon^{ijk}\varphi^j(y)\frac{\delta}{\delta\varphi^k(y)}\right) e^{iF[\psi,\varphi]} \\ &= -\left(\int_{-\infty}^{\infty} dy\,\varepsilon^{ijk}\psi^j(y)\frac{\delta}{\delta\psi^k(y)}\right) e^{iF[\psi,\varphi]}. \end{aligned} \tag{3.15}$$

To take the last step, one must integrate by parts and either cancel or discard surface contributions. The action of the charges on the eigenfunctionals, as related by (3.13) and its inverse, is obtained by functionally integrating by parts after acting on $e^{iF[\psi,\varphi]}$. [Once again, dimensional regularization is convenient here as it allows us to blithely discard $\delta(0)$ terms.]

The only vestige of the difference in vector and axial symmetries carried by the transformation functional lies in its field parity properties. Under $\psi \to -\psi$, the generator F trivially changes sign. But under $\varphi \to -\varphi$, not only is the sign of F changed, but also the right current for the CM (φ theory) is converted into the left current, and thus $F[-\psi,-\varphi]$ generates a canonical transformation which projects onto right invariants. Therefore, it is possible to obtain all the effects of interchanging right $\leftrightarrow$ left currents in the transformation functional by merely splitting all wave functionals into components even and odd under field parity.[7] Since this can always be done, using one current in F instead of the other results in no loss of information.

We now return to our investigation of the PCM it-

[6]Since this is a fixed-time expression, the Schrödinger functional integral $\int d\varphi$ is over all field configurations at each point in space, but not at any other times. That is, $\int d\varphi$ in (3.13) is *not* a path integral.

[7]The same statement applies to interchanging northern and southern hemispheres for the three sphere defined by $(\varphi^0)^2 + \varphi^2 = 1$, since this only flips the sign of the square root appearing in F which is again tantamount to $\varphi \to -\varphi$.

self. In particular, we systematically study the nonlocal currents and their charges which are guaranteed to exist for the PCM by virtue of the conservation and vanishing curvature for J_μ.

IV. NONLOCAL CURRENTS AND CHARGES FOR THE PSEUDODUAL MODEL

The full set of nonlocal conservation laws is neatly described using the methods in [10] (see also [11–14]). For any conserved, curvature-free currents such as J_μ, irrespective of the specific model considered, introduce the Pohlmeyer dual boost spectral parameter κ to define

$$C_\mu(x,\kappa) = -\frac{\kappa^2}{1-\kappa^2} J_\mu - \frac{\kappa}{1-\kappa^2}\,\tilde{J}_\mu, \tag{4.1}$$

where $\tilde{J}_\mu \equiv \varepsilon_{\mu\nu} J^\nu$. Thus

$$(\partial^\mu + C^\mu)\,\widetilde{C}_\mu = 0\ . \tag{4.2}$$

This serves as the consistency condition for the equations

$$\partial_\mu\, \chi^{ab}(x) = -C_\mu^{ac}\, \chi^{cb}(x) \tag{4.3}$$

or, equivalently,

$$\varepsilon_{\mu\nu}\partial^\nu \chi = \kappa\,(\partial_\mu + J_\mu)\,\chi\ , \tag{4.4}$$

which are solvable recursively in κ [11,10]. Equivalently, the solution χ can be expressed as a path-ordered exponential (Polyakov's path-independent disorder variable) [12,10]

$$\chi(x,\kappa) = P\exp\left(-\int_{-\infty}^{x} dy\ C_1(y,t)\right) \equiv \mathbb{1} + \sum_{n=0}^{\infty} \kappa^{n+1}\chi^{(n)}\ . \tag{4.5}$$

These ensure conservation of an *antisymmetrized* nonlocal "master current" constructed as follows:

$$\mathfrak{J}^\mu(x,\kappa) \equiv \frac{1}{2\kappa}\,\varepsilon^{\mu\nu}\partial_\nu\,[\chi(x,\kappa) - \chi^T(x,\kappa)] \equiv \sum_{n=0}^{\infty} \kappa^n\ J^\mu_{(n)}(x)\ . \tag{4.6}$$

The conserved master current acts as the generating functional of all currents $J^\mu_{(n)}$ (separately) conserved order by order in κ. E.g., the lowest four orders yield

$$\begin{aligned}\mathfrak{J}_\mu(x,\kappa) = J_\mu(x)\ &+\ \kappa\left(\tilde{J}_\mu(x) + \tfrac{1}{2}\left[J_\mu(x)\ ,\ \int_{-\infty}^{x} dy\ J_0(y)\right]\right)\\ &+ \kappa^2\left(\tilde{J}^{(1)}_\mu(x) + \tfrac{1}{2}(J_\mu(x)\ \chi^{(1)} + \chi^{(1)T}\ J_\mu(x))\right)\\ &+ \kappa^3\left(\tilde{J}^{(2)}_\mu(x) + \tfrac{1}{2}(J_\mu(x)\ \chi^{(2)} + \chi^{(2)T}\ J_\mu(x))\right)\ +\ O(\kappa^4)\ .\end{aligned} \tag{4.7}$$

Integrating the nonlocal master current yields a conserved "master charge"

$$\mathfrak{Q}(\kappa) = \int_{-\infty}^{+\infty} dx\ \mathfrak{J}_0(x,\kappa) \equiv \sum_{n=0}^{\infty} \kappa^n\ Q_{(n)}\ . \tag{4.8}$$

$Q_{(0)}$ is the conventional symmetry charge, while $Q_{(1)}, Q_{(2)}, Q_{(3)}, \ldots$ are the well-known nonlocal charges, best studied for σ models [8,7,11,12], the Gross-Neveu model [15], and supersymmetric combinations of the two [10].

For the PCM [1,2], however, it readily follows from

$$J^{(0)}_\mu = J_\mu = \varepsilon_{\mu\nu}\partial^\nu\phi \quad\Longrightarrow\quad \chi^{(0)}(x) = \phi(x) - \phi(-\infty) \tag{4.9}$$

that

$$J^{(1)}_\mu = \partial_\mu\phi + \tfrac{1}{2}\varepsilon_{\mu\nu}[\partial^\nu\phi,\phi] - \tfrac{1}{2}[J_\mu,\phi(-\infty)] = Z_\mu - \tfrac{1}{2}[J_\mu,\phi(-\infty)]. \tag{4.10}$$

Recall that, as stressed in Sec. II, $\phi(-\infty)$ is taken to be time independent, and thus each piece of this current is separately conserved. So the CM↔PCM transmutation has yielded a local current for the first "nonlocal." Continuing,

$$\chi^{(1)}(x) = \int_{-\infty}^{x} dy\ \Big[\partial_0\phi(y) + \partial_1\phi(y)\ \phi(y)\Big] - \phi(x)\phi(-\infty) + \phi(-\infty)^2\ . \tag{4.11}$$

Likewise,

$$J^{(2)}_\mu = \varepsilon_{\mu\nu}\partial^\nu\phi + [\partial_\mu\phi,\phi] - \phi\ \varepsilon_{\mu\nu}\partial^\nu\phi\ \phi + \tfrac{1}{2}\varepsilon_{\mu\nu}\partial^\nu\Big(\phi\ \chi^{(1)} + \chi^{(1)T}\phi\Big)$$
$$-\tfrac{1}{2}[Z_\mu,\phi(-\infty)] + \tfrac{1}{4}\varepsilon_{\mu\nu}\partial^\nu(\phi^2\phi(-\infty) + \phi(-\infty)\phi^2)$$
$$= J_\mu - R_\mu - \tfrac{1}{3}\varepsilon_{\mu\nu}\partial^\nu(\phi^3) + \tfrac{1}{2}\varepsilon_{\mu\nu}\partial^\nu\Big(\phi\chi^{(1)} + \chi^{(1)T}\phi\Big) - \tfrac{1}{2}[Z_\mu,\phi(-\infty)] + \tfrac{1}{4}\varepsilon_{\mu\nu}\partial^\nu(\phi^2\phi(-\infty) + \phi(-\infty)\phi^2). \quad (4.12)$$

On-shell properties of the currents have been used. However, this second "nonlocal" current is also effectively local: The skew-gradient term, which might appear to contribute a nonlocal piece to the charge via $\chi^{(1)}$, only contributes $[\phi(\infty), Q_Z]/2$, i.e., a trivial piece based on a local current.

In contrast to the first two steps, however, the third step in the recursive algorithm gives

$$J^{(3)}_\mu = \tfrac{1}{2}\Big(Z_\mu\chi^{(1)} + \chi^{(1)T}Z_\mu\Big) + \cdots\ , \quad (4.13)$$

where ellipses $(\cdots)$ indicate terms which contribute only local pieces to the corresponding charge, whereas the term written explicitly may be seen to contribute ineluctable nonlocal pieces to that charge. Thus $J^{(3)}_\mu$ appears, like all higher currents, genuinely nonlocal. In fact, we will see below that the action of $Q^{(3)}$ on the field changes the boundary condition at $x = \infty$ to a different one than at $-\infty$, and thereby switches its topological sector, which is quantified by $Q^{(0)}$.

In summary for the pseudodual model, the charge $Q^{(0)}$ is topological, while $Q^{(1)}$ generates shifts, $Q^{(2)}$ generates "right" rotations, and $Q^{(n\geq 3)}$ appear genuinely nonlocal.

It may well be objected that the above master current construction is best suited to the chiral model, but is not really handy for the PCM, since the procedure starts off with a non-Noether (topological) current, then "stalls" twice at the first two steps before finally producing genuine nonlocals at the third step and beyond. The construction also produces a clutter of total derivatives and other terms whose charges are extraneous. To overcome some of these objections, we offer here a refined algorithm which begins with the lowest nontopological (Noether) current Z_μ to produce an alternate but equally viable conserved master current. This refined construction helps to reduce some of the clutter and only stalls once. Following [14] as illustrated above, define

$$W_\mu(x,\kappa) \equiv Z_\mu + \kappa\tilde{Z}_\mu, \quad (4.14)$$

which is readily seen to be C-covariantly conserved:

$$\partial^\mu W_\mu + [C^\mu, W_\mu] = 0\ . \quad (4.15)$$

This condition then empowers W_μ to serve as the seed for a new and improved conserved master current

$$\mathfrak{W}_\mu(x,\kappa) = \chi^{-1}W_\mu\ \chi$$
$$= Z_\mu + \kappa\,(T_\mu - [Z_\mu,\phi(-\infty)]) + \kappa^2\left(N_\mu - [T_\mu,\phi(-\infty)] + \frac{1}{2}[[Z_\mu,\phi(-\infty)],\phi(-\infty)]\right) + O(\kappa^3), \quad (4.16)$$

where we have introduced the convenient combination

$$T_\mu \equiv J_\mu - \tfrac{3}{2}R_\mu = \tilde{Z}_\mu + [Z_\mu,\phi], \quad (4.17)$$

and where now the terms of second order and higher are genuinely nonlocal, e.g.,

$$N_\mu = [T_\mu,\phi] - \tfrac{1}{2}[[Z_\mu,\phi],\phi] + \left[Z_\mu, \int_{-\infty}^{x} dy\ Z_0(y)\right]\ . \quad (4.18)$$

This is a refined equivalent of $J^{(3)}_\mu$ above. Note that the terms in $\mathfrak{W}_\mu$ involving the constant matrices $\phi(-\infty)$ are separately conserved, as remarked previously.

In general, it is straightforward to see that the seeds for such improved master currents only need be conserved currents, such as Z_μ above, which also have a vanishing J-covariant curl of the type (2.10). For instance, the previous nonlocal currents themselves may easily be fashioned to satisfy (2.10), and thereby seed respective conserved master currents.

As an aside, in some models, such as the CM, master currents such as $\mathfrak{W}$ amount to the currents of nonlocal-similarity-transformed fields, which obey the classical equations of motion if the fields do [13], thereby allowing an interpretation of the similarity transform as an auto-Bäcklund transformation (quite distinct from the Bäcklund transformation of the previous section). Here, however, the corresponding transform of ϕ,

$$\Phi \equiv \chi^{-1}\ \phi\ \chi, \quad (4.19)$$

does not quite obey the original classical equations of motion. Rather,

$$\partial^\mu\partial_\mu\Phi - \varepsilon^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = \left(\tfrac{2\kappa}{1-\kappa^2}\right)^2\ \mathfrak{W}^\mu\mathfrak{W}_\mu, \quad (4.20)$$

which would seem to obviate any interpretation of (4.19) as an auto-Bäcklund transformation for the PCM. Nonetheless, (4.20) is still an evocative relation which encodes all solutions of the PCM.

As an additional aside, it may be worth recalling with Ref. [1] that there is also a *local* sequence of conserved currents predicated on conserved, curvature-free currents such as J_μ. This follows directly from rewriting

$$(\partial_0 \pm \partial_1)(J_0 \mp J_1) = \pm\tfrac{1}{2}[J_0 - J_1, J_0 + J_1] \ , \quad (4.21)$$

in light-cone coordinates

$$\partial_\pm J_\mp = \pm\tfrac{1}{2}[J_-, J_+] \ , \quad (4.22)$$

which leads to the following sequence of conservation laws for arbitrary integers m, n:

$$\partial_- \mathrm{Tr}\ J_+^n = 0 = \partial_+ \mathrm{Tr}\ J_-^m = 0 \ . \quad (4.23)$$

Such conservation laws are normally nontrivial ($n = m = 2$ is energy-momentum conservation), but for orthogonal groups such as exemplified here odd powers vanish identically by virtue of the cyclicity of the trace.

V. POISSON BRACKETS AND CURRENT ALGEBRA

Our goals in this section are to systematically work out the canonical bracket algebra of all the local currents for the pseudodual theory, and, through the use of these, to also demonstrate unequivocably that the action of the charge for the nonlocal current N_μ given above is genuinely nonlocal. Explicitly,

$$[\![Q_N, \phi^{ab}(y)]\!] = -[[M^{ab}, \phi(y)], \phi(y)] + 2\int_{-\infty}^{+\infty} dx\ \varepsilon(y-x)[Z_0(x), M^{ab}] \quad (5.1)$$

will eventuate. Evidently, then, Q_N changes the boundary condition on the field ϕ at $x = +\infty$ to a different one than at $x = -\infty$, and thus changes the topology of the field configuration upon which it acts, a change which is quantified by the charge Q.

First, observe that, for the PCM,

$$J_0(x) = \partial_x\phi(x), \qquad J_1(x) = \pi(x) - \frac{1}{3}[\partial_x\phi(x), \phi(x)] \ . \quad (5.2)$$

So the J current algebra follows immediately from the fundamental Poisson brackets

$$[\![\phi(x), \pi^{ab}(y)]\!] = M^{ab}\ \delta(x-y) = -[\![\pi(x), \phi^{ab}(y)]\!] \ , \quad (5.3)$$

where we have suppressed one pair of indices, and introduced the O(N) antisymmetrizer matrix

$$(M^{ab})_{cd} \equiv \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}. \quad (5.4)$$

Note the distinction between matrix commutators, as given by $[\ \cdots, \cdots\ ]$, and Poisson brackets, as given by $[\![\cdots, \cdots]\!]$. Combining the fundamental brackets with (5.2), we compute in succession

$$[\![J_0(x), \phi^{ab}(y)]\!] = 0, \qquad [\![J_0(x), \pi^{ab}(y)]\!] = M^{ab}\ \delta'(x-y), \qquad [\![J_0(x), [\partial_y\phi(y), \phi(y)]^{ab}]\!] = 0, \quad (5.5)$$

followed by

$$[\![J_1(x), \phi^{ab}(y)]\!] = -\delta(x-y)\ M^{ab} = -[\![\phi(x), J_1^{ab}(y)]\!] \ , \quad (5.6)$$

$$[\![J_1(x), \pi^{ab}(y)]\!] = -\frac{1}{3}[J_0(x), M^{ab}]\ \delta(x-y) + \frac{1}{3}[\phi(x), M^{ab}]\ \delta'(x-y), \quad (5.7)$$

$$[\![J_1(x), [\partial_y\phi(y), \phi(y)]^{ab}]\!] = [J_0(y), M^{ab}]\ \delta(x-y) + [\phi(y), M^{ab}]\ \delta'(x-y) = [\![J_1(x), [J_0(y), \phi(y)]^{ab}]\!] \ . \quad (5.8)$$

Now, substitution of these results into the expressions in Eq. (5.2) leads to the "topological" current algebra[8]

$$[\![J_0(x), J_0^{ab}(y)]\!] = 0, \quad (5.9)$$

$$[\![J_1(x), J_1^{ab}(y)]\!] = -[J_0(x), M^{ab}]\ \delta(x-y), \quad (5.10)$$

$$[\![J_0(x), J_1^{ab}(y)]\!] = M^{ab}\ \delta'(x-y) = [\![J_1(x), J_0^{ab}(y)]\!] \ . \quad (5.11)$$

[8]N.B. Recall the distribution lemma $f(x,y)\ \delta'(x-y) = -\delta(x-y)\ \partial_x f(x,y)$ when $f(x,x) = 0$. Also note, when ϕ is an antisymmetric matrix, $[M^{ab}, \phi] \equiv \phi^{ac}M^{cb} - M^{ac}\phi^{cb}$, etc. Alternatively, $[M^{ab}, \phi]_{cd} = -[M^{cd}, \phi]_{ab}$, and similarly, for antisymmetric ϕ and ψ, $[[M^{ab}, \phi], \psi]_{cd} = [[M^{cd}, \psi], \phi]_{ab}$.

Moving right along, bracket Z_0 with the local fields and topological currents. First, act on the field and its conjugate (recall $Z_0 = J_1 + \frac{1}{2}[J_0, \phi]$):

$$[\![Z_0(x), \phi^{ab}(y)]\!] = -M^{ab}\ \delta(x-y) = -[\![\phi(x), Z_0^{ab}(y)]\!] , \tag{5.12}$$

$$\begin{aligned}[\![Z_0(x), \pi^{ab}(y)]\!] &= \frac{1}{6}[J_0(x), M^{ab}]\ \delta(x-y) - \frac{1}{6}[\phi(x), M^{ab}]\ \delta'(x-y) \\ &= \frac{1}{3}\delta(x-y)\ [J_0(x), M^{ab}] - \frac{1}{6}\partial_x\{\delta(x-y)\ [\phi(x), M^{ab}]\} ,\end{aligned} \tag{5.13}$$

In the last step, we have isolated a total spatial derivative to make transparent the action of the Q_Z charge on π. We will often take such clarifying steps below.

Next act on the topological current components:

$$[\![Z_0(x), J_0^{ab}(y)]\!] = M^{ab}\ \delta'(x-y) = [\![J_0(x), Z_0^{ab}(y)]\!] , \tag{5.14}$$

$$\begin{aligned}[\![Z_0(x), J_1^{ab}(y)]\!] &= -\frac{1}{2}[J_0, M^{ab}]\ \delta(x-y) - \frac{1}{2}[\phi(x), M^{ab}]\ \delta'(x-y) \\ &= -\frac{1}{2}\partial_x\{\delta(x-y)\ [\phi(x), M^{ab}]\} = -\frac{1}{2}[J_0, M^{ab}]\ \delta(x-y) - [\![J_1(x), Z_0^{ab}(y)]\!] ,\end{aligned} \tag{5.15}$$

$$[\![J_1(x), Z_0^{ab}(y)]\!] = -\frac{1}{2}[J_0, M^{ab}]\ \delta(x-y) + \frac{1}{2}[\phi(y), M^{ab}]\ \delta'(x-y) = \frac{1}{2}[\phi(x), M^{ab}]\ \delta'(x-y). \tag{5.16}$$

Continuing this pattern, now bracket with Z_1 (recall $Z_1 = J_0 + \frac{1}{2}[J_1, \phi]$):

$$[\![Z_1(x), \phi^{ab}(y)]\!] = \frac{1}{2}\delta(x-y)\ [\phi(x), M^{ab}] = [\![\phi(x), Z_1^{ab}(y)]\!] , \tag{5.17}$$

$$\begin{aligned}[\![Z_1(x), \pi^{ab}(y)]\!] &= \left(\frac{1}{2}[J_1(x), M^{ab}] + \frac{1}{6}[\phi(x), [J_0(x), M^{ab}]]\right)\delta(x-y) + \left(M^{ab} - \frac{1}{6}[\phi(x), [\phi(x), M^{ab}]]\right)\ \delta'(x-y) \\ &= \delta(x-y)\ \left(\frac{1}{2}[J_1(x), M^{ab}] + \frac{1}{3}[\phi(x), [J_0(x), M^{ab}]] + \frac{1}{6}[J_0(x), [\phi(x), M^{ab}]]\right) \\ &\quad +\partial_x\left(\delta(x-y)\ \left(M^{ab} - \frac{1}{6}[\phi(x), [\phi(x), M^{ab}]]\right)\right) ,\end{aligned} \tag{5.18}$$

$$[\![Z_1(x), J_0^{ab}(y)]\!] = -\frac{1}{2}[\phi(x), M^{ab}]\ \delta'(x-y) = \frac{1}{2}\delta(x-y)\ [J_0(x), M^{ab}] - \frac{1}{2}\partial_x\{\delta(x-y)\ [\phi(x), M^{ab}]\}, \tag{5.19}$$

$$[\![Z_1(x), J_1^{ab}(y)]\!] = \left(\frac{1}{2}[J_1(x), M^{ab}] + \frac{1}{2}[\phi(x), [J_0(x), M^{ab}]]\right)\ \delta(x-y) + M^{ab}\ \delta'(x-y). \tag{5.20}$$

Combining these, we arrive at the pseudo-Abelian "shift" current algebra

$$[\![Z_0(x), Z_0^{ab}(y)]\!] = \frac{1}{2}\ [J_0, M^{ab}]\ \delta(x-y), \tag{5.21}$$

$$[\![Z_1(x), Z_1^{ab}(y)]\!] = \delta(x-y)\ \left(\frac{1}{2}[Z_1, M^{ab}] + \frac{1}{4}[\phi, [J_0, [\phi, M^{ab}]]]\right) , \tag{5.22}$$

$$\begin{aligned}[\![Z_0(x), Z_1^{ab}(y)]\!] &= \left(\frac{1}{2}[J_1, M^{ab}] - \frac{1}{4}[J_0, [\phi, M^{ab}]]\right)\ \delta(x-y) + \left(M^{ab} - \frac{1}{4}[\phi(x), [\phi(y), M^{ab}]]\right)\ \delta'(x-y) \\ &= \frac{1}{2}\delta(x-y)\ [J_1(x), M^{ab}] + \partial_x\left(\delta(x-y)\ \left(M^{ab} - \frac{1}{4}[\phi(x), [\phi(y), M^{ab}]]\right)\right) .\end{aligned} \tag{5.23}$$

$$[\![Z_1(x), Z_0^{ab}(y)]\!]$$
$$= \left(\frac{1}{2}[J_1, M^{ab}] + \frac{1}{4}[\phi, [J_0, M^{ab}]]\right) \delta(x-y) + \left(M^{ab} - \frac{1}{4}[\phi(x), [\phi(y), M^{ab}]]\right) \delta'(x-y)$$
$$= \delta(x-y) \left(\frac{1}{2}[J_1(x), M^{ab}] + \frac{1}{4}[\phi, [J_0, M^{ab}]] + \frac{1}{4}[J_0, [\phi, M^{ab}]]\right) + \partial_x \left(\delta(x-y) \left(M^{ab} - \frac{1}{4}[\phi(x), [\phi(y), M^{ab}]]\right)\right). \tag{5.24}$$

Pressing onward, consider the remaining local current, T_μ (recall $T_0 = Z_1 + [Z_0, \phi]$ and $T_1 = Z_0 + [Z_1, \phi]$):

$$[\![T_0(x), \phi^{ab}(y)]\!] = \frac{3}{2}[\phi, M^{ab}] \delta(x-y) = [\![\phi(x), T_0^{ab}(y)]\!], \tag{5.25}$$

$$[\![T_0(x), \pi^{ab}(y)]\!] = \frac{3}{2}[\pi, M^{ab}] \delta(x-y) + M^{ab} \delta'(x-y) = [\![\pi(x), T_0^{ab}(y)]\!], \tag{5.26}$$

$$[\![T_0(x), J_0^{ab}(y)]\!] = -\frac{3}{2}[\phi(x), M^{ab}] \delta'(x-y) = -[\![J_0(x), T_0^{ab}(y)]\!]$$
$$= \delta(x-y) \frac{3}{2}[J_0(x), M^{ab}] - \partial_x \left(\delta(x-y) \frac{3}{2}[\phi(x), M^{ab}]\right), \tag{5.27}$$

$$[\![T_0(x), J_1^{ab}(y)]\!] = \left([Z_0, M^{ab}] + \frac{1}{2}[J_1, M^{ab}] + [\phi, [J_0, M^{ab}]]\right) \delta(x-y) + \left(M^{ab} + \frac{1}{2}[\phi(x), [\phi(x), M^{ab}]]\right) \delta'(x-y)$$
$$= \delta(x-y) \frac{3}{2}[J_1, M^{ab}] + \partial_x \left(\delta(x-y) \left(M^{ab} + \frac{1}{2}[\phi(x), [\phi(x), M^{ab}]]\right)\right), \tag{5.28}$$

$$[\![T_0(x), Z_0^{ab}(y)]\!] = \left(\frac{3}{2}[Z_0, M^{ab}] - \frac{1}{4}[J_0, [\phi, M^{ab}]]\right) \delta(x-y) + \left(M^{ab} - \frac{1}{4}[\phi(x), [\phi(y), M^{ab}]]\right) \delta'(x-y)$$
$$= \delta(x-y) \frac{3}{2}[Z_0, M^{ab}] + \partial_x \left(\delta(x-y) \left(M^{ab} - \frac{1}{4}[\phi(x), [\phi(y), M^{ab}]]\right)\right), \tag{5.29}$$

$$[\![T_0(x), Z_1^{ab}(y)]\!] = \left(\frac{3}{2}[Z_1, M^{ab}] - [J_0, M^{ab}] + \frac{1}{4}[\phi, [J_0, [\phi, M^{ab}]]] + \frac{1}{4}[J_0, [\phi, [\phi, M^{ab}]]]\right) \delta(x-y)$$
$$+ \left(-[\phi(x), M^{ab}] + \frac{1}{4}[\phi(x), [\phi(x), [\phi(y), M^{ab}]]]\right) \delta'(x-y)$$
$$= \delta(x-y) \frac{3}{2}[Z_1, M^{ab}] + \partial_x \left(\delta(x-y) \left(-[\phi(x), M^{ab}] + \frac{1}{4}[\phi(x), [\phi(x), [\phi(y), M^{ab}]]]\right)\right), \tag{5.30}$$

$$[\![T_0(x), T_0^{ab}(y)]\!] = \frac{3}{2}[T_0(x), M^{ab}] \delta(x-y), \tag{5.31}$$

$$[\![T_0(x), T_1^{ab}(y)]\!]$$
$$= \left(\frac{3}{2}[T_1, M^{ab}] - \frac{5}{4}[J_0, [\phi, M^{ab}]] + \frac{1}{4}[\phi, [J_0, [\phi, [\phi, M^{ab}]]]] + \frac{1}{4}[J_0, [\phi, [\phi, [\phi, M^{ab}]]]]\right) \delta(x-y)$$
$$+ \left(M^{ab} - \frac{5}{4}[\phi(x), [\phi(y), M^{ab}]] + \frac{1}{4}[\phi(x), [\phi(x), [\phi(y), [\phi(y), M^{ab}]]]]\right) \delta'(x-y)$$
$$= \delta(x-y) \frac{3}{2}[T_1, M^{ab}] + \partial_x \left(\delta(x-y) \left(M^{ab} - \frac{5}{4}[\phi(x), [\phi(y), M^{ab}]] + \frac{1}{4}[\phi(x), [\phi(x), [\phi(y), [\phi(y), M^{ab}]]]]\right)\right). \tag{5.32}$$

Finally, consider the genuinely nonlocal current N_μ. (Recall $N_0 = \frac{1}{2}[Z_1, \phi] + \frac{1}{2}[T_0, \phi] + [Z_0, \chi_Z]$ and $N_1 = \frac{1}{2}[Z_0, \phi] + \frac{1}{2}[T_1, \phi] + [Z_1, \chi_Z]$.) It suffices here to consider only the time component

$$[\![N_0(x), \phi^{ab}(y)]\!] = \left([\chi_Z, M^{ab}] - [\phi, [\phi, M^{ab}]] \right) \delta(x-y) - [Z_0(x), M^{ab}] \, \theta(x-y). \tag{5.33}$$

This is sufficient to infer the action on ϕ of the nonlocal charge $Q_N \equiv \int_{-\infty}^{+\infty} dx \; N_0(x)$:

$$[\![Q_N, \phi^{ab}(y)]\!] = -[[M^{ab}, \phi(y)], \phi(y)] + 2 \int_{-\infty}^{+\infty} dx \; \varepsilon(y-x)[Z_0(x), M^{ab}], \tag{5.34}$$

as claimed at the beginning of this section.

Actually, it is not too difficult to extend this result for the Poisson brackets of the local field ϕ with the first genuine nonlocal charge to the full set of nonlocal charges as contained in the path-ordered generating functional χ of the previous section. To that end, we define the path-ordered exponential for an interval $[x, y]$ (suppressing the κ dependence which is understood to be carried by C_1):

$$\chi[x,y] = P \exp\left(- \int_y^x dz \; C_1(z,t) \right), \tag{5.35}$$

and note the general relation for the variation of χ induced by any variation of the matrix C_1, such as that obtained from a Poisson bracket,

$$\delta\chi[x,y] = - \int_y^x dz \; \chi[x,z] \; \delta C_1(z) \chi[z,y]. \tag{5.36}$$

To determine the full nonlocal transformations of ϕ and π we therefore utilize their Poisson brackets with χ, or in view of the last relation, with C_1. These are given by

$$[\![C_0(x), \phi^{ab}(y)]\!] = \frac{\kappa}{1-\kappa^2} \; \delta(x-y) \; M^{ab}, \tag{5.37}$$

$$[\![C_1(x), \phi^{ab}(y)]\!] = \frac{\kappa^2}{1-\kappa^2} \; \delta(x-y) \; M^{ab}, \tag{5.38}$$

$$[\![C_0(x), \pi^{ab}(y)]\!] = \frac{-\kappa}{1-\kappa^2} \left\{ -\frac{2}{3}\delta(x-y) \; [J_0(x), M^{ab}] + \partial_x \left(\delta(x-y) \; \left(\kappa M^{ab} + \frac{1}{3}[\phi(x), M^{ab}] \right) \right) \right\}, \tag{5.39}$$

$$[\![C_1(x), \pi^{ab}(y)]\!] = \frac{-\kappa}{1-\kappa^2} \left\{ -\frac{2}{3}\kappa\delta(x-y) \; [J_0(x), M^{ab}] + \partial_x \left(\delta(x-y) \; \left(M^{ab} + \frac{1}{3}\kappa[\phi(x), M^{ab}] \right) \right) \right\}. \tag{5.40}$$

Thus, the full nonlocal transformations of the field and its conjugate momentum are given by

$$\delta_\kappa \phi^{ab}(x) \equiv \frac{1}{2} [\![\mathrm{Tr} \; (\Omega\chi[\infty, -\infty]), \phi^{ab}(x)]\!] = \frac{-\frac{1}{2}\kappa^2}{1-\kappa^2} \mathrm{Tr} \; (\Omega\chi[\infty, x] \; M^{ab} \; \chi[x, -\infty]), \tag{5.41}$$

$$\delta_\kappa \pi^{ab}(x) = \frac{-\frac{1}{2}\kappa^2}{1-\kappa^2} \mathrm{Tr} \left\{ \Omega\chi[\infty, x] \; \left(\left[\frac{2}{3} J_0(x) + \frac{1}{\kappa} C_1(x), M^{ab} \right] - \frac{1}{3}[C_1(x), [M^{ab}, \phi(x)]] \right) \; \chi[x, -\infty] \right\}. \tag{5.42}$$

Also note that

$$[\![C_0(x), [\partial_y \phi(y), \phi(y)]^{ab}]\!] = \frac{-\kappa}{1-\kappa^2} \; (\delta(x-y) \; [J_0(x), M^{ab}] + \partial_x \{ \delta(x-y) \; [\phi(x), M^{ab}] \}), \tag{5.43}$$

$$[\![C_1(x), [\partial_y \phi(y), \phi(y)]^{ab}]\!] = \kappa \; [\![C_0(x), [\partial_y \phi(y), \phi(y)]^{ab}]\!], \tag{5.44}$$

which, when substituted into $\delta\chi$, yields the nonlocal transformation of $[\partial_x \phi, \phi]$.

We may then combine all these results to obtain the full nonlocal transformation of the local currents. We find the simple result (note that $C_1 + \kappa J_0 = \kappa C_0$)

$$\delta_\kappa J_\mu^{ab}(x) = \frac{1}{2} [\![\mathrm{Tr} \; (\Omega\chi[\infty, -\infty]), J_\mu^{ab}(x)]\!] = \frac{-\frac{1}{2}\kappa^2}{1-\kappa^2} \mathrm{Tr} \; \{ \Omega\chi[\infty, x] \; [\tilde{C}_\mu(x), M^{ab}] \; \chi[x, -\infty] \}, \tag{5.45}$$

where all fields are evaluated at the same time.

VI. CONCLUDING REMARKS

The canonical machinery of the last section permits us to address one final point in conclusion. Davies *et al.* [17] have criticized the use of Noetherian methods to generate the nonlocal currents through nonlocal variations of fields, such as used in [15] for the Gross-Neveu model. They argue that the proper demonstration that the nonlocals constitute meaningful conservation laws, even for the classical theory, must consist in showing through canonical methods that they are constants of the motion. That is, for the classical theory, the nonlocal charges must have vanishing Poisson brackets with the Hamiltonian, while for the quantum theory, they must have vanishing commutators.

At least for the classical theory, the result (5.45) leads to this result, and more. It immediately shows that the Hamiltonian *density* is invariant under the full nonlocal symmetry:

$$\delta_\kappa \mathcal{H} = 0, \tag{6.1}$$

where

$$\mathcal{H} = \tfrac{1}{2}\, \mathrm{Tr}\, \Big(J_0 J_0 + J_1 J_1\Big), \tag{6.2}$$

since

$$[J_0, C_1] = -[J_1, C_0] = \frac{-\kappa^2}{1-\kappa^2}[J_0, J_1]. \tag{6.3}$$

Furthermore, (5.45) also shows that the momentum density is invariant under the nonlocal transformations:

$$\delta_\kappa \mathcal{P} = 0, \tag{6.4}$$

where

$$\mathcal{P} = \mathrm{Tr}\, \Big(J_0 J_1\Big), \tag{6.5}$$

since

$$[J_0, C_0] = -[J_1, C_1] = \frac{-\kappa}{1-\kappa^2}[J_0, J_1]. \tag{6.6}$$

Moreover, because the energy-momentum tensor is symmetric and traceless for the classical theory, these two results lead to the conclusion that all components of the classical local energy-momentum tensor are invariant under the nonlocal symmetry:

$$\delta_\kappa \theta_{\mu\nu} = 0, \tag{6.7}$$

$$\theta_{\mu\nu} = \mathrm{Tr}\Big(J_\mu J_\nu - \tfrac{1}{2} g_{\mu\nu} J_\lambda J^\lambda\Big). \tag{6.8}$$

Thus we have established within the canonical framework the conservation of all the classical nonlocal charges as contained in the master charge generating functional.

A next logical step would be to consider the status of this classical result in the context of the quantum theory. This step will be taken in a subsequent investigation. Suffice it to say here that quantum corrections are indeed expected in the transformation of the local energy-momentum tensor. In particular, since the trace of that tensor, θ^μ_μ, probably does not vanish in quantum theory, as suggested by the nonvanishing one-loop renormalization flow, the nonlocal symmetry will probably not leave the untraced local energy-momentum tensor invariant in the quantum theory [18]. However, since the PCM is *not* asymptotically free, a reliable short-distance expansion is not within the grasp of direct perturbative methods for this model. Therefore, a convincing analysis of nonlocal transformations for the pseudodual quantum theory may take quite some time.

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