Improving the effective potential, multimass problem, and modified mass-dependent scheme

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We present a new procedure for improving the effective potential by using the renormalization group equation (RGE) in the presence of several mass scales. We propose a modification of the mass-dependent (MD) renormalization scheme ($\overline{\text{MD}}$ scheme) so that the scalar mass parameter runs at most logarithmically on the one hand and the decoupling of heavy particles is naturally incorporated in the RGE's on the other. Thanks to these properties, the procedure in the $\overline{\text{MD}}$ scheme turns out to be very simple compared with the regionwise procedure in the modified minimal subtraction scheme proposed previously. The relation with other schemes is also discussed both analytically and numerically.

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I. INTRODUCTION

Recently, there has been renewed interest on how to sum up large logarithms in the effective potential to investigate the standard model and beyond. Basically, large logarithms such as $\ln(M/\mu)$, which makes the perturbation expansion unreliable, appear when one deals with a system possessing a large mass scale M compared with the scale μ at which one discusses the physics. In this situation one considers resumming the perturbation series by using the renormalization group equation [1]. When one is concerned with the functional form of the effective potential, one considers its renormalization-group (RG) improvement. This has been well known since the work by Coleman and Weinberg [2] for massless $\lambda\phi^4$ theory, although the complete description even for massive $\lambda\phi^4$ theory has been given only recently [3–5].

In many realistic applications, one often has to deal with an additional mass scale m with the hierarchy $\mu \ll m \ll M$. In the supersymmetric standard model, for instance, one can regard μ, m, and M as the weak, supersymmetry-breaking, and unification scales, respectively. When we discuss such a system, we face the problem of multimass scales [6]: There appear several types of logarithms, $\ln(M/\mu)$ and $\ln(m/\mu)$, while we are able to sum up just a single logarithm by using the RG equation (RGE).

In Ref. [7] one way to improve the effective potential in the presence of multimass scales was described in the modified minimal subtraction ($\overline{\text{MS}}$) renormalization scheme. The point was to make use of the decoupling theorem [8–12] and to divide the energy region (region of field space) so that in each region there remains essentially a single logarithmic factor. Although there is nothing wrong in principle, such a regionwise procedure may be cumbersome in practice. So it is desirable to have an alternative way to handle multimass-scale systems.

In this paper we propose a simple modification of the conventional mass-dependent (MD) renormalization scheme, which we call the *modified MD scheme* ($\overline{\text{MD}}$ scheme), and apply it to improving the effective potential in the presence of several mass scales. Basically, in the MD scheme, the RG coefficient functions (β and γ functions) depend on mass parameters and hence the decoupling of heavy particles is taken into account in the form of RG runnings [13]. In addition, the proposed $\overline{\text{MD}}$ scheme has a property that mass parameters run at most logarithmically while keeping the "automatic" decoupling in the RGE's; namely, it enjoys simultaneously that (i) the quadratic running of the scalar mass parameters is absent and (ii) the decoupling effects of heavy particles are naturally built in. Based on these properties (i) and (ii), we show, by adopting a simple model with two mass scales, that the *same condition as in the single-mass-scale case* [4] *is enough to achieve the RG improvement of the effective potential over the whole region of field space.*

We should remark that property (i) is crucial to prove the above statement. Generally, in the MD scheme, there appear nonlogarithmic and powerlike corections proportional to μ^2, which are potentially large in the high-energy region. Such nonlogarithmic corrections cause trouble in summing up the leading logarithms. We modify the renormalization scheme in order to cure this point.

The existence of nonlogarithmic corrections is related to the scheme dependence of the RG-improved potential. (To examine this point is another motivation of the present work.) Note that it is not trivial at all that the RG-improved potentials in the $\overline{\text{MS}}$ and $\overline{\text{MD}}$ schemes coincide with each other. Of course, the *full* effective potential is independent of the renormalization scheme: The effective potentials in various schemes are related to each other simply by changes of variables. The effective potential correctly calculated up to a certain loop order is also scheme independent since the loop expansion has a scheme-independent parameter, the Planck constant $\hbar$. In general, however, once one makes an approximation to the full theory, it is quite possible that the results are different scheme by scheme; some schemes give better approximations than the others.

In our case we approximate the full effective potential by resumming "logarithmic" parts of the perturbation series so that it satisfies the RGE. Then the scheme independence becomes nontrivial: The RGE relates "loga-

rithmic factors" at different loop orders, but the "logarithmic" structure will differ scheme by scheme. Moreover, there may appear nonlogarithmic corrections as mentioned above. This is why there is no *a priori* relation between the RG-improved potentials in various schemes. Do they give the same approximations? This is the problem of the scheme dependence. Our result will support to some extent the naive expectation that it *is* scheme independent.

This paper is organized as follows. In the next section, we briefly review the basic ingredients for improving the effective potential by the RGE. In Secs. III and IV, we define the $\overline{\text{MD}}$ renormalization scheme and discuss its basic features. We show that the RGE's in the $\overline{\text{MD}}$ scheme inherit the property (i) as in the mass-independent (MI) scheme [14] as well as property (ii) as in the MD one. The absence of the quadratic running is proved directly from the renormalization conditions, and the automatic decoupling is established by utilizing the decoupling theorem. A detailed study of the structure of the effective potential in the $\overline{\text{MD}}$ scheme is given in Sec. V. We first define the leading logarithmic series expansion in the $\overline{\text{MD}}$ scheme and describe how to sum up the leading logarithm. It will be shown, by examining the high- and low-energy regions separately, that we can correctly sum up all logarithmic factors over the whole region and that nonlogarithmic corrections are in fact small. After establishing the procedure in the $\overline{\text{MD}}$ scheme, we compare the leading logarithmic potential in the $\overline{\text{MD}}$ scheme with those in other schemes such as the $\overline{\text{MS}}$ and MD ones in Sec. VI. By numerically solving the RGE's, a good coincidence will be found. The final section is devoted to conclusions and further comments. Some one-loop results can be found in the Appendixes.

II. IMPROVING THE EFFECTIVE POTENTIAL IN THE $\overline{\text{MS}}$ SCHEME AND PROBLEMS OF MULTIMASS-SCALES

In order to explain the basic ingredients needed later, let us first make a review of the procedure [4,7] for improving the effective potential by using the RGE in the $\overline{\text{MS}}$ scheme. We also describe why the problem of multimass scales arises in the context of RG improvement of the effective potential.

Following Ref. [7], let us consider the Yukawa model

$$\mathcal{L}=\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4+\bar{\psi}(i\partial\!\!\!/-g\phi)\psi-\omega\ , \tag{2.1}$$

where ϕ is a massive real scalar field and $\psi=(\psi_1,\ldots,\psi_N)^T$ are massless Dirac fields. We take the Dirac field to be N component in order to indicate which correction comes from the fermion loop. For simplicity, we impose "chiral-parity" invariance, $\phi\to-\phi$, $\psi\to\gamma_5\psi$, to forbid the bare mass of fermion. The last term ω ($=hm^4$ in the notation in Ref. [4]) is a vacuum-energy term, which is usually omitted, but plays an important role [4] in the $\overline{\text{MS}}$ scheme. In this paper we assume that both coupling constants $g^2/(16\pi^2)$ and $\lambda/(16\pi^2)$ are small and of the same order.

In order to compute the effective potential $V(\varphi)$ for the scalar field vacuum expectation value (VEV), $\varphi=\langle\phi\rangle$, we make a field shift $\phi\to\phi+\varphi$ in (2.1) and obtain

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}(\partial\phi)^2-\frac{1}{2}M_B^2\phi^2-\frac{1}{3!}\lambda\varphi\phi^3-\frac{1}{4!}\lambda\phi^4\\&+\bar{\psi}(i\partial\!\!\!/-M_F-g\phi)\psi-V^{(0)}(\varphi)\\&+[\phi-\text{linear terms}]\ ,\end{aligned} \tag{2.2}$$

where the last term is the tree potential $V^{(0)}(\varphi)\equiv\omega+m^2\varphi^2/2+\lambda\varphi^4/4!$ and we have introduced the masses for the boson ϕ and fermion ψ, respectively,

$$M_B^2\equiv m^2+\frac{\lambda}{2}\varphi^2\ ,\quad M_F\equiv g\varphi\ , \tag{2.3}$$

in the presence of the scalar background φ. When φ is small, the field ϕ may be regarded as a heavy field and ψ as a light field.

The starting point is that the effective potential is independent of the renormalization point μ and thus satisfies the RGE

$$0=\mu\frac{d}{d\mu}V=\mathcal{D}V(\varphi,\lambda,g^2,m^2,\omega;\mu)\ , \tag{2.4}$$

$$\begin{aligned}\mathcal{D}\equiv&\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_g\frac{\partial}{\partial g^2}-\gamma_m\frac{\partial}{\partial\ln m^2}\\&-\gamma_\phi\frac{\partial}{\partial\ln\varphi}-\gamma_\omega m^4\frac{\partial}{\partial\omega}\ ,\end{aligned} \tag{2.5}$$

supplemented with the RGE's for parameters such as

$$\mu\frac{d}{d\mu}g^2=\beta_g\ ,\quad \mu\frac{d}{d\mu}m^2=-\gamma_m m^2\ ,\quad \mu\frac{d}{d\mu}\omega=-\gamma_\omega m^4\ . \tag{2.6}$$

One can immediately write down the general solution to (2.4) as

$$\begin{aligned}&V(\varphi,\lambda,g^2,m^2,\omega;\mu)\\&\quad=V(\bar{\varphi}(t),\bar{\lambda}(t),\bar{g}^2(t),\bar{m}^2(t),\bar{\omega}(t);e^t\mu)\ ,\end{aligned} \tag{2.7}$$

where the barred quantities $\bar{\varphi}(t)$, etc., denote the solutions of running equations with a running distance t from the initial values φ, etc., at the renormalization point μ. (Here we are regarding the RGE's as a differential equation with respect to an independent "time" t, not μ.)

The RG improvement of the effective potential consists in solving the RGE (2.4). The RGE (2.4) by itself, however, does not determine the RG-improved effective potential since it is a first-order *homogeneous* differential equation. We should impose the suitable boundary condition on the functional form of V at a certain time t. We call the boundary "value" of the potential the *boundary function*. The RG-improved potential is fixed by requiring that the right-hand side (RHS) of Eq. (2.7) coincide at a certain time t with the boundary function. [The RGE (2.4) guarantees that we can make a convenient choice of t.] It is the choice of boundary functions that determines how well the obtained potential approximates the exact one.

How can we find a suitable boundary function?

Let us work in the $\overline{\text{MS}}$ scheme for a moment and examine the detailed structure of loop corrections to the effective potential. [The following arguments are valid in any mass-independent (MI) schemes since the structure of loop corrections does not change.] The simplest way [4] to see this is to rewrite our Lagrangian (2.2) by rescaling the fields by a factor g as $\Phi=g\phi$ and $\Psi=g\psi$ into

$$\mathcal{L}=\frac{1}{g^2}\left[\frac{1}{2}(\partial\Phi)^2-\frac{1}{2}M_B^2\Phi^2+\bar{\Psi}(i\partial\!\!\!/-M_F)\Psi -\frac{1}{3!}\left[\frac{\lambda}{g^2}M_F\right]\Phi^3-\frac{1}{4!}\left[\frac{\lambda}{g^2}\right]\Phi^4-\bar{\Psi}\Phi\Psi\right] -V^{(0)}(\varphi) \tag{2.8}$$

and to regard Φ and Ψ as our basic quantum fields. In this form the parameter g^2 is an overall factor in front of the action just like the Planck constant $\hbar$. So the L-loop contribution ($L\geq 1$) to the effective potential clearly takes the form

$$V^{(L)}=g^2\varphi^4\left[\frac{g^2}{16\pi^2}\right]^L \times\left[\text{function in } \ln\frac{M_F^2}{\mu^2},\ln\frac{M_B^2}{\mu^2},\frac{M_F^2}{\mu^2},\frac{\lambda}{g^2}\right]. \tag{2.9}$$

In the $\overline{\text{MS}}$ scheme, we have two types of logarithms $\ln(M_F^2/\mu^2)$ and $\ln(M_B^2/\mu^2)$ in our two-mass-scale system, both of which can become large. Since we know the logarithms appear at most to the Lth power at the L-loop level, we can rewrite Eq. (2.9) as

$$V^{(L)}=\frac{M_F^4}{g^2}\sum_{i,j\geq 0}^{i+j\leq L}\left[\frac{g^2}{16\pi^2}\right]^{L-(i+j)}v_{i,j}^{(L)}(x,y)s_F^i s_B^j \tag{2.10}$$

by introducing the variables

$$s_F\equiv\frac{g^2}{16\pi^2}\ln\frac{M_F^2}{\mu^2},\quad s_B\equiv\frac{g^2}{16\pi^2}\ln\frac{M_B^2}{\mu^2}, \tag{2.11}$$

$$x\equiv\frac{M_F^2}{M_B^2},\quad y\equiv\frac{\lambda}{g^2}. \tag{2.12}$$

Although we are assuming that the coupling constant $g^2/(16\pi^2)$ is small, we should regard the "Kastening variables" s_F and s_B as of order 1 since the logarithms may be large. The other variables x,y and hence the coefficient functions $v_{i,j}^{(L)}(x,y)$ are also of order 1. Then we sum up $V^{(L)}$ with respect to L and further rewrite it as a summation over $l\equiv L-(i+j)$,

$$V=\sum_{L=0}^{\infty}V^{(L)}=\omega+\frac{M_F^4}{g^2}\sum_{l=0}^{\infty}\left[\frac{g^2}{16\pi^2}\right]^l f_l, \tag{2.13}$$

$$f_l(s_F,s_B;x,y)\equiv\sum_{i,j\geq 0}^{\infty}v_{i,j}^{(l+i+j)}(x,y)s_F^i s_B^j, \tag{2.14}$$

where we have included the tree part $f_0=x^{-1}/2-5y/24$ into the summation. This form of the expansion of the effective potential, first introduced by Kastening [3] for the single-mass-scale case, is called the *leading logarithmic series expansion.* When expressed in terms of the variables (2.11) and (2.12), it is the power series expansion in the small coupling constant $g^2/(16\pi^2)$. The coefficients $f_0,f_1,\ldots,f_l,\ldots$ corresponds to the leading, next-to-leading, . . . , lth-to-leading, . . . , logarithmic terms, respectively. Of course, it does not matter whether one uses λ instead of g^2 as the expansion parameter.

Now let us return to the question of how to specify the boundary function. The summation in (2.14) for the lth-to-leading logarithmic term f_l involves the quantities at the $(L=l,l+1,\ldots)$-loop level. If one could set $s_F=s_B=0$ in (2.14), then only the first term with $i=j=0$ would survive and the summation would terminate at finite-loop order, $L=l$:

$$f_l(0,0;x,y)=v_{0,0}^{(L=l)}(x,y). \tag{2.15}$$

Since $v_{0,0}^{(L)}$ can be obtained by computing the L loops, this would imply that one could use the l-loop potential $V_l=V^{(0)}+\cdots+V^{(l)}$ evaluated at $s_F=s_B=0$ as the boundary function for the lth-to-leading logarithmic potential. In other words, if one could find a time t_0 such that

$$\bar{s}_F(t_0)=\bar{s}_B(t_0)=0, \tag{2.16}$$

then the desired lth-to-leading logarithmic potential would be given[1] by requiring that the RHS of Eq. (2.7) be the boundary function $V_{L=l}|_{s_F=s_B=0}$:

$$V(\varphi,\lambda,g^2,m^2,\omega;\mu)=V_{L=l}(\bar{\varphi}(t),\bar{\lambda}(t),\bar{g}^2(t),\bar{m}^2(t),\bar{\omega}(t);e^t\mu)\Big|_{\substack{\bar{s}_F(t)=0\\ \bar{s}_B(t)=0}}+\frac{\bar{M}_F^4(t_0)}{\bar{g}^2(t_0)}\times O\left(\left[\frac{\bar{g}^2(t_0)}{16\pi^2}\right]^{l+1}\right). \tag{2.17}$$

Actually, the condition (2.16) is sufficient, but not a necessary one. For our purpose it would be enough to find a time t_0 at which the logarithmic factors $\bar{s}_F(t_p)$ and $\bar{s}_B(t_0)$ are of $O(\bar{g}^2/16\pi^2)$, instead of zero:

$$\bar{s}_F(t_0)\simeq\bar{s}_B(t_0)=O\left(\frac{\bar{g}^2(t_0)}{16\pi^2}\right). \tag{2.18}$$

As can be seen from Eq. (2.10), under this condition, these logarithmic factors contained at the L-loop level reduce to precisely Lth-to-leading logarithmic order

[1] As was *proved* in Ref. [4], one should use the RGE's at $(l+1)$-loop order. Note also that, strictly speaking, the error in this equation is $O((\bar{g}^2/16\pi^2)^{l+1})$, not $O((g^2/16\pi^2)^{l+1})$, but the difference will be small unless some coupling blows up (where our approximation itself does not make sense).

quantities. So to obtain the boundary function for an lth-to-leading logarithmic potential, it would be necessary and sufficient [7] to retain these logarithmic factors up to l loops, just as in Eq. (2.17).

Unfortunately, such a condition (2.16), or an even weaker one (2.18), cannot always be satisfied simultaneously since the difference $s_B - s_F$ becomes of order 1 when $g^2\varphi^2 \ll m^2$. The RGE (2.4) enables us to set just a single variable to the desired value, but not several variables. So one cannot find a solution to (2.16) or (2.18) and is left with the infinite summation in (2.14). This is the problem of multimass scales in the context of the RG improvement of the effective potential.

Now we examine whether the MD scheme provides us with a solution to this problem. Even in the MD scheme, one will have a similar structure of the leading logarithmic series expansion as in Eq. (2.13). However, there explicitly appears the renormalization point μ^2 in the effective potential, other than $\ln\mu$. Such explicit μ dependence is closely related to the existence of the quadratic running of the scalar mass and makes perturbation theory unreliable. In particular, in the context of the leading logarithmic series expansion, it may make the coefficient functions $v^{(L)}$ arbitrarily large. In fact, as we shall show in the following sections, we remedy this point by modifying the renormalization conditions in the MD scheme.

III. MODIFIED MASS-DEPENDENT SCHEME

In this section we give a definition of the modified MD scheme (MD scheme). For the theory (2.1), we define the MD renormalization scheme by the following renormalization conditions. For the scalar two-point vertex $\Gamma_\phi^{(2)}$, we impose

$$\Gamma_\phi^{(2)}\Big|_{\substack{p^2=0\\ m^2=0}} \equiv \lim_{m^2\to 0}(\Gamma_\phi^{(2)}|_{p^2=0})=0 , \tag{3.1}$$

$$\frac{\partial}{\partial m^2}\Gamma_\phi^{(2)}|_{p^2=-\mu^2}=-1 , \tag{3.2}$$

$$\frac{\partial}{\partial p^2}\Gamma_\phi^{(2)}|_{p^2=-\mu^2}=1 . \tag{3.3}$$

The fermion two-point vertex takes the form $\Gamma_\psi^{(2)}=A(p^2)\not{p}$ due to the chiral-parity symmetry, for which we require

$$A|_{p^2=-\mu^2}=1 . \tag{3.4}$$

Finally, for the Yukawa vertex $\Gamma_g^{(3)}$ and the scalar four-point vertex $\Gamma_\phi^{(4)}$, we impose

$$\begin{aligned}&\Gamma_g^{(3)}(p,-p;0)|_{p^2=-\mu^2}=-g ,\\ &\Gamma_\phi^{(4)}|_{p_ip_j=-\mu^2\delta_{ij}+\mu^2(1-\delta_{ij})/3}=-\lambda ,\end{aligned} \tag{3.5}$$

where we see the boson external momentum equal to zero in $\Gamma_g^{(3)}$.

To be precise these renormalization conditions (3.1)–(3.5) should be supplemented with that for the zero-point vertex $\Gamma^{(0)}$:

$$\begin{aligned}&\Gamma^{(0)}|_{m^2=0}=\frac{\partial}{\partial m^2}\Gamma^{(0)}|_{m^2=0}=0 ,\\ &\frac{1}{2}\left[\frac{\partial}{\partial m^2}\right]^2\Gamma^{(0)}=-h .\end{aligned} \tag{3.6}$$

Clearly, the vacuum-energy term $\omega=hm^4$, which played an important role in the MS scheme, is completely independent of the renormalization point and is irrelevant for later discussions. (Instead, one can simply impose $\Gamma^{(0)}=-\omega$.)

The new set of the renormalization conditions (3.1)–(3.5) is a modified version of the MD ones. We have modified the conditions on the scalar two-point vertex, Eqs. (3.1) and (3.2), which take the place of a single condition

$$\Gamma_\phi^{(2)}|_{p^2=-\mu^2}=-\mu^2-m^2 \tag{3.7}$$

in the conventional MD scheme. With this modification the MD scheme enjoys the properties announced in the Introduction; to be precise, (i) in the high-energy region $\mu^2 \gg m^2$, the RGE's in the MD scheme approach those in a certain mass-independent scheme—in particular, the mass parameter runs at most logarithmically, and (ii) in the low-energy region $\mu^2 \ll m^2$, decoupling effects are automatically taken into account in the RGE's and the vertex functions. As we shall see, it is crucial to separate the condition (3.7) into Eqs. (3.1) and (3.2) in order to realize property (i), which will play an important role in Sec. V.

Before showing properties (i) and (ii), let us take a close look at the renormalization conditions (3.1)–(3.5). First, they differ from those in the MI scheme [14]. In the MI scheme, one treats a one parameter family of theories with different values of mass and renormalizes them at a certain value such as $m^2=0$, $m^2=\mu^2$; one imposes, for instance for $\Gamma_\phi^{(2)}$, the condition (3.1) and

$$\frac{\partial}{\partial m^2}\Gamma_\phi^{(2)}\Big|_{\substack{p^2=-\mu^2\\ m^2=0}}=-1 , \quad \frac{\partial}{\partial p^2}\Gamma_\phi^{(2)}\Big|_{\substack{p^2=-\mu^2\\ m^2=0}}=1 . \tag{3.8}$$

Clearly all the renormalization constants are independent of the renormalized mass parameter m^2. On the other hand, in our MD scheme, we are still treating, in a sense, a one parameter family of theories with different values of mass in order to impose the conditions (3.1) and (3.2). With the renormalization conditions (3.2)–(3.5), however, the renormalization constants Z_X for $X=\phi$, ψ, m, g, λ generally depend on the ratio μ^2/m^2:

$$Z_X=Z_X\left[\lambda,g^2,\ln\frac{m^2}{\mu^2};\frac{m^2}{\mu^2}\right] . \tag{3.9}$$

The RG coefficient functions (β and γ functions), which are calculated from the Z's, also depend on the mass parameter.

Second, the renormalization constants Z_X are consistently determined in the MD scheme. A complication occurs only in the scalar two-point vertex, while other vertices can be treated in the same manner as in the MD scheme. Let us write the scalar two-point vertex as

$$\Gamma_{\phi}^{(2)}(p,-p;m^2)=Z_{\phi}p^2-Z_m m^2+\Pi(p^2;m^2)\ . \quad (3.10)$$

As usual, the wave-function factor Z_{ϕ} is determined by the condition (3.3):

$$Z_{\phi}=1-\frac{\partial}{\partial p^2}\Pi|_{p^2=-\mu^2}\ . \quad (3.11)$$

As for Z_m, the renormalization conditions (3.1) and (3.2) yield, respectively,

$$\lim_{m^2\to 0} m^2 Z_m=\Pi|_{p^2=m^2=0}\ , \quad (3.12)$$

$$\left[1+m^2\frac{\partial}{\partial m^2}\right]Z_m = 1+\left[\left[1-p^2\frac{\partial}{\partial p^2}\right]\left[\frac{\partial}{\partial m^2}\Pi\right]\right]\Bigg|_{p^2=-\mu^2}\ , \quad (3.13)$$

where we have used Eq. (3.11) in deriving Eq. (3.13). Observe that Eq. (3.13), being a differential equation, does not completely determine Z_m. This is most evident by noting that a piece $Z_m\sim\mu^2/m^2$ drops from the LHS of Eq. (3.13). What determines this piece is precisely Eq. (3.12). Thus we see that the renormalization condition (3.1) provides the condition (3.2) with a boundary condition and that all the renormalization constants are uniquely determined in the $\overline{\text{MD}}$ scheme.

Now, let us look at one-loop examples and confirm that properties (i) and (ii) actually hold. (See the Appendixes for more details.) The β and γ functions are given by

$$\begin{aligned}
&16\pi^2\gamma_{\phi}=2Ng^2\ ,\quad 16\pi^2\gamma_{\psi}=\frac{g^2}{2}K_{\psi}\left[\frac{\mu^2}{m^2}\right]\\
&16\pi^2\gamma_m=-4Ng^2\ ,\quad 16\pi^2\beta_g=6g^4K_g\left[\frac{\mu^2}{m^2}\right]+4Ng^4\ ,\\
&16\pi^2\gamma_{\omega}=0\ ,\quad 16\pi^2\beta_{\lambda}=3\lambda^2K_{\lambda}\left[\frac{\mu^2}{m^2}\right]+8Ng^2(\lambda-6g^2)\ .
\end{aligned} \quad (3.14)$$

The functions $K_X(z)$ $(X=\psi,g,\lambda)$ are defined by

$$\begin{aligned}
K_{\psi}(z)&=1+\frac{2}{z}-\frac{2}{z}\left[1+\frac{1}{z}\right]\ln(z+1)\ ,\\
K_g(z)&=1+\frac{2}{3z}-\frac{4}{3z}\left[1+\frac{1}{2z}\right]\ln(z+1)\ ,\\
K_{\lambda}(z)&=1-\frac{3}{2z}\frac{1}{\sqrt{1+3/z}}\ln\frac{\sqrt{1+3/z}+1}{\sqrt{1+3/z}-1}\ ,
\end{aligned} \quad (3.15)$$

which are normalized to be 1 in the high-energy limit $z(=\mu^2/m^2)\to\infty$ and, remarkably, vanish in the low-energy limit $z\to 0$ (see Fig. 1):

$$K_X(z)\to\begin{cases}1 & \text{as } z\to\infty\ ,\\ 0 & \text{as } z\to 0\ .\end{cases} \quad (3.16)$$

Recall that the terms proportional to N come solely from the light-particle (fermion) loops. Others come from the heavy-particle (boson) loops. The latter terms are accompanied by the functions K_X, which have the property

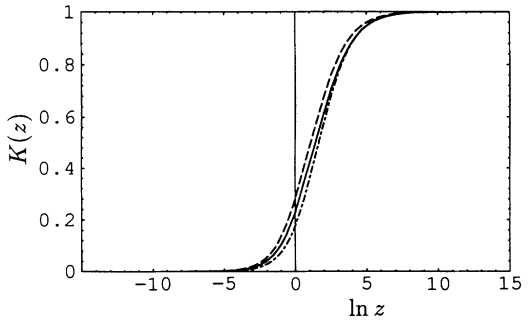

FIG. 1. "Interpolating" functions K_{ψ} (solid line), K_g (dashed line), and K_{λ} (dot-dashed line).

(3.16). This is nothing but the decoupling of heavy-particle loops, as claimed in (ii).

In the MI (or $\overline{\text{MS}}$) scheme, the RGE's do not have such a property of the automatic decoupling. Instead, one has to switch from the full theory to the low-energy effective theory. In the MI scheme,[2] the RGE's for the full theory are

$$\begin{aligned}
&16\pi^2\gamma_{\phi}=2Ng^2\ ,\quad 16\pi^2\gamma_{\psi}=\frac{g^2}{2}\ ,\\
&16\pi^2\gamma_m=-4Ng^2\ ,\quad 16\pi^2\beta_g=6g^4+4Ng^4\ ,\\
&16\pi^2\gamma_{\omega}=-\tfrac{1}{2}\ ,\quad 16\pi^2\beta_{\lambda}=3\lambda^2+8Ng^2(\lambda-6g^2)\ .
\end{aligned} \quad (3.17)$$

In the low-energy effective theory, we keep only the terms proportional to N in Eqs. (3.17) and have the RGE's

$$\begin{aligned}
&16\pi^2\gamma_{\phi}=2Ng^2\ ,\quad 16\pi^2\gamma_{\psi}=0\ ,\\
&16\pi^2\gamma_m=-4Ng^2\ ,\quad 16\pi^2\beta_g=4Ng^4\ ,\\
&16\pi^2\gamma_{\omega}=0\ ,\quad 16\pi^2\beta_{\lambda}=8Ng^2(\lambda-6g^2)\ .
\end{aligned} \quad (3.18)$$

By comparing Eqs. (3.14) with Eqs. (3.17) and (3.18), one clearly sees that the RGE's in the $\overline{\text{MD}}$ scheme interpolate those in the MI scheme for the high- and low-energy regions.

At one-loop order, the γ_m in the $\overline{\text{MD}}$ scheme is the same as in the MI scheme:

$$\mu\frac{d}{d\mu}m^2=\frac{4Ng^2}{16\pi^2}m^2\ , \quad (3.19)$$

which means that the mass parameter in the $\overline{\text{MD}}$ scheme runs logarithmically, as claimed in (i). (See Fig. 2.) This is the result of our modification of the renormalization conditions. This is in sharp contrast to the case of the conventional MD scheme. Indeed, with the MD renormalization condition (3.7), a fermion one-loop contribution to $\Gamma_{\phi}^{(2)}$ produces [15,16] a piece proportional to

[2]Here we adopt the renormalization condition as in Eq. (3.8). If we renormalize $\Gamma_{\phi}^{(2)}$ at $m^2=\mu^2$, instead of $m^2=0$, then the γ_m in such a MI scheme coincides with that in the $\overline{\text{MS}}$ scheme: $16\pi^2\gamma_m=-4Ng^2-\lambda$. But the difference is not so important here.

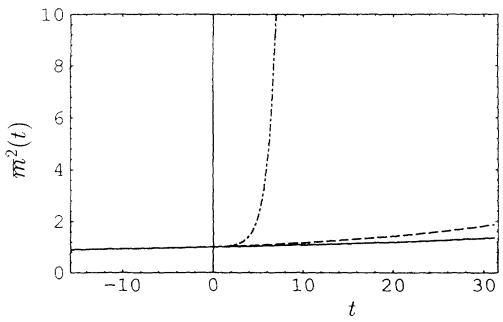

FIG. 2. Typical behavior of $\bar{m}^2$ in the $\overline{\text{MD}}$ (solid line), $\overline{\text{MS}}$ (dashed line), and MD (dot-dashed line) schemes. All schemes as well as the full and effective theories in the $\overline{\text{MS}}$ scheme are matched at $\ln(\mu/m)=0$.

μ^2/m^2 in γ_m:

$$\mu\frac{d}{d\mu}m^2=\frac{4Ng^2}{16\pi^2}(m^2+\mu^2)\ . \tag{3.20}$$

One sees that the running of the mass parameter is completely different from that in Eq. (3.17) since the second term on the RHS of Eq. (3.20) dominates in the high-energy region [while it approaches that in Eq. (3.18) in the low-energy region]. If such quadratic running is present, the RGE's never interpolate the MI ones in the high- and low-energy regions.

IV. LOGARITHMIC RUNNING AND AUTOMATIC DECOUPLING IN THE RGE

In the last section, we illustrated properties (i) and (ii) of the $\overline{\text{MD}}$ scheme by one-loop examples. We now present the general argument to show that these properties hold to any loop order.

Basically, property (i) follows from the fact that we introduce the μ dependence only through the *dimensionless* combinations of vertex functions $\Gamma^{(n)}$ $(n\neq 0)$, such as $(\partial/\partial m^2)\Gamma^{(2)}$. In other words, we never introduce the μ dependence in the renormalization condition (3.1) on $\Gamma_\phi^{(2)}$ which has dimension 2. Since the dependence on μ is introduced only through quantities which are at most logarithmically divergent, we do not meet the quadratic dependence on μ.

This property (i) can be confirmed directly as follows. First, condition (3.3) determines the momentum dependence of the two-point vertex to be

$$\Gamma_\phi^{(2)}=p^2-m^2-\mu^2 c\left[\frac{m^2}{\mu^2}\right]+\frac{(p^2+\mu^2)^2}{\mu^2}f\left[\frac{-p^2}{\mu^2},\frac{m^2}{\mu^2}\right]. \tag{4.1}$$

Condition (3.2) implies that the unknown function c is independent of m^2, and condition (3.1) determines it to be $f(0,0)$. Note that the function f should be nonsingular in the limit $m^2\to 0$ since $\Gamma_\phi^{(2)}$ has a massless limit. Then the renormalized scalar two-point vertex in the $\overline{\text{MD}}$ scheme takes the form

$$\Gamma_\phi=p^2-m^2+\mu^2\left[\left[\frac{p^2+\mu^2}{\mu^2}\right]^2 f\left[\frac{-p^2}{\mu^2},\frac{m^2}{\mu^2}\right]-f(0,0)\right]. \tag{4.2}$$

Now we use the RGE for $\Gamma_\phi^{(2)}$:

$$0=\left[\hat{\mathcal{D}}-2\gamma_\phi-\gamma_m m^2\frac{\partial}{\partial m^2}\right]\Gamma_\phi^{(2)}\ , \tag{4.3}$$

$$\hat{\mathcal{D}}\equiv\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_g\frac{\partial}{\partial g^2}\ . \tag{4.4}$$

Inserting the general form (4.2) into Eq. (4.3) and taking a limit $m^2\to 0$ after setting $p^2=0$, we obtain that

$$0=\lim_{m^2\to 0} m^2\gamma_m\left[1-\mu^2\frac{\partial}{\partial m^2}f\left[0,\frac{m^2}{\mu^2}\right]\right]. \tag{4.5}$$

Since the quantity in the square brackets does not vanish (at least perturbatively), Eq. (4.5) implies that

$$0=\lim_{m^2\to 0} m^2\gamma_m\ . \tag{4.6}$$

Since the quadratic running of the mass parameter corresponds to the behavior $\gamma_m\sim\mu^2/m^2$, Eq. (4.6) proves the absence of the quadratic running in the $\overline{\text{MD}}$ scheme.

Next, we turn to property (ii). Let us examine the relation between the full theory in the low-energy region and the low-energy effective theory. Here the low-energy effective theory is obtained from the full theory by regarding heavy fields as external fields (instead of quantum fields), i.e., by taking out heavy-field internal lines. What we want to prove is that, in the $\overline{\text{MD}}$ scheme, the full theory in the low-energy region will automatically go over into the low-energy effective theory.

In order to find such a relation, we make use of the decoupling theorem [8–10]: The contributions due to heavy particles, aside from those which are suppressed by the inverse power of the heavy mass, can be renormalized into the parameters of the low-energy effective theory. Let $\Gamma^{(n)}$ be the n-point vertex in the full theory and $\tilde{\Gamma}^{(n)}$ the corresponding vertex in the low-energy effective theory. (We denote the quantities in the low-energy effective theory by the tilde.) Then, according to the decoupling theorem, when all the external momenta p_i as well as the renormalization point μ are small compared with the mass m; i.e., for $|p_ip_j|,\mu^2\ll m^2$,

$$\Gamma^{(n)}(p_i,g,\lambda,m^2;\mu)=Z_\phi^{b/2}Z_\psi^f\tilde{\Gamma}^{(n)}(p_i,\tilde{g},\tilde{\lambda},\tilde{m}^2;\mu)+O\left[\frac{p_ip_j}{m^2},\frac{\mu^2}{m^2}\right], \tag{4.7}$$

where b and $2f$ stands for the number of external bosons and fermions, respectively: $n=b+2f$. $\Gamma^{(n)}$ does not depend on $\tilde{m}^2$ except for $n=2$.

Originally, the low-energy effective theory is not completely fixed by specifying the Lagrangian itself. So we fix it by imposing the *same* $\overline{\text{MD}}$ renormalization conditions as (3.1)–(3.5). Then, let us look at the scalar two-point vertex

$$\Gamma_\phi^{(2)}(p,-p)=Z_\phi\tilde{\Gamma}_\phi^{(2)}(p,-p)+O\left[-\frac{p^2}{m^2},\frac{\mu^2}{m^2}\right]\times[-p^2 \text{ or } \mu^2]\ , \tag{4.8}$$

where we have retained a factor $-p^2$ or μ^2. (m^2 never appears here.) By differentiating this equation with respect to p^2 and setting $p^2=-\mu^2$, conditions (3.3) for $\Gamma_\phi^{(2)}$ and $\tilde{\Gamma}_\phi^{(2)}$ lead to

$$Z_\phi=1+O\left[\frac{\mu^2}{m^2}\right]\ . \tag{4.9}$$

Similarly, one can use the conditions (3.4) to show that

$$Z_\psi=1+O\left[\frac{\mu^2}{m^2}\right]\ . \tag{4.10}$$

Thus the relation (4.7) between $\Gamma^{(n)}$ and $\tilde{\Gamma}^{(n)}$ reduces simply to

$$\Gamma^{(n)}(p_i,g^2,\lambda,m^2;\mu)=\tilde{\Gamma}^{(n)}(p_i,\tilde{g}^2,\tilde{\lambda},\tilde{m}^2;\mu)+O\left[\frac{p_ip_j}{m^2},\frac{\mu^2}{m^2}\right]\ . \tag{4.11}$$

As for the dimensionless couplings g and λ, we set $p^2=-\mu^2$ in Eqs. (4.11) with $n=3,4$ and use conditions (3.5) in the full and the low-energy effective theories to obtain

$$g=\tilde{g}+O\left[\frac{\mu^2}{m^2}\right]\ ,\quad \lambda=\tilde{\lambda}+O\left[\frac{\mu^2}{m^2}\right]\ . \tag{4.12}$$

In this way the finite renormalizations are not necessary also in the coupling constants. It remains to show that the same is true for the mass parameters

$$m^2=\tilde{m}^2+O(\mu^2)=\tilde{m}^2\left[1+O\left[\frac{\mu^2}{m^2}\right]\right]\ . \tag{4.13}$$

We use the general form of the scalar two-point vertex $\Gamma_\phi^{(2)}$ [Eq. (4.2)], from which we have, by setting $p^2=-\mu^2$,

$$\Gamma_\phi^{(2)}|_{p^2=-\mu^2}=-\mu^2-m^2-\mu^2f(0,0)\ . \tag{4.14}$$

We also have the same expression for $\tilde{\Gamma}_\phi^{(2)}$. Inserting both expressions into Eq. (4.11) with $n=2$ and $p^2=-\mu^2$, we have

$$-\mu^2-m^2-\mu^2f(0,0)=-\mu^2-\tilde{m}^2-\mu^2\tilde{f}(0,0)+\mu^2O\left[\frac{\mu^2}{m^2}\right]\ ,$$

which is nothing but the desired result (4.13). Thus, once the low-energy effective theory is renormalized by the same $\overline{\text{MD}}$ conditions, we no longer need the finite renormalization relating the full theory in the low-energy region to the low-energy effective theory; all parameters in the $\overline{\text{MD}}$ scheme automatically go over into those in the low-energy effective theory.

Since the parameters in high- and low-energy theories are related in a way described above, it is easy to see that the RG coefficient functions in both theories are the same modulo $O(\mu^2/m^2)$ corrections: For $X=g,\lambda$ and $Y=\phi,\psi,m^2$, we have

$$\beta_X=\tilde{\beta}_X+O\left[\frac{\mu^2}{m^2}\right]\ ,\quad \gamma_Y=\tilde{\gamma}_Y+O\left[\frac{\mu^2}{m^2}\right]\ . \tag{4.15}$$

This completes the proof of property (ii).

It is instructive here to see how the $\overline{\text{MD}}$ scheme modifies the conventional MD one. Let us apply the same reasoning as above in the proof of (i) to the conventional MD scheme. Again, because of condition (3.3), the renormalized two-point vertex takes the form (4.1). Now, the renormalization condition (3.7) determines the unknown function $c(g^2,\lambda;m^2/\mu^2)$ to be zero:

$$\Gamma_{\phi\text{MD}}^{(2)}=p^2-m^2+O((p^2+\mu^2)^2)\ . \tag{4.16}$$

Inserting this into the RGE (4.3) and setting $p^2=-\mu^2$, we obtain

$$\gamma_m=-2\left[1+\frac{\mu^2}{m^2}\right]\gamma_\phi\ . \tag{4.17}$$

This clearly shows that the mass parameter in the MD scheme runs quadratically in the high-energy region (as long as the wave-function renormalization γ_ϕ does not vanish).

Let us recapitulate this in a different way. In the MD scheme, the scalar two-point vertex (4.16) satisfies all the $\overline{\text{MD}}$ renormalization conditions except the condition (3.1), and we meet the quadratic running of the mass parameter m^2_{MD}. Now, we finitely renormalize the two-point vertex so that the condition (3.1) is satisfied. By requiring the equality between the expressions (4.16) and (4.2) at $p^2=-\mu^2$, we have

$$m^2_{\text{MD}}=m^2_{\overline{\text{MD}}}+\mu^2f(0,0)\ . \tag{4.18}$$

This shows that the mass parameter $m^2_{\overline{\text{MD}}}$ is just the logarithmic part of m^2_{MD}: We have succeeded in separating the logarithmic and quadratic parts in m^2_{MD} by dividing the MD renormalization condition (3.7) into the $\overline{\text{MD}}$ ones (3.1) and (3.2). This is property (i).

In this way we conclude that the present $\overline{\text{MD}}$ scheme simultaneously enjoys the "automatic" decoupling of heavy particles (as in the conventional MD scheme) and logarithmic RG running (as in the MI scheme).

V. IMPROVING THE EFFECTIVE POTENTIAL IN THE $\overline{\text{MD}}$ SCHEME

We now turn to our main task of how to improve the effective potential in the presence of several mass scales by using the RGE in the $\overline{\text{MD}}$ scheme. Let us first describe the structure of the effective potential for the system (2.8) in the $\overline{\text{MD}}$ scheme. This can be done by applying almost the same reasoning as reviewed in Sec. II for the $\overline{\text{MS}}$ case. A difference arises from the finite part of

counterterms:[3] There appears another type of logarithmic factor $\ln(m^2/\mu^2)$ and nonlogarithmic dependence on μ. So the L-loop contribution now takes the form

$$V^{(L)}=\frac{M_F^4}{g^2}\left[\frac{g^2}{16\pi^2}\right]^L\left[L\text{th order polynomial in } \ln\frac{M_F^2}{\mu^2},\ln\frac{M_B^2}{\mu^2},\ln\frac{m^2}{\mu^2}\right.$$
$$\left.\text{whose coefficients depend on } \frac{M_F^2}{M_B^2},\frac{\lambda}{g^2},\frac{\mu^2}{m^2}\right]. \tag{5.1}$$

Introducing the variables

$$s_F=\frac{g^2}{16\pi^2}\ln\frac{M_F^2}{\mu^2}\ ,\quad s_B=\frac{g^2}{16\pi^2}\ln\frac{M_B^2}{\mu^2}\ ,$$
$$s_m\equiv\frac{g^2}{16\pi^2}\ln\frac{m^2}{\mu^2}\ , \tag{5.2}$$

$$x=\frac{M_F^2}{M_B^2}\ ,\quad y=\frac{\lambda}{g^2}\ ,\quad z\equiv\frac{\mu^2}{m^2}\ , \tag{5.3}$$

we rewrite Eq. (5.1) ($L\geq 1$) as

$$V^{(L)}=\frac{M_F^4}{g^2}\sum_{i,j,k\geq 0}^{i+j+k\leq L}\left[\frac{g^2}{16\pi^2}\right]^{L-(i+j+k)}$$
$$\times v_{i,j,k}^{(L)}(x,y,z)s_F^i s_B^j s_m^k\ . \tag{5.4}$$

Then we *define* the leading logarithmic series expansion in the $\overline{\text{MD}}$ scheme by

$$V=\sum_{L=0}^{\infty}V^{(L)}=\omega+\frac{M_F^4}{g^2}\sum_{l=0}^{\infty}\left[\frac{g^2}{16\pi^2}\right]^l f_l\ , \tag{5.5}$$

$$f_l(s_F,s_B,s_m;x,y,z)$$
$$=\sum_{i,j,k\geq 0}v_{i,j,k}^{(l+i+j+k)}(x,y,z)s_F^i s_B^j s_m^k\ , \tag{5.6}$$

where we have again included the tree part into the summation and we have defined the order l of this expansion as $l\equiv L-(i+j+k)$.

The leading logarithmic series expansion (5.5) is the power series expansion in a small coupling constant $g^2/(16\pi^2)$, as before. Compared with (2.14) in the $\overline{\text{MS}}$ scheme, however, the coefficient functions $v_{i,j,k}^{(L)}$ have a dependence on the new variable $z=\mu^2/m^2$, which potentially makes $v_{i,j,k}^{(L)}$ large. So we do not know, at this stage, whether or not the expansion (5.5) is a sensible one; we do not know whether or not terms in the $(l+1)$th-to-leading logarithmic order are smaller than those in the lth-to-leading logarithmic order. Furthermore, we have a logarithmic factor $s_m\sim\ln(m^2/\mu^2)$ in addition to the "original" ones (2.11) s_F and s_B. So it is not obvious how one can sum up these logarithms simultaneously by using the RGE, which can eliminate just a single variable.

At first sight the situation in the $\overline{\text{MD}}$ scheme appears to be worse than in the $\overline{\text{MS}}$ scheme due to the new variables s_m and z. What saves the day are the properties established in the last section. Based on properties (i) and (ii) as well as the fact that the model has a well-defined massless limit, we claim that the *single* condition

$$\bar{s}_F(t)=0 \tag{5.7}$$

is enough to determine the correct boundary function *over the whole region of field space.* The potentially dangerous variable $z=\mu^2/m^2$ is in fact harmless and the remaining logarithms $\bar{s}_B$ and $\bar{s}_m$ are automatically summed up; otherwise, they decouple.[4] As a result, with the l-loop effective potential $V_{L=l}$ and $(l+1)$-loop RGE's in the $\overline{\text{MD}}$ scheme at hand, the lth-to-leading logarithmic potential is given simply by

$$V(\varphi,\lambda,g^2,m^2,\omega;\mu)=V_{L=l}(\bar{\varphi}(t),\bar{\lambda}(t),\bar{g}^2(t),\bar{m}^2(t),\bar{\omega}(t);e^t\mu)|_{\bar{s}_F(t)=0}+\frac{\bar{M}_F^4(t_0)}{\bar{g}^2(t_0)}\times O\left(\left[\frac{\bar{g}^2(t_0)}{16\pi^2}\right]^{l+1}\right). \tag{5.8}$$

The proof of the statement proceeds in a regionwise manner; we divide the field space into the larger φ (high-energy) region $g^2\varphi^2\gg m^2$, the small φ (low-energy) region $g^2\varphi^2\ll m^2$, and the intermediate region $g^2\varphi^2\sim m^2$, and prove that the statement holds region by region. We should stress here that the final answer (5.8) does not require us to divide the region of field space unlike the procedure in the $\overline{\text{MS}}$ scheme.

First, the proof is rather simply in the intermediate region $g^2\varphi^2\sim m^2$. In this region, $\bar{s}_B$ and $\bar{s}_m$ are of $O(\bar{g}^2/16\pi^2)$ from Eq. (5.7) and the weak condition as in Eq. (2.18) is satisfied. So $\bar{s}_B$ and $\bar{s}_m$ are already summed together with $\bar{s}_F$ by the single condition (5.7) and the correct choice of the boundary function is the same as in the single-mass-scale case, i.e., $V_{L=l}|_{s_F=0}$.

[3]Loop corrections themselves depend on m^2 only through the combination $M_B^2=m^2+\lambda\varphi^2/2$ since we evaluate the potential in the background $\varphi=\langle\phi\rangle$. But since the renormalization constants are determined in the symmetric phase $\varphi=0$, counter terms produce the dependence on $\ln(m^2/\mu^2)$ and μ^2/m^2.

[4]This is why we have used the logarithmic factor of the *lightest* particle in the condition (5.7).

Concerning the asymptotic regions ($g^2\varphi^2 \gg m^2$ and $g^2\varphi^2 \ll m^2$), we make an observation needed for the proof. The boundary function is defined by setting $\mu^2 = M_F^2$ ($=g^2\varphi^2$) and depends on m^2 only through the ratio m^2/μ^2. It follows that *as far as the boundary function is concerned,* taking the high-energy limit $\mu^2 = g^2\varphi^2 \to \infty$ is equivalent to considering the massless limit $m^2 \to 0$, while the low-energy limit $\mu^2 = g^2\varphi^2 \to 0$ corresponds to the limit $m^2 \to \infty$.

It should be noted that the validity of this equivalence between the high-energy and massless limits heavily depend on property (i). If m^2 run quadratically, then the ratio m^2/μ^2 would approach a finite value in the high-energy limit, and the equivalence would break down.

A. High-energy region ($g^2\varphi^2 \gg m^2$)

Now, let us begin with the large φ region, $g^2\varphi^2 \gg m^2$. The second logarithmic factors $\bar{s}_B$ becomes asymptotically equal to the first logarithm $\bar{s}_F$:

$$\bar{s}_B - \bar{s}_F = \frac{\bar{g}^2}{16\pi^2}\left[\ln\frac{\bar{\lambda}}{2\bar{g}^2} + \ln\left[1 + \frac{2\bar{m}^2}{\bar{\lambda}\bar{\varphi}^2}\right]\right] = O\left[\frac{\bar{g}^2}{16\pi^2}\right], \tag{5.9}$$

as is the case in the $\overline{\text{MS}}$ scheme [7]. (Recall that $y = \lambda/g^2 \sim 1$.) So setting $\bar{s}_F$ equal to zero is equivalent to setting $\bar{s}_B$ equal to zero, modulo the quantities of $O(\bar{g}^2/16\pi^2)$, which is of higher order in the leading logarithmic series expansion. Physically, we can regard the massive particle (here ϕ) as massless. Thus the condition (5.7) automatically sums up the second logarithmic factor s_B as well as the first one s_F.

Next, let us make sure that the variable $z = \mu^2/m^2$ does not make the coefficient functions $v^{(L)}$ large in the high-energy limit $z \to \infty$. To this end it is enough to see that there is no positive power term in z when the potential is expanded asymptotically in z^{-1} ($\ll 1$). As noted above, this limit is equivalent to the massless limit $m^2 \to 0$ in our boundary function and we know that there arises no singularity in the latter limit. This establishes that powerlike pieces of μ^2/m^2 vanish in this region.

As for the third logarithmic factor s_m, we show that it does not contribute to the boundary function. Since we know that the massless limit is regular, the dangerous variable s_m in this limit disappears in our boundary function like

$$\frac{m^2}{\mu^2}\ln\frac{m^2}{\mu^2} \to 0 .$$

Thus we no longer need to sum up this logarithmic factor in the high-energy limit $g^2\varphi^2 = \mu^2 \gg m^2$. This establishes our claim in the large φ region.

B. Low-energy region ($g^2\varphi^2 \ll m^2$)

Next, we turn to the small φ region. Now, $\bar{s}_F = 0$ no longer implies $\bar{s}_B = 0$ since their difference $\bar{s}_B - \bar{s}_F$ becomes of order 1. Furthermore, $\bar{s}_m$ is also large. So we cannot sum up these two logarithmic factors $\bar{s}_B$ and $\bar{s}_m$ simultaneously. Instead, we shall show that in the low-energy limit $g^2\varphi^2 \to 0$, the logarithmic factors s_B and s_m as well as $z = \mu^2/m^2$ decouple so that we no longer need to sum them up.

This can be seen by using the decoupling theorem. As in Eqs. (4.12), (4.13), and (4.15), all the parameters as well as the β and γ functions of the full theory approach, in the low-energy limit, those of the low-energy effective theory. In particular,

$$s_F = \tilde{s}_F + O\left[\frac{\mu^2}{m^2}\right] . \tag{5.10}$$

This property of the automatic decoupling holds also for the effective potential itself. In particular, the boundary function $V|_{s_F=0}$ satisfies

$$V|_{s_F=0} = \tilde{V}|_{s_F=0} + M_F^4 \times O\left[\frac{g^2\varphi^2}{m^2}\right] . \tag{5.11}$$

Here $\tilde{V}$ is the effective potential in the low-energy effective theory fixed by the same $\overline{\text{MD}}$ renormalization conditions and we have replaced $s_F = 0$ with $\tilde{s}_F = 0$ by using Eq. (5.10). (A one-loop example can be found in Appendix A.) From this expression we see that there is no contribution from the potentially large variables s_B and s_m as well as negative power terms in μ^2/m^2. The first term $\tilde{V}$ does not contain loop effects due to heavy particles by definition. Furthermore, the remaining terms are small by themselves. Thus we need not sum up s_B and s_m. This establishes the correctness of our boundary function in the low-energy asymptotic region $g^2\varphi^2 = \mu^2 \ll m^2$. Together with the automatic decoupling (ii) in β and γ functions, this completes the proof of our assertion in this region.

In this way we establish the procedure to improve the effective potential. The final answer is given by Eq. (5.8).

VI. COMPARISON WITH OTHER SCHEMES: NUMERICAL STUDY

In this section we work with the leading logarithmic order and demonstrate the results of the RG improvement. At the leading logarithmic order, we use the tree potential $V^{(0)}$ as the boundary function and one-loop RGE's.

Note that our procedure is such that, at an individual point φ in field space, the value $V(\varphi)$ of the improved potential is evaluated by Eq. (5.8). Actually, this is not economical since we solve the running equations at each φ. As explained in Ref. [4], one can avoid this duplication by finding the value of φ corresponding to the solution to $\bar{s}_F(t) = 0$ for each value of the running distance t. Namely, as we solve the running equations, we simultaneously obtain the value of the effective potential at

$$\varphi^2 = e^{2t}\mu^2 \frac{1}{\bar{g}^2(t)}\left[\frac{\bar{\varphi}(0)}{\bar{\varphi}(t)}\right]^2 . \tag{6.1}$$

The last factor $\bar{\varphi}(0)/\bar{\varphi}(t)$ does not depend on the initial value $\bar{\varphi}(0)$. We set $\mu = m$ for later convenience. Then,

Eq. (6.1) becomes

$$\ln\frac{g^2\varphi^2}{m^2}=2t+\ln\left[\frac{\bar{g}^2(0)}{\bar{g}^2(t)}\frac{\bar{\varphi}^2(0)}{\bar{\varphi}^2(t)}\right]\sim 2t\ , \tag{6.2}$$

so that the region $g^2\varphi^2\gtrsim m^2$ corresponds to $t\gtrsim 0$.

We compare the improved potential in the $\overline{\text{MD}}$ scheme with those in the $\overline{\text{MS}}$ and MD schemes. For this purpose we should match the renormalized parameters in different schemes in order to guarantee that we are treating the same system. For the leading logarithmic potential, the parameter relations should be exact also in the leading logarithmic order. By using the fact that the improved potential exactly satisfies the RGE (2.4) with one-loop β and γ functions, we match the parameters at $\mu=m$. Then the parameter relations reduce to the tree-level ones since $s_m\sim\ln(m^2/\mu^2)=0$.

We now show the result of our numerical calculations. We present it in a moderate case (a) $g^2=0.55$, $\lambda=2.3$ and in extreme cases (b) $g^2=0.5$, $\lambda=2.5$, in which $\bar{\lambda}$ blows up, and (c) $g^2=0.7$, $\lambda=2.0$, in which the vacuum becomes unstable. We set $m=1$ (as the mass unit), $N=1$, and the vacuum energy to $\omega=0$ so that $V(\varphi=0)=0$. Figure 3 shows the asymptotic behavior of the leading logarithmic potentials in the $\overline{\text{MD}}$, $\overline{\text{MS}}$, and MD schemes for case (a). (Other cases are similar and so are omitted.) The horizontal and vertical axes are $\ln(\varphi^2/m^2)$ and $\ln|V/m^4|$, respectively. We find that the scheme dependence is quite small.[5] This can be understood from the fact that since we are using the tree potential as the boundary function, the asymptotic behaviors in the large and small φ regions are mainly determined by the quartic coupling $\bar{\lambda}$ and mass $\bar{m}^2$, respectively.

The behaviors of $\bar{g}^2(t)$ and $\bar{\lambda}(t)$ in the $\overline{\text{MD}}$ scheme are shown in Fig. 4. The horizontal axis is the same as in Fig. 3. The scheme dependence is mild even in the extreme case (b) where the quartic coupling $\bar{\lambda}$ blows up in the high-energy region and the case (c) where $\bar{\lambda}$ becomes negative and a vacuum instability occurs.

In order to examine how much the RG improvement is obtained, we evaluate the difference of the improved potential V from the tree one $V^{(0)}$ normalized by $V^{(0)}$:

$$\chi(\varphi)\equiv\frac{V(\varphi)-V^{(0)}(\varphi)}{V^{(0)}(\varphi)}\ . \tag{6.3}$$

Figure 5 show the results of $\chi_{\overline{\text{MD}}}$, $\chi_{\overline{\text{MS}}}$, and χ_{MD}, respectively, for each case of the parameter choices. As expected, the longer distance we run from $t=0$, the larger improvement is obtained. The large improvement $|\chi|\sim 1$ is obtained for $2t\sim 30$ since the variables s_F and s_B become of order 1 for our parameter choices. In extreme cases, an even larger improvement is obtained, but the leading logarithmic approximation itself breaks down for these cases.

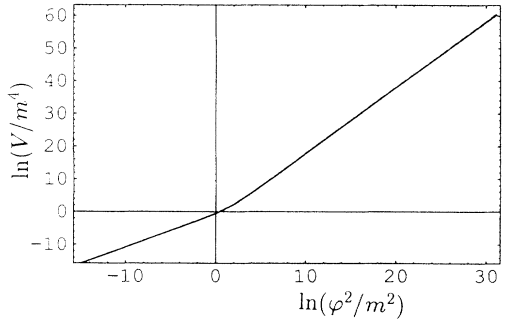

FIG. 3. Asymptotic behavior of the leading logarithmic potential in the $\overline{\text{MD}}$ (solid line) $\overline{\text{MS}}$ (dotted line), and MD schemes (dot-dashed line). The inputs at $\mu=m$ are $g^2=0.55$, $\lambda=2.3$ [case (a)]. $V\sim\varphi^4$ in the large φ region and $V\sim\varphi^2$ in the small φ region.

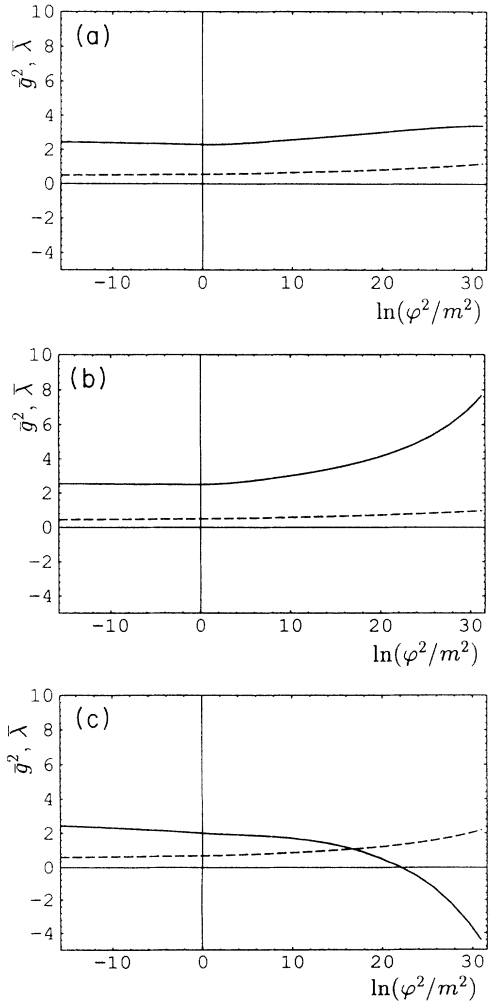

FIG. 4. (a) Behavior of $\bar{\lambda}$ (solid line) and $\bar{g}^2$ (dashed line) in the $\overline{\text{MD}}$ scheme. The input is moderate one: $g^2=0.55$, $\lambda=2.3$. (b) Same as (a) for the extreme case $g^2=0.5$, $\lambda=2.5$. $\bar{\lambda}$ hits Landau singularity in the high-energy region. (c) Same as (a) for the extreme case $g^2=0.7$, $\lambda=2.0$. $\bar{\lambda}$ becomes negative to cause the vacuum instability.

[5]A rather good coincidence is found between the $\overline{\text{MS}}$ and MD schemes. It is not clear to us that this persists to higher leading logarithmic order since the reasoning presented in Sec. V will not apply to the conventional MD scheme.

Finally, we add a remark. We can further improve the approximation by requiring that the potential be correct not only in the leading logarithmic order, but also in the one-loop level [4,7]. For this combined approximation, we should use the one-loop potential as the boundary function while one-loop RGE's are enough. Also, the parameter should be matched at the one-loop level by performing a finite renormalization at $\mu=m$.[6]

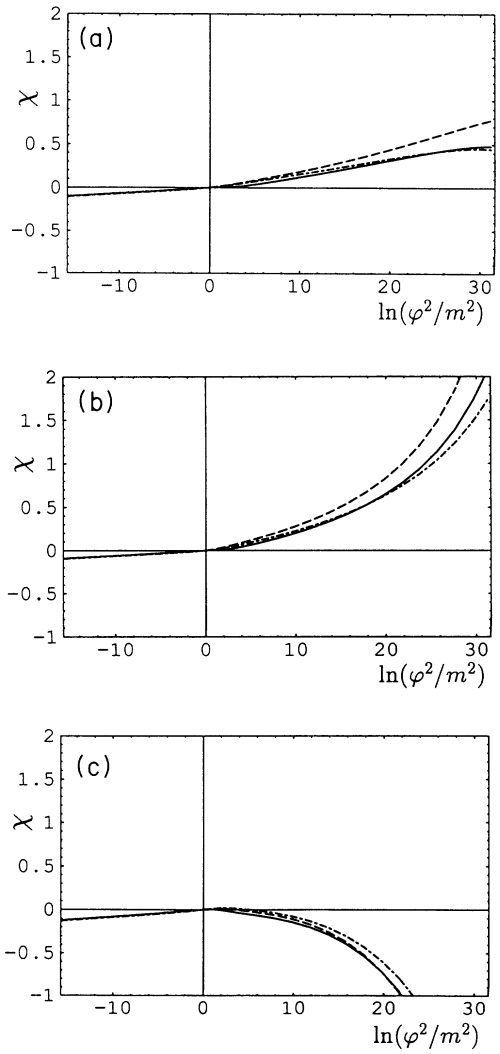

FIG. 5. (a) "Character" $\chi_{\overline{\mathrm{MD}}}$ (solid line), $\chi_{\overline{\mathrm{MS}}}$ (dashed line), and χ_{MD} (dot-dashed line) for the moderate case (a). The large improvement is obtained in asymptotic regions. (b) Same as (a) for the extreme case (b). Because of K_λ, a sizable scheme dependence is observed near Landau singularity. (c) Same as (a) for the extreme case (c). Since the mass term dominates the quartic term in V, a sizable difference from the $\overline{\mathrm{MS}}$ and $\overline{\mathrm{MD}}$ schemes is observed in the MD scheme.

[6]By matching at $\mu=m$, higher-order terms are small in the $\overline{\mathrm{MS}}$ and $\overline{\mathrm{MD}}$ schemes. This might not be the case in the conventional MD scheme.

VII. CONCLUSIONS AND DISCUSSIONS

We have discussed the issues concerning the RG improvement of the effective potential in the presence of several mass scales. Originally, the coexistence of multimass scales causes trouble in determining the boundary function needed for the general solution of the RGE. By adopting a simple model possessing two mass scales, we have seen in this paper that the $\overline{\mathrm{MD}}$ scheme provides us with a suitable choice of boundary functions without dividing the field space: The correct boundary function for the lth-to-leading logarithmic potential is just the l-loop potential evaluated at $s_F=0$.

The $\overline{\mathrm{MD}}$ scheme is the new renormalization scheme proposed in this paper. Its crucial properties are the automatic decoupling of heavy particles and the absence of quadratic running of scalar mass. These properties enable us to show that the leading logarithmic series expansion is well defined even in the presence of nonlogarithmic corrections. Then the procedure can be stated by a single condition over the entire region, which is the same as in the single-mass-scale case.

We make a comment on other possible methods to handle multimass-scale systems. First, we already have the procedure in the $\overline{\mathrm{MS}}$ scheme [7]. Such a regionwise procedure will be cumbersome especially when there are many mass thresholds. In the $\overline{\mathrm{MD}}$ scheme, calculations of the RGE's become harder, but once they are calculated, then various threshold effects are automatically taken care of. Our method will be more useful when some intensive investigations will be needed, such as scanning for large parameter space. On the other hand, Einhorn and Jones proposed [17] to introduce several renormalization points μ_i by which the multilogarithmic factors are simultaneously summed up. Their method is interesting, but the RGE's become partial differential equations. Our method described here involves solving ordinary differential equations, which will be a much easier task.

Finally, we comment on possible applications of the method in this paper. Among them, it is interesting to apply our procedure to the analysis of the Higgs potential in the supersymmetric standard model, in which many mass scales are present. When the supersymmetry-breaking scale is rather high, we expect a large improvement to the usual analyses which make use of at most a one-loop potential [18,19].

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APPENDIX A

We gather one-loop results by dimensional regularization. We first determine the renormalization constants $Z_X=1+\hbar Z_X^{(1)}+O(\hbar^2)$ from the $\overline{\mathrm{MD}}$ renormalization conditions (3.1)–(3.5). We then calculate one-loop RGE's and the one-loop contribution $V^{(1)}$ to the effective

potential. Our conventions are $d=4-2\epsilon$, $1/\bar{\epsilon}\equiv 1/\epsilon-\gamma+\ln 4\pi$, and $\mathrm{Tr}\mathbb{1}=4$. A mass scale μ_0 is introduced so that the dimensionality becomes correct. μ_0 always appears as $\mu_0^{2\epsilon}\int d^dk/(2\pi)^d i$ and is identified with the renormalization point μ in the $\overline{\mathrm{MS}}$ scheme. Otherwise, μ_0 has nothing to do with μ, which is introduced through the renormalization conditions.

As usual, we renormalize the theory (2.1) in the symmetric phase, $\langle\phi\rangle=0$. First, the vacuum energy at one loop is

$$\Gamma^{(0)}=-\omega_{\mathrm{bare}}+\frac{1}{4}\frac{1}{16\pi^2}m^4\left[\frac{1}{\bar{\epsilon}}+\frac{3}{2}-\ln\frac{m^2}{\mu_0^2}\right]. \quad \text{(A1)}$$

From $\omega_{\mathrm{bare}}=\omega+\hbar\omega^{(1)}+O(\hbar^2)$, the simplified condition $\Gamma^{(0)}=-\omega$ leads to

$$\omega^{(1)}=\frac{1}{4}\frac{1}{16\pi^2}m^4\left[\frac{1}{\bar{\epsilon}}+\frac{3}{2}+\ln\frac{\mu_0^2}{m^2}\right]. \quad \text{(A2)}$$

Second, the boson self-energy

$$\Gamma_\phi^{(2)}(p,-p)=Z_\phi p^2-Z_m m^2+\Pi(p^2)$$

is given by

$$\Pi(p^2)=\frac{2Ng^2}{16\pi^2}p^2\left[\frac{1}{\bar{\epsilon}}+2+\ln\frac{\mu_0^2}{-p^2}\right]+\frac{\lambda/2}{16\pi^2}m^2\left[\frac{1}{\bar{\epsilon}}+1+\ln\frac{\mu_0^2}{m^2}\right]. \quad \text{(A3)}$$

The renormalization constant $Z_\phi^{(1)}$ is determined as usual by Eq. (3.11), while Z_m is determined by solving the differential equation (3.13) and imposing the boundary condition (3.12) to be

$$Z_\phi^{(1)}=-\frac{2Ng^2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+1+\ln\frac{\mu_0^2}{\mu^2}\right],$$
$$Z_m^{(1)}=\frac{\lambda/2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+1+\ln\frac{\mu_0^2}{m^2}\right]. \quad \text{(A4)}$$

The fermion self-energy $\Gamma_\psi^{(2)}(p,-p)=Z_\psi\not{p}-\Sigma(p)$ is similar:

$$Z_\psi^{(1)}=-\frac{g^2/2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+\ln\frac{\mu_0^2}{m^2}-I_\psi\left[\frac{\mu^2}{m^2}\right]\right], \quad \text{(A5)}$$

where

$$I_\psi(z)\equiv(1+1/z)^2\ln(z+1)-(1/z)-2\ .$$

The vertex correction to the Yukawa coupling is $\Gamma_g^{(3)}(p,-p;0)=-Z_g g+\Lambda_g(p,-p;0)$,

$$\Lambda_g^{(3)}(p,-p;0)=g\frac{g^2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+\ln\frac{\mu_0^2}{m^2}-I_g\left[\frac{-p^2}{m^2}\right]\right], \quad \text{(A6)}$$

where $I_g(z)\equiv(1+1/z)\ln(z+1)$. From the condition (3.5), we obtain

$$Z_g^{(1)}=\frac{g^2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+\ln\frac{\mu_0^2}{m^2}-I_g\left[\frac{\mu^2}{m^2}\right]\right]. \quad \text{(A7)}$$

The vertex correction to the quartic scalar coupling, $\Gamma_\phi^{(4)}=-Z_\lambda\lambda+\Lambda_\lambda$, is a little bit complicated. We separate the boson- and fermion-loop contributions: $\Lambda_\lambda=\Lambda_B+\Lambda_F$. Let p_i be the incoming external momenta ($i=1$ to 4) and $s=(p_1+p_2)^2$, $t=(p_1+p_4)^2$ and $u=(p_1+p_3)^2$. The boson contribution consists of the s-, t-, and u-channel ones as

$$\Lambda_B(-s)+\Lambda_B(-t)+\Lambda_B(-u)\ ,$$

where

$$\Lambda_B(-s)\equiv\frac{\lambda^2/2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+\ln\frac{\mu_0^2}{m^2}-I_\lambda\left[\frac{-s}{m^2}\right]\right], \quad \text{(A8)}$$

$$I_\lambda\left[\frac{-s}{m^2}\right]\equiv\int_0^1 d\alpha\ln\left[1+\frac{-s}{m^2}\alpha(1-\alpha)\right]$$
$$=\ln\frac{-s}{4m^2}+L\left[\left[1+\frac{4m^2}{-s}\right]^{1/2}+1\right]-L\left[\left[1+\frac{4m^2}{-s}\right]^{1/2}-1\right], \quad \text{(A9)}$$

where $L(\zeta)\equiv\zeta(\ln\zeta-1)$. The fermion contribution is evaluated in Appendix B,

$$\Lambda_F(p_i)=-\frac{Ng^4}{16\pi^2}I_0(p_i)+[\text{permutation in } p_i]\ , \quad \text{(A10)}$$

$$I_0(p_i)\equiv\frac{1}{\bar{\epsilon}}+2+\ln\frac{\mu_0^2}{-(p_1+p_2)^2}+J(-p_1^2,-p_2^2;-(p_1+p_2)^2)-\frac{1}{4}J(p_1^2p_3^2,p_2^2p_4^2;(p_1+p_2)^2(p_2+p_3)^2)\ , \quad \text{(A11)}$$

and the function $J(\xi,\eta;\zeta)$ is defined in the Appendix B. Both Λ_B and Λ_F are completely symmetric in p_i ($i=1$–4). Then, we impose the condition (3.5) at the symmetric point, $p_i^2=-\mu^2$ and $s=t=u=-\frac{4}{3}\mu^2$, to obtain

$$Z_\lambda^{(1)}\lambda=\frac{3\lambda^2/2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+\ln\frac{\mu_0^2}{m^2}-I_\lambda\left[\frac{4\mu^2}{3m^2}\right]\right]-\frac{4!Ng^4}{16\pi^2}\left[\frac{1}{\bar{\epsilon}}+\ln\frac{\mu_0^2}{\mu^2}+F\right], \quad \text{(A12)}$$

with a constant $F=3.021\,98\ldots$.

Some remarks are in order. First, observe that the functions I_X (contained in $Z_X^{(1)}$ for $X=\psi,g,\lambda$) behave in the high-energy (massless) limit $\mu^2/m^2\to\infty$ as

$$I_x\left[\frac{\mu^2}{m^2}\right]\to\ln\frac{\mu^2}{m^2}\ ,\quad \mu^2\frac{d}{d\mu^2}I_X\left[\frac{\mu^2}{m^2}\right]\to 1\ . \quad \text{(A13)}$$

An important point is that in the low-energy limit $\mu^2/m^2\to 0$ they all approach constant values, $I_\psi(0)=-\frac{1}{2}$, $I_g(0)=1$, and $I_\lambda(0)=0$, as

$$I_X\left[\frac{\mu^2}{m^2}\right]\to I_X(0)\ ,\quad \mu^2\frac{d}{d\mu^2}I_X\left[\frac{\mu^2}{m^2}\right]\to 0\ . \quad \text{(A14)}$$

Note also that we have to renormalize $\Gamma_\phi^{(4)}$ at the symmetric point in order that all the s-, t-, and u-channel boson loops equally contribute to I_λ.

Now, we turn to the RG coefficient functions and the effective potential. The β and γ functions are obtained by noting that the bare parameters are independent of μ. For instance, from $g^2_{\rm bare}=Z_g^2 Z_\phi^{-1} Z_\psi^{-2} g^2$,

$$\begin{aligned}\frac{\beta_g}{g^2}&=-\mu\frac{d}{d\mu}\ln Z_g^2 Z_\phi^{-1} Z_\psi^{-2}\\&=2\mu^2\frac{d}{d\mu^2}\left[-2Z_g^{(1)}+Z_\phi^{(1)}+2Z_\psi^{(1)}\right]+O(\hbar^2)\\&=\frac{4Ng^2}{16\pi^2}+\frac{2g^2}{16\pi^2}\mu^2\frac{d}{d\mu^2}\left[2I_g\left[\frac{\mu^2}{m^2}\right]+I_\psi\left[\frac{\mu^2}{m^2}\right]\right]\\&\quad+O(\hbar^2)\ .\end{aligned}$$

By looking at the asymptotic behavior (A13) and (A14), we introduce the "interpolating" functions (3.15) with the property (3.16) by

$$K_\psi(z)\equiv z\frac{d}{dz}I_\psi(z)\ ,\quad K_\lambda(z)\equiv z\frac{d}{dz}I_\psi\left[\frac{4z}{3}\right]\ ,$$
$$K_g(z)\equiv\frac{1}{3}z\frac{d}{dz}[2I_g(z)+I_\psi(z)]\ . \tag{A15}$$

With these definitions we finally obtain the RGE's (3.14).

Next, we discuss the effective potential $V(\varphi)$ for $\varphi=\langle\phi\rangle$. The one-loop contribution to the effective potential takes the form $V^{(1)}=V^{(1)}_{\rm loop}+\Delta V^{(1)}$, where

$$\begin{aligned}V^{(1)}_{\rm loop}=&-\frac{4N}{64\pi^2}M_F^4\left[-\frac{1}{\bar\epsilon}-\frac{3}{2}+\ln\frac{M_F^2}{\mu_0^2}\right]\\&+\frac{1}{64\pi^2}M_B^4\left[-\frac{1}{\bar\epsilon}-\frac{3}{2}+\ln\frac{M_B^2}{\mu_0^2}\right]\ ,\end{aligned}\tag{A16}$$

while the contribution from counterterms are, from Eqs. (A2), (A4), and (A12),

$$\begin{aligned}\Delta V^{(1)}\equiv&\,\omega^{(1)}+Z_m^{(1)}\frac{1}{2}m^2\varphi^2+Z_\lambda^{(1)}\frac{1}{4!}\lambda\varphi^4\\=&-\frac{4N}{64\pi^2}M_F^4\left[\frac{1}{\bar\epsilon}+\ln\frac{\mu_0^2}{\mu^2}\right]-\frac{4N}{64\pi^2}g^4\varphi^4F+\frac{1}{64\pi^2}M_B^4\left[\frac{1}{\bar\epsilon}+\ln\frac{\mu_0^2}{\mu^2}+\ln\frac{\mu^2}{m^2}\right]\\&+\frac{1}{64\pi^2}\left[\frac{3}{2}m^4+m^2\lambda\varphi^2-\left[\frac{\lambda}{2}\varphi^2\right]^2 I_\lambda\left[\frac{4\mu^2}{3m^2}\right]\right]\ .\end{aligned}\tag{A17}$$

Thus the final form of $V^{(1)}$ in the $\overline{\rm MD}$ scheme is

$$\begin{aligned}V^{(1)}=&-\frac{4N}{64\pi^2}M_F^4\left[\ln\frac{M_F^2}{\mu^2}-\frac{3}{2}\right]+\frac{1}{64\pi^2}M_B^4\left[\ln\frac{M_B^2}{\mu^2}-\frac{3}{2}\right]+\frac{1}{64\pi^2}m^4\left[\ln\frac{\mu^2}{m^2}+\frac{3}{2}\right]\\&+\frac{1}{64\pi^2}m^2\lambda\varphi^2\left[\ln\frac{\mu^2}{m^2}+1\right]+\frac{1}{64\pi^2}\left[\frac{\lambda}{2}\varphi^2\right]^2\left[\ln\frac{\mu^2}{m^2}-I_\lambda\left[\frac{4\mu^2}{3m^2}\right]\right]-\frac{4N}{64\pi^2}g^4\varphi^4F\ .\end{aligned}\tag{A18}$$

Note that the first two terms in Eq. (A18) take just the same form as $V^{(1)}$ in the $\overline{\rm MS}$ scheme, while the remaining terms are extra contributions to it in the $\overline{\rm MD}$ scheme.

From expression (A18) we can explicitly confirm what we have generally argued in Sec. V. Expression (A18) is suitable in the large φ region. In the low-energy region $\mu^2\ll m^2$, we rewrite it, by combining $\ln(\mu^2/m^2)$ with $\ln(M_B^2/\mu^2)$, into the form

$$V^{(1)}=-\frac{4N}{64\pi^2}M_F^4\left[\ln\frac{M_F^2}{\mu^2}-\frac{3}{2}+F\right]+\frac{1}{64\pi^2}M_B^4\left[\ln\frac{M_B^2}{m^2}-\frac{3}{2}\right]+\frac{1}{64\pi^2}\left[\frac{3}{2}m^4+m^2\lambda\varphi^2-\left[\frac{\lambda}{2}\varphi^2\right]^2\times O\left[\frac{\mu^2}{m^2}\right]\right]\ ,\tag{A19}$$

where we used $I_\lambda(0)=0$. When the second term is expanded in the small φ region $\varphi^2\ll m^2$, terms proportional to m^4 and $m^2\varphi^2$ cancel, so that we have

$$V^{(1)}=\tilde V^{(1)}+\frac{1}{64\pi^2}\left[\frac{\lambda}{2}\varphi^2\right]^2\times O\left[\frac{\mu^2}{m^2},\frac{\lambda\varphi^2}{m^2}\right]\ ,\tag{A20}$$

where we denote the first term in Eq. (A19) as $\tilde V^{(1)}$, which is just the potential of the low-energy effective theory. This is an example of the "automatic" decoupling.

APPENDIX B

We evaluate the fermion one-loop contribution Λ_F to the four-point vertex $\Gamma_\phi^{(4)}=-Z_\lambda\lambda+\Lambda_B+\Lambda_F$. The generic one-loop integral can be reduced to the scalar-loop integral [20] and expressed in terms of a Spence function, but we give another expression for Λ_F.[7]

[7]The authors are grateful to T. Kugo for discussions on this calculation.

The fermion one-loop contribution to the four-point vertex $\Gamma_{\phi}^{(4)}$ is given by

$$\Lambda_F(p_1,p_2,p_3,p_4)=-\frac{Ng^4}{16\pi^2}I(p_1,p_2,p_3,p_4) +[\text{permutation in } p_2,p_3,p_4]\ , \tag{B1}$$

$$I(p_1,p_2,p_3,p_4) \equiv 16\pi^2\mu_0^{2\epsilon}\int\frac{d^dk}{(2\pi)^di}\mathrm{Tr}\left[\frac{1}{\not k(\not k+\not p)(\not k+\not q)(\not k+\not r)}\right]\ ,$$

where $p=p_1$, $q=p_1+p_2$, and $r=p_1+p_2+p_3=-p_4$. This integral can be reduced to the scalar-loop integral by carrying out the trace and making use of the identity

$$2(k+a)(k+b)=(k+a)^2+(k+b)^2-(a-b)^2$$

for any momenta a and b. As a result, the integral I decomposes into three parts $I=I_2+I_3+I_4$, where I_n contains n propagators. We denote as $D_a\equiv(k+a)^2$.

The first integral I_2 is evaluated in cyclically symmetric way by making a suitable shift in the loop momentum as

$$\begin{aligned}I_2&=16\pi^2\mu_0^{2\epsilon}\int\frac{d^dk}{(2\pi)^di}2\left[\frac{1}{D_pD_r}+\frac{1}{D_0D_q}\right]\\&=16\pi^2\mu_0^{2\epsilon}\int\frac{d^dk}{(2\pi)^di}\frac{1}{D_0D_{p_1+p_2}}+[\text{cyclic in } p_i]\\&=\left[\frac{1}{\bar\epsilon}+2+\ln\frac{\mu_0^2}{-(p_1+p_2)^2}\right]+[\text{cyclic in } p_i]\ .\end{aligned} \tag{B2}$$

Similarly, the second one I_3 is evaluated symmetrically as

$$\begin{aligned}I_3&=16\pi^2\int\frac{d^4k}{(2\pi)^4i}\frac{-p_1^2-p_2^2+(p_1+p_2)^2}{D_{-p_1}D_0D_{p_2}}\\&\quad+[\text{cyclic in } p_i]\\&=J(-p_1^2,-p_2^2;-(p_1+p_2)^2)+[\text{cyclic in } p_i]\ .\end{aligned} \tag{B3}$$

We have introduced the function $J(\xi,\eta;\zeta)=J(\eta,\xi;\zeta)$ defined for $\xi,\eta,\zeta>0$ as

$$J(\xi,\eta;\zeta)\equiv\int_0^1 d\alpha\frac{\xi+\eta-\zeta}{\xi\alpha+\eta(1-\alpha)-\zeta\alpha(1-\alpha)} \times\ln\frac{\xi\alpha+\eta(1-\alpha)}{\zeta\alpha(1-\alpha)}\ , \tag{B4}$$

in terms of which we have the useful formula

$$16\pi^2\int\frac{d^4k}{(2\pi)^4i}\frac{-a^2-b^2+(a+b)^2}{D_{-a}D_0D_b} =J(-a^2,-b^2;-(a+b)^2)\ . \tag{B5}$$

Finally, the I_4 is

$$I_4=16\pi^2\int\frac{d^4k}{(2\pi)^4i} \times\frac{r^2(p-q)^2+p^2(q-r)^2-q^2(r-p)^2}{D_0D_pD_qD_r}\ , \tag{B6}$$

which can also be expressed in terms of J by making a conformal change [20] of the integration variable $k_u=(\mu_0^2/\bar k^2)\bar k_\mu$. ($\mu_0$ is an arbitrary scale.) Under the conformal transformation $\bar a_\mu\equiv(\mu_0^2/a^2)a_\mu$, $(k+a)^2=(a^2/\bar k^2)(\bar k+\bar a)^2$, we have

$$\int\frac{d^4k}{(2\pi)^4i}\frac{1}{D_0D_pD_qD_r}=-\frac{\mu_0^4}{p^2q^2r^2}\int\frac{d^4k}{(2\pi)^4i}\frac{1}{D_{\bar p}D_{\bar q}D_{\bar r}}\ .$$

By applying formula (B5), we evaluate the integral (B6) as

$$\begin{aligned}I_4&=16\pi^2\int\frac{d^4k}{(2\pi)^4i}\frac{-(\bar p-\bar q)^2-(\bar q-\bar r)^2+(\bar r-\bar p)^2}{D_{-(\bar q-\bar p)}D_0D_{\bar r-\bar q}}\\&=J(-\frac{\mu_0^4}{p^2q^2}(p-q)^2,-\frac{\mu_0^4}{q^2r^2}(q-r)^2;-\frac{\mu_0^4}{r^2p^2}(r-p)^2)\\&=-\frac{1}{4}J(p_1^2p_3^2,p_2^2p_4^2;(p_1+p_2)^2(p_2+p_3)^2)\\&\quad+[\text{cyclic in } p_i]\ ,\end{aligned} \tag{B7}$$

where we have used the definition (B4) and the definition of p, q, and r in the last equality so that the cyclic symmetry becomes manifest.

Thus the final forms of the integral $I=I_2+I_3+I_4$ and the fermion one-loop contribution Λ_F are, from Eqs. (B2), (B3), and (B7),

$$I(p_i)=I_0(p_1,p_2,p_3,p_4)+[\text{cyclic in } p_i]\ , \tag{B8}$$

$$\Lambda_F(p_i)=-\frac{Ng^4}{16\pi^2}I_0(p_1,p_2,p_3,p_4)+[\text{permutation in } p_i]\ , \tag{B9}$$

with I_0 given in Eq. (A11). At the symmetric point, I_0 reduces to the second term in Eq. (A12) with

$$\begin{aligned}F&\equiv2+\ln\tfrac{3}{4}+J(1,1;\tfrac{4}{3})-\tfrac{1}{4}J(1,1;\tfrac{16}{9})\\&=3.021\,98\ldots\ ,\end{aligned} \tag{B10}$$

where we numerically integrate the function $J(\xi,\eta;\zeta)\equiv(\xi+\eta-\zeta)J_0(\xi,\eta,\zeta)$, [Eq. (B4)] by using $E(\alpha)\equiv\xi\alpha+\eta(1-\alpha)-\zeta\alpha(1-\alpha)$ and

$$\begin{aligned}J_0(\xi,\eta,\zeta)&=\frac{\xi+\eta}{\xi\eta}+\int_0^1 d\alpha\frac{1}{E(\alpha)}\ln\frac{\xi\alpha+\eta(1-\alpha)}{\zeta}\\&\quad-\int_0^1 d\alpha\left\{\left[\frac{1}{E(\alpha)}-\frac{1}{E(0)}\right]\ln\alpha\right.\\&\quad\left.+\left[\frac{1}{E(\alpha)}-\frac{1}{E(1)}\right]\ln(1-\alpha)\right\}\ .\end{aligned} \tag{B11}$$

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