# Subcritical closed string field theory in less than 26 dimensions

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In this paper we construct the second-quantized action for subcritical closed string field theory with zero cosmological constant in dimensions $2 \leq D < 26$, generalizing the nonpolynomial closed string field theory action proposed by the author and the Kyoto and MIT groups for $D = 26$. The proof of gauge invariance is considerably complicated by the presence of the Liouville field $\phi$ and the nonpolynomial nature of the action. However, we explicitly show that the polyhedral vertex functions obey BRST invariance to all orders. By point-splitting methods we calculate the anomaly contribution due to the Liouville field, and show in detail that it cancels only if $D - 26 + 1 + 3Q^2 = 0$, in both the bosonized and unbosonized polyhedral vertex functions. We also show explicitly that the four-point function generated by this action reproduces the shifted Shapiro-Virasoro amplitude found from $c = 1$ matrix models and Liouville theory in two dimensions. This calculation is nontrivial because the conformal transformation from the $z$ to the $\rho$ plane requires rather complicated third elliptic integrals and is hence much more involved than the ones found in the usual polynomial theories.

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## I. INTRODUCTION

At present matrix models [1–3] give us a simple and powerful technique for constructing the $S$ matrix of two-dimensional string theory. However, all string degrees of freedom are missing, and hence many of the successes of the theory are intuitively difficult to interpret in terms of string degrees of freedom. Features such as the discrete states [4–7] and the $w(\infty)$ algebra arise in a rather obscure fashion.

By contrast, Liouville theory [8,9] manifestly includes all string degrees of freedom, but the theory is notoriously difficult to solve, even for the free case.

In order to further develop the Liouville approach, we present the details of a second-quantized field theory of closed strings defined in $2 \leq D < 26$ dimensions with $\mu = 0$. (See Refs. [10,11] for work on $c$=1 open string field theory.)

There are several advantages to presenting a second-quantized field formulation of Liouville theory.

(a) The $c = 1$ barrier, which has proved to be insurmountable for matrix models, is trivially breached for Liouville theory (although we no longer expect the model to be exactly solvable beyond $c = 1$).

(b) In principle, it should be possible to present a supersymmetric Liouville theory in field theory form, which is difficult for the matrix model approach.

(c) For $c = 1$, the rather mysterious features appearing in matrix models, which are intuitively difficult to understand, have a standard field theoretical interpretation. For example, "discrete states" arise naturally as string degrees of freedom with discrete momenta when we calculate the physical states of the theory. In other words, the $\Phi(X, b, c, \phi)$ field contains three sets of states. Symbolically, we have

$$|\Phi(X,b,c,\phi)\rangle = |\mathrm{tachyon}\rangle + |\mathrm{discrete\,states}\rangle + |\mathrm{BRST\,trivial\,states}\rangle \ . \qquad (1)$$

Also, the structure constants of $w(\infty)$ arise as the coefficients of the three-string vertex function, analogous to the situation in Yang-Mills theory. We see that $w(\infty)$ is just a subalgebra of the full string field theory gauge algebra. For example, if $\langle j, m|$ labels the SU(2) quantum numbers of the discrete states, then we can show that the three-string vertex function $\langle \Phi^3 \rangle$, taken on discrete states, reproduces the structure constants of $w(\infty)$:

$$\langle j_1, m_1|\langle j_2, m_2|\langle j_3, m_3|V_3\rangle \sim \Big\langle \Psi_{j_1,m_1}(0)\Psi_{j_2,m_2}(1)\Psi_{j_3,m_3}(\infty)\Big\rangle \sim (j_1 m_2 - j_2 m_1)\delta_{j_3,j_1+j_2-1}\delta_{m_3,m_1+m_2} \ , \qquad (2)$$

where we have made a conformal transformation from the three-string world sheet to the complex plane, and where the charges $Q_{j,m} = \oint \frac{dz}{2\pi i}\Psi_{j,m}(z)$ generate the standard $w(\infty)$ algebra:

$$[Q_{j_1,m_1}, Q_{j_2,m_2}] = (j_1 m_2 - j_2 m_1) Q_{j_1+j_2+1,m_1+m_2} \ . \qquad (3)$$

To construct the string field theory action for noncritical strings, we first begin with the nonpolynomial closed string action of the 26-dimensional string theory, first written down by the author [12] and the Kyoto and MIT groups [13–15]:

$$\mathcal{L} = \langle \Phi|Q|\Phi\rangle + \sum_{n=3}^{\infty} \alpha_n \langle \Phi^n \rangle \ , \qquad (4)$$

 

where $Q = Q_0(b_0 - \bar{b}_0)$, $Q_0$ is the usual Becchi-Rouet-Stora-Tyutin (BRST) operator, and the field $\Phi$ transforms as

$$\delta|\Phi\rangle = |Q\Lambda\rangle + \sum_{n=1}^{\infty} \beta_n |\Phi^n \Lambda\rangle \ , \tag{5}$$

where $n$ labels the number of faces of the polyhedra, and there are more than one distinct polyhedra at each level. For example, there are 2 polyhedra at $N = 6$, 5 polyhedra at $N = 7$, and 14 polyhedra at $N = 8$ [12].

If we insert $\delta|\Phi\rangle$ into the action, we find that the result does not vanish, unless

$$(-1)^n \langle\Phi||Q\Lambda\rangle + n\langle Q\Phi||\Phi^{n-1}\Lambda\rangle + \sum_{p=1}^{n-2} C_p^n \langle\Phi^{n-p}||\Phi^p\Lambda\rangle = 0 \ , \tag{6}$$

where the double bars mean that when we join two polyhedra, the common boundary has circumference $2\pi$. The meaning of this equation is rather simple. The first two terms on the left-hand side represent the action of $\sum_i Q_i$ on the vertex function. Naively, we expect the sum of these two terms to vanish. However, naive BRST invariance is broken by the third term, which has an important interpretation. This third term consists of polyhedra with rather special parameters; i.e., they are polyhedra which are at the end points of the modular region. Thus, these polyhedra are actually composites; they can be split in half, into two smaller polyhedra, such that the boundary of contact is $2\pi$. This is the meaning of the double bars.

[This action also has additional quantum corrections because the measure of integration $D\Phi(X)$ is not gauge invariant. These quantum corrections can be explicitly solved in terms of a recursion relation. These corrections can be computed either by calculating these loop corrections to the measure [16], or by using the Batalin-Vilkovisky (BV) quantization method [17].]

If strings have equal parametrization length $2\pi$, then we must triangulate moduli space with cylinders of equal circumference but arbitrary extension, independent of the dimension of space-time. Thus, the triangulation of moduli space on Riemann surfaces remains the same in any dimension $D$. Therefore, the basic structure of the action remains the same for subcritical strings with equal parametrization length.

What is different, of course, is that the string degrees of freedom have changed drastically, and a Liouville field $\phi$ must be introduced. The addition of the Liouville theory complicates the proof of gauge invariance considerably, however, since this field must be inserted at curvature singularities within the vertex functions, i.e., at the corners of the polyhedra. This means that the standard proof of gauge invariance formally breaks down, and hence must be redone.

This raises a problem, since the explicit cancellation of these anomalies has only been performed for polynomial string field theory actions, not the nonpolynomial one. In particular, the anomaly cancellation of the Witten field theory depends crucially on knowledge of the specific numerical value of the Neumann functions. However, the Neumann functions of the nonpolynomial field theory are only defined formally. Explicit forms for them are not known. Thus, it appears that the cancellation of anomalies seems impossible.

However, we will use point-splitting methods, pioneered in [18–21], which have the advantage that we can isolate those points on the world sheet where these insertion operators must be placed, and hence only need to calculate the anomaly at these insertion points. Thus, we do not need to have an explicit form for the Neumann functions; we need only certain identities which these Neumann functions obey. The great advantage of the point-splitting method, therefore, is that we can show BRST invariance to all orders in polyhedra, without having to have explicit expressions for the Neumann functions. As an added check, we will calculate the anomaly in two ways, using both bosonized and unbosonized ghost variables.

Thus, we will first calculate the anomaly contribution, isolating the potential divergences coming from the insertion points, and show that they sum to zero. Then we will show that our theory reproduces the standard shifted Shapiro-Virasoro amplitude.

## II. BRST INVARIANCE OF VERTICES

We will specify our conventions by introducing a field which combines the string variable $X^i$ (where $i$ labels the Lorentz index) and the Liouville field $\phi$. We introduce $\phi^\mu$ where $\mu = 0, 1, 2, ..., D$ and where $\phi^D$ corresponds to the Liouville field, so that $\phi^\mu = \{X^i, \phi\}$.

The first-quantized action is given by

$$S = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left\{ g^{ab} \left( \partial_a X^i \partial_b X_i + \partial_a \phi \partial_b \phi \right) + Q\hat{R} \right\} \ . \tag{7}$$

The holomorphic part of the energy-momentum tensor is therefore

$$T^{\phi}_{zz} = -\frac{1}{2} \left(\partial_z \phi^\mu\right)^2 - \frac{Q_\mu}{2} \left(\partial_z^2 \phi^\mu\right) \ ,$$
$$T^{\rm gh}_{zz} = \frac{1}{2} \left(\partial_z \sigma\right)^2 + \frac{3}{2} \left(\partial_z^2 \sigma\right) \ , \tag{8}$$

where we have bosonized the ghost fields via $c = e^\sigma$ and $b = e^{-\sigma}$ and where $Q^\mu = (0, Q)$. Demanding that the central charge of the Virasoro algebra vanish implies that

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0} \ , \tag{9}$$

with total central charge

$$c = D + 1 + 3Q^2 - 26 = 0 \ , \tag{10}$$

so that $Q = 2\sqrt{2}$ for $D = 1$ (or for two dimensions if we promote $\phi$ to a dimension). Notice that the ghost field has a background charge of $-3$ and the $\phi^\mu$ field has a background charge of $Q^\mu = (0, Q)$. This allows us to collectively place the bosonized ghost field and the $\phi^\mu$ field together into one field. We will use the index $M$ when referring to the collective combination of the string variable, the Liouville field, and the bosonized field. We will define

$$Q^M = \{0, Q, -3\} ,$$
$$\phi^M = \{X^i, \phi, \sigma\} . \tag{11}$$

To calculate the insertion factors in the vertex function, we must analyze the terms in the first-quantized action proportional to the background charge:

$$\frac{Q^M}{8\pi} \int \sqrt{g} R \, \phi^M d^2\xi , \tag{12}$$

where we have normalized the curvature on the world sheet such that $\int \sqrt{g} R d^2\xi = 4\pi\chi$, where $\chi$ is the Euler number. In general, the curvature on the string world sheet is zero, except at isolated points where the strings join. At these interior points, the curvature is a $\delta$ function, such that $\int \sqrt{g} R d^2\xi = -2\pi$ around these points. This means that $N$-point vertex functions, in general, must have insertions (after rescaling $\phi$)

$$\prod_{j=1}^{2(N-2)} \left(e^{-Q^M \phi^M/2}\right)_j , \tag{13}$$

where $j$ labels the $2(N-2)$ sites where we have curvature singularities on the string world sheet. These insertions, in fact, are the principal complication facing us in calculating the anomalies of the various vertex functions.

The vertex is then defined as

$$|V_N\rangle = \int B_N |V_N\rangle_0 , \tag{14}$$

where $B_N$ consists of line integrals of $b$ operators defined over Beltrami differentials (see the Appendix for conventions for the vertex function) and $|V_N\rangle_0$ is the standard vertex function given as an overlap condition on the string and ghost degrees of freedom which must satisfy the usual BRST condition

$$\sum_{i=1}^{N} Q_i |V_N\rangle_0 = 0 . \tag{15}$$

Notice that it is the presence of this factor $B_N$ which prevents the vertex function from being trivially BRST invariant. The reason for this is that $B_N$ contains line integrals of the $b$ operators, defined over Beltrami differentials $\mu_k$, such that

$$T_{\mu_k} = \{Q, b_{\mu_k}\} . \tag{16}$$

Whenever $Q$ is commuted past a term in $B_N$, it creates an expansion or contraction of some of the modular parameters within the polyhedral vertex function. The deformation generated by $T_{\mu_k}$ is given as a total derivative in the modular parameter $\tau_k$, i.e.,

$$\int d\tau_k T_{\mu_k} \sim \int d\tau_k \frac{\partial}{\partial \tau_k} . \tag{17}$$

When this deformation is integrated over the modular parameter, we find only the end points of the modular region. However, the end points of the modular region are where the polyhedra splits into two smaller polyhedra, connected by a common boundary of $2\pi$. This, in turn, reproduces the residual terms $\langle \Phi^{n-p} || \Phi^p \Lambda \rangle$ appearing in Eq. (6) which violate naive BRST invariance. Thus, the importance of this $B_N$ term is that it gives the corrections to the naive BRST invariance equations. (The terms which violate naive BRST invariance, corresponding to the boundary of the region of integration, were constructed in [12], as shown in Eq. (6). However, these $B_N$ factors were first constructed in detail in Ref. [14], thereby completing the proof of classical gauge invariance of the theory.)

Fortunately, the factor $B_N$ remains the same even for the subcritical case independent of the dimension of space-time. Therefore, we can ignore this term and shall concentrate instead on the properties of $|V_N\rangle_0$, which is defined as

$$|V_N\rangle_0 = \left( \prod_{j=1}^{2(N-2)} e^{-(Q^M \phi^M/2)_j} \right) \int \delta\left( \sum_{i=1}^{N} p_i^M + Q^M \right) \prod_{i=1}^{N} P_i$$
$$\times \exp\left\{ \sum_{r,s}^{N} \sum_{n,m=0}^{\infty} \frac{1}{2} N_{nm}^{rs} \alpha_{-n}^{Mr} \alpha_{-m}^{Ms} \right\} \exp\left\{ \sum_{r,s}^{N} \sum_{n,m=0}^{\infty} \frac{1}{2} N_{nm}^{rs} \tilde{\alpha}_{-n}^{Mr} \tilde{\alpha}_{-m}^{Ms} \right\} \left( \prod_{i=1}^{N} d^M p_i |p_i^M\rangle \right) , \tag{18}$$

where $P_i$ represents the operator which rotates the string field by $\theta$ and then averages it from zero to $2\pi$, where $j$ labels the insertion points, where we have deliberately dropped an uninteresting constant, and where the state vector $|p_i^M\rangle$ and the Neumann functions are defined in the Appendix.

For our calculation, we would like to commute the insertion operator directly into the vertex function. Performing the commutation, we find (for $N = 3$)

$$|V_3\rangle_0 = \int \delta(p_1^M + p_2^M + p_3^M + Q^M) \prod_{i=1}^{3} P_i \exp\left\{ \sum_{r,s}^{3} \sum_{n,m=0}^{\infty} \frac{1}{2} N_{nm}^{rs} \alpha_{-n}^{Mr} \alpha_{-m}^{Ns} - \frac{1}{3} \frac{\bar{Q}^M}{2} \left[ \sum_{r=1}^{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \alpha_{-2n}^{Mr} + \sum_{r,s}^{3} N_{00}^{rs} \alpha_0^{Mr} \right.\right.$$
$$\left.\left. + \sum_{r,s}^{3} \sum_{n=1}^{\infty} N_{n0}^{rs} \alpha_{-n}^{Mr} - \sum_{n=1}^{\infty} \sum_{r,s}^{3} N_{nm}^{rs} \alpha_{-n}^{Mr} \cos m\pi/2 \right] \right\} \left( \prod_{i=1}^{3} d^M p_i |p_i\rangle \right) , \tag{19}$$

where $\bar{Q}^M = \{0, -iQ, -3\}$. The factor 1/3 appearing before the background charge arises because we have broken up the insertion operator into three equal pieces, each defined in terms of the three different harmonic oscillators. (For simplicity, we have only presented the holomorphic part of the vertex function, and deleted $\tilde{\alpha}$ operators for convenience. It is understood that all vertex functions contain both the $\alpha$ and $\tilde{\alpha}$ operators.)

With these conventions, we now wish to show that the vertices of the nonpolynomial theory are BRST covariant. For the three-string vertex, this means $\sum_{i=1}^{3} Q_i|V_3\rangle_0 = 0$.

Naively, this calculation appears to be trivial, since the vertex function simply represents a $\delta$ function across three overlapping strings. Hence, we expect that the three contributions to $Q$ cancel exactly. However, this calculation is actually rather delicate, since there are potentially anomalous contributions at the joining points.

Previous calculations of this identity were limited by the fact that they used specific information about the three-string vertex function. We would like to use a more general method which will apply for the arbitrary $N$-string vertex function. The most general method uses point splitting.

We wish to construct a conformal map from the multisheeted, three-string world sheet configuration in the $\rho$ plane to the flat, complex $z$ plane. Fortunately, this map was constructed in [12]:

$$\frac{d\rho(z)}{dz} = C\frac{\prod_{i=1}^{N-2}\sqrt{(z-z_i)(z-\tilde{z}_i)}}{\prod_{i=1}^{N}(z-\gamma_i)} , \tag{20}$$

where the $N$ variables $\gamma_i$ map to points at infinity (the external lines in the $\rho$ plane) and the $N-2$ pair of variables $(z_i, \tilde{z}_i)$ map to the points where two strings collide, creating the $i$th vertex (which are interior points in the $\rho$ plane).

The set of complex numbers $\{C, z_i, \tilde{z}_i, \gamma_i\}$ constitutes an initial set of $2 + 4(N-2) + 2N = 6N - 6$ unknowns. In order to achieve the correct counting, we must impose a number of constraints. First, we must set the length of the external strings at infinity to be $\pm\pi$. In the limit where $z \to \gamma_i$, we have

$$\lim \frac{d\rho(z)}{dz} \to \frac{\pm 1}{z-\gamma_i} . \tag{21}$$

This gives us $2N$ constraints. However, by projective invariance we have the freedom of selecting three of the $\gamma_i$ to be $\{0, 1, \infty\}$. Then we must subtract two, because of overall charge conservation (taking into account that there are charges due to the Riemann cuts as well as charges located at $\gamma_i$.) Thus we have $2N+6-2 = 2N+4$ constraints.

Next, we must impose the fact that the overlap of two colliding strings at the $i$th vertex is given by $\pi$, such that the interaction takes place instantly in proper time $\tau$. This gives us

$$\pm i\pi = \rho(z_i) - \rho(\tilde{z}_i) . \tag{22}$$

This gives us $2(N-2)$ additional constraints, for a total of $4N$ constraints. Thus, the number of variables minus the number of constraints is given by $2N-6$. But this is precisely the number of Koba-Nielsen variables necessary to describe $N$-string scattering, or the number of moduli necessary to describe a Riemann surface with $N$ punctures consisting of cylinders of equal circumference and arbitrary extension.

These moduli can be described in terms of the proper time $\tau$ separating the $i$th and $j$th vertices, as well as the angle $\theta$ separating them.

We can define

$$\hat{\tau}_{ij} = \tau_{ij} + i\theta_{ij} = \rho(z_i) - \rho(z_j) , \tag{23}$$

where $\tau_{ij}$ is the proper time separating the $i$th and $j$th vertices, and $\theta_{ij}$ is the relative angle between them.

There are precisely $2N-6$ independent variables contained within the $\hat{\tau}_{ij}$, as expected. (Not all the $\hat{\tau}_{ij}$ are independent.)

In summary, we find that

$$\begin{aligned} \{C, z_i, \tilde{z}_i, \gamma_i\} &\to 6N-6\,\text{unknowns} , \\ \{\rho'(\gamma_i), \rho(z_i) - \rho(\tilde{z}_i)\} &\to 4N\,\text{constraints} , \\ \{\tau_{ij} + i\theta_{ij}\} &\to 2N-6\,\text{moduli} . \end{aligned} \tag{24}$$

The conformal map, with these constraints, describes $N$-point scattering consisting of three-string vertices only. This is not enough to cover all of moduli space. In addition, we find a "missing region" [22]. For example, we must include the $2N-6$ moduli necessary to describe the lengths of the sides of an $N$-sided polyhedra. The moduli describing the various polyhedra are specified by setting $\tau_{ij}$ all equal to each other. In other words, on the world sheet, the polyhedral interaction takes place instantly in $\tau$ space. Then the $2N-6$ variables necessary to describe the polyhedra can be found among the $\theta_{ij}$.

Now that we have specified the conformal map, we can begin the calculation of the BRST invariance of the vertex functions.

First, we will find it convenient to transform the BRST operator $Q$ into a line integral over the $\rho$ plane. For the three-point vertex function, we have three line integrals which, for the most part, cancel each other out (because of the continuity equations across the vertex function). The only terms which do not vanish are the ones which encircle the joining points $z_i$ and $\tilde{z}_i$.

Written as a line integral, the BRST condition becomes

$$\begin{aligned} \sum_{i=1}^{3} Q_i|V_3\rangle_0 = \oint_{C_1+C_2+C_3} & \frac{d\rho}{2\pi} c(\rho) \\ & \times\left\{-\frac{1}{2}(\partial_z\phi^\mu)^2 + \frac{dc}{d\rho}b(\rho)\right. \\ & \left.+\frac{Q}{2}(\partial_z^2\phi)\right\}|V_3\rangle_0 , \end{aligned} \tag{25}$$

where $C_i$ are infinitesimal curves which together comprise circles which go around $\rho(z_i)$ and $\rho(\tilde{z}_i)$. Notice that this expression is, strictly speaking, divergent because they are defined at the joining point $z_i$, where these quantities, in general, diverge. To isolate the anomaly, we will now make a conformal transformation from the $\rho$ plane to the $z$ plane. When two operators are defined at the same point, as in $(\partial_z\phi^\mu)^2$, we will point split them by introducing another variable $z'$ which is infinitesimally close to $z$. Then our expression becomes

$$\sum_{i=1}^{3} Q_i|V_3\rangle_0 = \oint_{C_1+C_2+C_3} \frac{dz}{2\pi i} c(z) \times\left\{ -\frac{1}{2}\left(\frac{dz'}{dz}\right)\partial\phi_\mu(z')\partial\phi^\mu(z) + \left(\frac{dz'}{dz}\right)^2 \frac{dc}{dz} b(z') + \frac{Q}{2}\partial^2\phi(z)\right\}\left|V_3\right\rangle_0 , \tag{26}$$

where $z'$ is infinitesimally close to $z$, where $\mu$ ranges over the $D$-dimensional string modes as well as the $\phi$ mode, where $b$ and $c$ are the usual reparametrization ghosts, and the $C_i$ are now infinitesimally small curves in the $z$ plane which encircle the joining point, which we call $z_0$. In making the transition from the $\rho$ plane to the $z$ plane, we have made the redefinition

$$c(\rho) = \frac{d\rho}{dz}c(z), \quad b(\rho) = \left(\frac{d\rho}{dz}\right)^{-2} b(z) . \tag{27}$$

The major complication to this calculation is that the Liouville $\phi$ field does not transform as a scalar. Instead, it transforms as

$$\phi(\rho) \to \phi(z) + \frac{Q}{2}\ln\left|\frac{dz}{d\rho}\right| . \tag{28}$$

This means that the energy-momentum tensor $T$ transforms as

$$T_{\rho\rho} \to \left(\frac{dz}{d\rho}\right)^2 T_{zz} + \left(\frac{Q}{2}\right)^2 S , \qquad S = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 , \tag{29}$$

where $S$ is called the Schwartzian. The form of the Schwartzian that is most crucial for our discussion will be

$$\left(\frac{Q}{2}\right)^2 S = T^\phi_{zz}\left(\partial_z\phi \to \frac{Q}{2}\partial_z \ln\left|\frac{dz}{d\rho}\right|\right) . \tag{30}$$

This complicates the calculation considerably, since it means that there are subtle insertion factors located at $\delta$-function curvature singularities in the vertex function. These add nontrivial $\phi$ contributions to the calculation.

## III. POINT SPLITTING

In order to perform this sensitive calculation, we will use the method of point splitting.

Let us examine the behavior of the various variables near the splitting point $z_0$ using the original conformal map in Eq. (20). Near this point, we have

$$\frac{d\rho}{dz} = a\sqrt{z-z_0} + b(z-z_0)^{3/2} + \cdots ,$$
$$\rho(z) = \rho(z_0) + \frac{2}{3}a(z-z_0)^{3/2}\left(1 + \frac{3}{5}\frac{b}{a}(z-z_0) + \cdots\right) . \tag{31}$$

Now let us define $\epsilon = z - z_0$ and power expand these functions for small $\epsilon$. For the purpose of point splitting, we introduce the variable $z'$, which is infinitesimally close to both $z$ and $z_0$, and is defined implicitly through the equation

$$\rho(z') = \rho(z) + \frac{2}{3}a\delta , \tag{32}$$

where $\delta$ is an infinitesimally small constant, which we will later set equal to zero.

We will find it convenient to introduce the following function:

$$f(\epsilon) = z' - z_0 = \epsilon\left\{1 + \sum_{n=1}^{\infty} f_n(\epsilon)\delta^n\right\} . \tag{33}$$

We can easily solve for the coefficients $f_n$ by power expanding the following equation:

$$\begin{aligned}\rho(z') - \rho(z) &= \frac{2}{3}a\delta \\ &= \frac{2}{3}a\left[f^{3/2}(\epsilon) - \epsilon^{3/2}\right] + \frac{2b}{5}\left[f^{5/2}(\epsilon) - \epsilon^{5/2}\right] \\ &\quad + \cdots .\end{aligned} \tag{34}$$

By equating the coefficients of $\delta$, we find

$$\begin{aligned} f_1 &= \frac{2}{3}\epsilon^{-3/2}(1 - p\epsilon) + \cdots , \\ f_2 &= -\frac{1}{9}\epsilon^{-3} + \cdots , \\ f_3 &= \frac{4}{81}\epsilon^{-9/2}(1-p\epsilon) + \cdots , \end{aligned} \tag{35}$$

where $p = b/a$.

In our calculation, we will find potentially divergent quantities, such as $1/(z'-z)$ and $dz'/dz$, and so we will power expand all these quantities in terms of $f_n$ in a double power expansion in $\epsilon$ and $\delta$.

Then we easily find

$$\begin{aligned} \frac{1}{z'-z} &= \frac{1}{f(\epsilon) - \epsilon} \\ &= \frac{1}{\epsilon f_1 \delta}\left[1 - \frac{f_2}{f_1}\delta + \delta^2\left(\frac{f_2^2}{f_1^2} - \frac{f_3}{f_1}\right) + \cdots\right] , \\ \frac{1}{(z'-z)^2} &= \frac{1}{\epsilon^2 f_1^2 \delta^2}\left[1 - 2\delta\frac{f_2}{f_1} + \delta^2\left(-2\frac{f_3}{f_1} + 3\frac{f_2^2}{f_1^2}\right) + \cdots\right] . \end{aligned} \tag{36}$$

Also,

$$\frac{dz'}{dz} = \frac{df(\epsilon)}{dz} = 1 + \sum_{n=1}^{\infty} \delta^n (\epsilon f_n)' . \tag{37}$$

We also find

$$\frac{dz'}{dz}\frac{1}{(z'-z)^2} = \epsilon^{-2} f_1^{-2}\Big[(\epsilon f_2)' - 2f_3 f_1' + 3f_2^2 f_1^{-2} + (\epsilon f_1)'(-2f_2 f_1^{-1}) + \cdots\Big] = \epsilon^{-2}\left(\frac{5}{48} + \frac{p\epsilon}{12}\right) + \cdots , \tag{38}$$

$$\left(\frac{dz'}{dz}\right)^2 \frac{1}{(z'-z)^2} = \epsilon^{-2} f_1^{-2}\Big[-4(\epsilon f_1)' f_2 f_1^{-1} + (\epsilon f_1)' + 2(\epsilon f_2)' - 2f_3 f_1^{-1} + 3f_2^2 f_1^{-2} + \cdots\Big] = \epsilon^{-2}\left(\frac{29}{48} + \frac{13}{12}p\epsilon\right) + \cdots , \tag{39}$$

$$\left(\frac{dz'}{dz}\right)^2 \frac{1}{z'-z} = \epsilon^{-1} f_1^{-1}\left[-f_2 f_1^{-1} + 2(\epsilon f_1)'\right] = -\frac{3}{4}\epsilon^{-1} + \cdots . \tag{40}$$

The terms contained in the ellipses correspond to terms which can be discarded in our approximation. We will take the limit such that $\epsilon$ is taken to be small but finite, such that $\delta < \epsilon$. [We maintain this constraint because otherwise, the points $z$ and $z'$, which originally belong to a single string, will belong to two different strings. In this limit, we will find terms like $\delta^{-2}$, which apparently diverge. However, these divergent terms can be shown to cancel if we carefully normal order the BRST operator. Once we adopt this normal ordering, we find that such terms vanish, as expected. For details, see Ref. [20]. Also, there is a Riemann cut in the map in Eq. (20), and so we will choose the regularization scheme in Ref. [21]. We can do this by altering Eq. (32) slightly. We can define our point splitting by reexpressing our operators in terms of two new variables, $z_1$ and $z_2$, such that $\rho(z_1) = \rho(z) + (2/3)a\delta$ and $\rho(z_2) = \rho(z) - (2/3)a\delta$. Then operators are defined in terms of averaging over $z_1$ and $z_2$. This averaging corresponds to choosing $\rho(z') = \rho(z) + (2/3)a\delta$ and then discarding odd powers of $\delta$.]

Now that we have defined all our regularized expressions, we can begin the process of calculating the anomaly. Let us first analyze the anomaly coming from the term $\partial_{z'}\phi(z')\partial\phi(z)$. We will commute this expression into the Neumann functions. We will then extract from this a $c$-number expression which represents the anomaly.

When we put this term into the vertex function, we pick up quantities which look like $nmN^{rs}_{nm}\omega_r^n\tilde{\omega}_s^m$, where $\omega = e^{\zeta}$, where, following Mandelstam, we take $\zeta$ to be a local variable defined on the closed string, such that $\zeta = \tau + i\sigma$. $\zeta$ and $\rho$ coincide for the closed string lying on the real axis. Fortunately, we know how to calculate this term in terms of $z$ variables.

Let us differentiate the expression in Eq. (A14) in the Appendix:

$$\frac{d}{d\zeta_s}N(\rho_r,\tilde{\rho}_s) = \delta_{rs}\left\{\frac{1}{2}\sum_{n\geq 1}\omega_r^{-n}(\tilde{\omega}_s^n + \tilde{\omega}_s^{*n}) + 1\right\} + \frac{1}{2}\sum_{n,m\geq 0} nN^{rs}_{nm}\omega_r^n(\tilde{\omega}_s^m + \tilde{\omega}_s^{*m}) = \frac{1}{2}\frac{dz}{d\zeta_r}\left(\frac{1}{z-\tilde{z}} + \frac{1}{z-\tilde{z}^*}\right) \tag{41}$$

and (by letting $\tilde{z}_s$ go to $\gamma_s$)

$$\delta_{rs} + \sum_{n\geq 1} nN^{rs}_{n0}\omega_r^n = \frac{dz}{d\zeta_r}\frac{1}{z-\gamma_s} . \tag{42}$$

A double differentiation leads to:

$$\frac{d}{d\zeta_r}\frac{d}{d\tilde{\zeta}_s}N(\rho_r,\tilde{\rho}_s) = \delta_{rs}\frac{1}{2}\sum_{n\geq 1} n\omega_r^{-n}\tilde{\omega}_s^n + \frac{1}{2}\sum_{n,m\geq 1} nmN^{rs}_{nm}\omega_r^s\tilde{\omega}_s^m = \frac{1}{2}\frac{dz_r}{d\zeta_r}\frac{d\tilde{z}_s}{d\tilde{\zeta}_s}\frac{1}{(z-\tilde{z})^2} . \tag{43}$$

We will now perform the calculation in two ways, using unbosonized ghost variables $b$ and $c$, and then using the bosonized ghost variable $\sigma$.

### A. Method I: Unbosonized ghosts

With these identities, it is now an easy matter to calculate the action of the BRST operator on the vertex function in terms of unbosonized ghost variables $b$ and $c$. Let the brackets $\langle\ \rangle$ represent the $c$-number expression that we obtain when we perform this commutation. Then we can show that the background-independent terms yield

$$\left\langle \partial_z'\phi^\mu(z')\partial_z\phi(z)^\nu\right\rangle = -\frac{1}{(z'-z)^2}\delta^{\mu\nu} ,$$
$$\left\langle c(z)b(z')\right\rangle = -\frac{1}{z'-z} . \tag{44}$$

With these expressions, we can now calculate the contribution to the anomaly due to $\partial_{z'}\phi(z')\partial_z\phi(z)$ and $\partial_z c(z)b(z')$. This calculation is simplified because the ghost insertion factor disappears in the $b$-$c$ formalism.

We find (dropping the background-dependent terms, for the moment)

$$\sum_{i=1}^{3} Q_i|V_3\rangle_0 = \oint_{C_1+C_2+C_3} \frac{dz}{4\pi i}\Bigg\{ c(z)\Big[ -\frac{dz'}{dz}\Big\langle \partial_{z'}\phi^\mu(z')\partial_z\phi^\mu(z)\Big\rangle + 2\left(\frac{dz'}{dz}\right)^2 \Big\langle \frac{dc}{dz}(z)b(z')\Big\rangle\Big] - 2\frac{dc}{dz}\left(\frac{dz'}{dz}\right)^2 \Big\langle c(z)b(z')\Big\rangle \Bigg\}|V_3\rangle_0 + \cdots$$
$$= -\oint_{C_1+C_2+C_3} \frac{dz}{4\pi i}\Bigg\{ c(z)\Big[ -\frac{dz'}{dz}\frac{1}{(z'-z)^2} + 2\left(\frac{dz'}{dz}\right)^2 \partial_z \frac{1}{z'-z}\Big] - 2\frac{dc}{dz}\left(\frac{dz'}{dz}\right)^2 \frac{1}{z'-z}\Bigg\}|V_3\rangle_0 + \ldots$$
$$= -\oint_{C_1+C_2+C_3} \frac{dz}{4\pi i}\Bigg\{ c(z)\Big[ -\epsilon^{-2}\left(\frac{5}{48}+\frac{p\epsilon}{12}\right) + 2\epsilon^{-2}\left(\frac{29}{48}+\frac{13}{12}p\epsilon\right)\Big] - 2\frac{dc}{dz}\left(-\frac{3}{4}\epsilon^{-1}\right)\Bigg\}|V_3\rangle_0 + \cdots , \tag{45}$$

where the ellipses are terms which are background dependent. Now let us combine the three arcs $C_i$ into one circle which goes around the joining point $z_0$. Integrating by parts, we find

$$\sum_{i=1}^{3} Q_i|V_3\rangle_0 = \oint_{z_0} \frac{dz}{2\pi i}\Bigg\{ \frac{pc(z)}{z-z_0}\left[\frac{D}{24}+\frac{1}{24}-\frac{13}{24}\right] + \frac{dc(z)}{dz}\frac{1}{z-z_0}\left[\frac{5D}{96}+\frac{5}{96}-\frac{65}{48}\right]\Bigg\}|V_3\rangle_0 + \cdots . \tag{46}$$

The last part of the calculation is perhaps the most crucial, i.e., calculating the contribution of the term $\partial_z\phi$ to the anomalies which are background dependent. Normally, this term does not contribute at all. However, in the presence of the insertion operator at the joining points $z_i$ and $\tilde{z}_i$, this term does in fact contribute an important part to the anomaly.

Our task is to put the operator $\partial_z\phi$ into the vertex function and calculate terms proportional to the background charge $Q$. We find

$$\partial_\rho\phi_r(\rho)|V_3\rangle_0 = (-i)^2\frac{Q}{2}\left(\sum_{n,m\geq 0}\sum_r \omega_r^n N_{nm}^{rs}\cos(m\pi/2) - \sum_{n\geq 1}\sum_r \omega_r^n N_{n0}^{rs}\right)|V_3\rangle_0 + \cdots . \tag{47}$$

We immediately recognize the terms on the right as being functions of $1/(z-z_i)$ and $1/(z-\gamma_i)$ in Eqs. (41) and (42) for the case $r \neq s$.

The contribution of the anomaly from the insertion operator is therefore given by

$$\Big\langle \partial_\rho\phi(\rho)\Big\rangle = -\frac{Q}{2}\left[\frac{1}{2}\sum_{i=1}^{M-2}\left(\frac{1}{z-z_i}+\frac{1}{z-\tilde{z}_i}\right) - \sum_{i=1}^{M}\frac{1}{z-\gamma_i}\right] = \frac{dz}{d\rho}\frac{Q}{2}\partial_z \ln\left|\frac{dz}{d\rho}\right| . \tag{48}$$

[As an added check on the correctness of this calculation, notice that the last step reproduces the desired transformation property of the $\phi$ field in Eq. (28), which has an additional contribution due to the background charge $Q$. Thus, when we insert this term into the expression for the energy-momentum tensor, we simply reproduce the Schwartzian.]

Given this expression, we can now calculate the contribution of the background-dependent terms to the anomaly. This contribution is

$$\cdots = \oint_{z_0}\frac{dz}{2\pi i}c(z)\left[-\frac{1}{2}\Big\langle\partial_z\phi\Big\rangle^2 + \frac{1}{2}Q\frac{d\rho}{dz}\partial_z\left(\frac{dz}{d\rho}\Big\langle\partial_z\phi\Big\rangle\right)\right]|V_3\rangle_0$$
$$= \oint_{z_0}\frac{dz}{2\pi i}c(z)\frac{Q^2}{4}\left(\frac{5}{8}\frac{1}{(z-z_0)^2}+\frac{1}{2}p\frac{1}{z-z_0}\right)|V_3\rangle_0 . \tag{49}$$

The last step is to put all terms together. Combining the results of Eqs. (46) and (49), we now easily find

$$\Bigg\{ pc(z_0)\left[\frac{D}{24}-\frac{13}{12}+\frac{1}{24}+\frac{1}{8}Q^2\right] + \frac{dc(z_0)}{dz}\left[\frac{5D}{96}-\frac{65}{48}+\frac{5}{96}+\frac{5}{32}Q^2\right]\Bigg\}|V_3\rangle_0 , \tag{50}$$

which cancels if

$$D - 26 + 1 + 3Q^2 = 0 , \tag{51}$$

which is precisely the consistency equation for Liouville theory in $D$ dimensions. Thus, the vertex is BRST invariant.

### B. Method II: Bosonized ghosts

Next, we will show that the calculation can also be performed using the bozonized ghost variable $\sigma$. We exploit the fact that the $X$, $\phi$, and $\sigma$ fields can be arranged in the same composite field $\phi^M$.

When we commute $\partial_z\phi^M$ into the vertex function, we find that the $\sigma$ ghost variables contribute an almost identical contribution as the $\phi$ variable.

Let us redo the calculation in two parts. We will calculate the background-independent terms first. This means dropping the $b$ and $c$ terms in Eq. (45) and replacing the $\phi^\mu$ field by $\phi^M$. The calculation is straightforward, and yields

$$\sum_{i=1}^{3} Q_i|V_3\rangle_0 = \oint_{z_0}\frac{dz}{2\pi i}\left\{\frac{pe^\sigma(z)}{z-z_0}\left[\frac{(D+2)}{24}\right] + \frac{de^\sigma(z)}{dz}\frac{1}{z-z_0}\left[\frac{5(D+2)}{96}\right]\right\}|V_3\rangle_0 + \cdots . \tag{52}$$

Next, we must calculate the background-dependent terms. We can generalize the equation which determines how the fields change when they are commuted past the insertion operators:

$$\left\langle \partial_\rho\phi^M(\rho)\right\rangle = \frac{dz}{d\rho}\frac{Q^M}{2}\partial_z \ln\left|\frac{dz}{d\rho}\right| . \tag{53}$$

The crucial complication is that the quadratic term in the energy-momentum tensor in Eq. (8) for the $\phi$ field and the $\sigma$ field differs by a factor of $-1$. This means that when we insert this expression into the BRST operator $Q$, we pick up an extra $-1$ factor, and so the contribution to the anomaly from the background-dependent terms now becomes

$$\begin{aligned}\cdots &= \oint_{z_0}\frac{dz}{2\pi i}e^{\sigma(z)}\left[\pm\frac{1}{2}\left\langle\partial_z\phi^M\right\rangle^2 + \frac{1}{2}Q^M\frac{d\rho}{dz}\partial_z\left(\frac{dz}{d\rho}\left\langle\partial_z\phi^M\right\rangle\right)\right]|V_3\rangle_0 \\ &= \oint_{z_0}\frac{dz}{2\pi i}e^\sigma(z)\frac{Q^2-3^2}{4}\left(\frac{5}{8}\frac{1}{(z-z_0)^2} + \frac{1}{2}p\frac{1}{z-z_0}\right)|V_3\rangle_0 ,\end{aligned} \tag{54}$$

where the $-$ $(+)$ sign appears with the $\phi$ $(\sigma)$ operator.

Now, let us put all the terms together in the calculation. We find

$$\left\{pe^{\sigma(z_0)}\left[\frac{D+2}{24}+\frac{1}{8}(Q^2-3^2)\right] + \frac{de^{\sigma(z_0)}}{dz}\left[\frac{5(D+2)}{96} + \frac{5}{32}(Q^2-3^2)\right]\right\}|V_3\rangle_0 . \tag{55}$$

Once again, we find that the anomaly cancels if we set

$$D+2+3(Q^2-3^2)=0 , \tag{56}$$

as desired. Thus, the anomaly cancels in both the bosonized and the unbosonized expressions, although each expression is qualitatively quite dissimilar from the other. This is a check on the correctness of our calculations.

Similarly, the anomaly can be canceled for all nonpolynomial vertices. For an $N$-sided polyhedral vertex, we first notice that the BRST operator $Q$, once it is commuted past the various $b$ operators, vanishes on the bare vertex because of the continuity equations, except at the $2(N-2)$ joining points $z_i$ and $\tilde{z}_i$.

Second, we notice that the conformal map around each joining point in Eq. (31) is virtually the same, no matter how complicated the polyhedral vertex function may be. All the messy dependence on the various constraints are buried within $\rho(z_0)$ and $p$. Fortunately, the dependence on these unknown factors cancels out of the calculation. This is why the calculation can be generalized to all polyhedral vertices.

Thus, the calculation of the anomaly cancellation can be performed on each of the various joining points $z_i$ and $\tilde{z}_i$ separately. But since the calculation is basically the same for each of these joining points, we have now shown that all possible polyhedral vertex functions are all anomaly free.

Notice that this proof does not need to know the specific value of the Neumann functions. The entire calculation just depended on knowing the derivatives of Eq. (A14) and how various operators behaved when commuted into the vertex function.

There is one last point. In the final proof of BRST invariance, one has to show that, when sewing $n$-sided polyhedra with $m$-sided polyhedra, we obtain a $(n+m-2)$-sided polyhedra. When performing the functional integration, we obtain rather complicated determinants, which must be shown to cancel. In Ref. [23], it was shown in detail how these determinants cancel for the critical string. We will not present all the details of how this result can be generalized to the subcritical string, but only sketch heuristically that it is correct.

The key observation is that these determinants are nothing but the determinants of the functional integration of the conformal Laplacian $\nabla^2$ over the conformal sheet representing the various polyhedra, raised to a certain power. These functional determinants arise when we integrate over the action given in Eq. (7). For the critical string, the exponent arising from the $X_\mu$ integration is $D$, while the exponent arising from the ghost integration is $-26$, and so the determinant vanishes if $D-26=0$. This is a formal result, but we know from the work of [23] that this heuristic argument, using functional integrals, can be shown to be rigorously correct, using harmonic oscillators.

Similarly, for the subcritical string we can perform the same heuristic argument. If we bosonize the $b, c$ ghosts, then the functional integration over the $\sigma$ field is nearly identical to the functional integration of the $\phi$ field, since both fields appear with a similar conformal action with the additional insertion factor in Eq. (13). The important observation is that the determinant factor for the ghosts (which was already calculated in [23]) is now almost identical to the determinant factor arising from the $\phi$ field. In particular, the exponent now occurs with factor $1 \pm 3(Q^M)^2$. For the $X_\mu$ field, this means a factor of 1 for each of the $D$ degrees of freedom. For the ghost determinant, this means an exponent of $1-3(3^2)$ or $-26$, while for the $\phi$ field this means an exponent of $1+3Q^2$. Adding the $X_\mu$ contribution, the $\sigma$ contribution, and the $\phi$ contribution, we now have a total exponent given by $D+1-3(3^2)+1+3Q^2$. For the determinant term to cancel, this sum must equal zero. But this just reproduces Eq. (56), as expected. Thus, we can ignore the determinant factor in the proof of BRST invariance.

We stress, however, that this argument is heuristic and certainly not a rigorous one.

## IV. SHIFTED SHAPIRO-VIRASORO AMPLITUDE

The next major test of the theory is whether it reproduces the shifted Shapiro-Virasoro amplitude. This calculation is highly nontrivial, since the conformal map between the multisheeted string scattering Riemann sheet to the complex plane is very involved. Unlike the conformal maps found in light cone theory, or even the maps found in Witten's open string theory [24,25], the nonpolynomial theory yields very complicated conformal maps.

Fortunately, for the four-point function, all conformal maps are known exactly, in terms of elliptic functions, and the calculation can be performed [26,27].

For the four-point function, the map in Eq. (20) can be integrated exactly. We use the identity

$$\frac{(z-z_1)(z-\tilde{z}_1)(z-z_2)(z-\tilde{z}_2)}{\prod_{i=1}^4(z-\gamma_i)} = 1 + \sum_{i=1}^4 \frac{A_i}{z-\gamma_i} , \quad (57)$$

where we define $z_i = ia_i + b_i$ and $\tilde{z}_i = -ia_i + b_i$ for *complex* $a_i$ and $b_i$, and

$$A_i = \frac{\left[(\gamma_i - b_1)^2 + a_1^2\right]\left[(\gamma_i - b_2)^2 + a_2^2\right]}{\prod_{j=1, j\neq i}^4 (\gamma_i - \gamma_j)} . \quad (58)$$

Then we can split the integral into two parts, with the result

$$\begin{aligned}
\rho(z) &= \rho_1(z) + \rho_2(z) , \\
\rho_1(z) &= \int_{y_1}^{y} \frac{N dz}{\sqrt{(z-z_1)(z-\tilde{z}_1)(z-z_2)(z-\tilde{z}_2)}} , \\
\rho_2(z) &= \sum_{i=1}^4 \int_{y_1}^{y} \frac{N A_i dz}{(z-\gamma_i)\sqrt{(z-z_1)(z-\tilde{z}_1)(z-z_2)(z-\tilde{z}_2)}} .
\end{aligned} \quad (59)$$

Written in this form, we can now perform all integrals exactly, using third elliptic integrals in Eq. (A18) and Eq. (A20) in the Appendix. It is then easy to show

$$\begin{aligned}
\rho_1(z) &= NgF(\phi, k') = Ng\,\mathrm{tn}^{-1}[\tan\phi, k'] , \\
\rho_2(z) &= \sum_{i=1}^4 \frac{gNA_i}{a_1 + b_1 g_1 - g_1\gamma_i}\left(g_1 F(\phi,k') + \frac{\omega_i - g_1}{1+\omega_i^2}\left[F(\phi,k') + \omega_i^2\Pi(\phi, 1+\omega_i^2, k') + \omega_i(\omega_i^2+1)f_i\right]\right) ,
\end{aligned} \quad (60)$$

where

$$\begin{aligned}
\omega_i &= \frac{a_1 + b_1 g_1 - \gamma_i g_1}{b_1 - a_1 g_1 - \gamma_i} , \\
f_i &= \frac{1}{2}(1+\omega_i^2)^{-1/2}(k^2+\omega_i^2)^{-1/2} \\
&\quad \times \ln\frac{(k^2+\omega_i^2)^{1/2} - (1-\omega_i^2)^{1/2}\mathrm{dn}u}{(k^2+\omega_i^2)^{1/2} + (1+\omega_i^2)^{1/2}\mathrm{dn}u} , \\
\phi &= \arctan\left(\frac{y - b_1 + a_1 g_1}{a_1 + g_1 b_1 - g_1 y}\right) ,
\end{aligned} \quad (61)$$

and where

$$\begin{aligned}
A^2 &= (b_1+b_2)^2 + (a_1+a_2)^2 , \\
B^2 &= (b_1-b_2)^2 + (a_1-a_2)^2 , \\
g_1^2 &= [4a_1^2 - (A-B)^2]/[(A+B)^2 - 4a_1^2] , \\
g &= 2/(A+B) , \\
y_1 &= b_1 - a_1 g_1, \quad k'^2 = 1 - k^2 = 4AB/(A+B)^2 , \\
u &= \mathrm{dn}^{-1}(1 - k'^2 \sin^2\phi) .
\end{aligned} \quad (62)$$

After a certain amount of algebra, this expression simplifies considerably to

$$\begin{aligned}
\rho(z) &= \sum_{i=1}^4 \frac{gNA_i}{a_1 + b_1 g_1 - g_1\gamma_i}\frac{\omega_i - g_1}{1+\omega_i^2} \\
&\quad \times \left[\omega_i^2\Pi(\phi, 1+\omega_i^2, k') + \omega_i(\omega_i^2+1)f_i\right] .
\end{aligned} \quad (63)$$

Now that we have an explicit form for the conformal map from the flat $z$ plane to the $\rho$ plane, in which string scattering takes place; we must next impose the constraint that the overlap between two colliding strings is given by $\pi$. This is satisfied by imposing

$$\begin{aligned}
\pi &= \mathrm{Im}\left[\rho(z_1) - \rho(y_1)\right] \\
&= -\frac{\pi}{2}gN\frac{(\omega_i - g_1)\omega_i}{a_1 + b_1 g_1 - g_1\gamma_i}\sum_{i=1}^4 \frac{A_i\Lambda_0(\beta_i, k)}{\sqrt{(1+\omega_i^2)(k^2+\omega_i^2)}} \\
&= \frac{-\pi}{2}\sum_{i=1}^4 \alpha_i\Lambda_0(\beta_i, k) ,
\end{aligned} \quad (64)$$

where $\alpha_i = NA_i[(\gamma_i - b_1)^2 + a_1^2]^{-1/2}[(\gamma_i - b_2)^2 + a_2^2]^{-1/2}$, where we have used Eq. (A23), where we have set $y = z_1$, so that $\tan\phi = i$, and where $\sin^2\beta_i = (1+\omega_i^2)^{-1}$. We have also used the fact that

$$\Pi(\phi, 1+\omega_i^2, k') = -\frac{1}{2}\pi i \frac{\sqrt{1+\omega_i^2}}{\sqrt{k^2+\omega_i^2}} \frac{\Lambda_0(\beta_i, k) - 1}{\omega_i} \ . \quad (65)$$

Next, we must calculate the separation between the two vertices and the relative angle of rotation between them. The proper time separating the two interactions is given by

$$\begin{aligned}\tau &= \mathrm{Re}\,[\rho(z_2) - \rho(z_1)] \\ &= g\sum_{i=1}^{4} NA_i \frac{\omega_i^2(\omega_i - g_1)\Pi(\pi/2, 1+\omega_i^2, k')}{(a_1 + b_1 g_1 - g_1\omega_i)(1+\omega_i^2)} \\ &= -K(k')\sum_{i=1}^{4}\alpha_i Z(\beta_i, k') \ ,\end{aligned} \quad (66)$$

where we have used Eq. (A24) and the fact that

$$\begin{aligned}\Pi(\phi, 1+\omega_i^2, k') &= \Pi(\phi_2, 1+\omega_i^2, k') - \Pi(\phi_1, 1+\omega_i^2, k') \ , \\ \Pi(\alpha^2, k) &= -\frac{\alpha K Z(\arcsin\alpha^{-1}, k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}} \ ,\end{aligned} \quad (67)$$

and

$$\begin{aligned}&\tan\phi_1 = i, \ \phi_1 = i\infty \ , \\ &\tan\phi_2 = \frac{i}{k}, \ \phi_2 = \arcsin\frac{1}{k'} \ ,\end{aligned} \quad (68)$$

which we can show by setting $y = z_1, z_2$.

Now that we have an explicit form for $\tau$, the next problem is to differentiate it and find the Jacobian of the transformation of $\tau$ to $x$.

By differentiating, we find

$$\begin{aligned}d\tau &= -\sum_{i=1}^{4}\alpha_i \frac{r^2(\beta_i, k')K(k') - E(k')}{r(\beta_i, k')} d\beta_i \\ &= \frac{\pi}{2}K(k)^{-1}\sum_{i=1}^{4}\frac{\alpha_i d\beta_i}{r(\beta_i, k')} \\ &= \frac{\pi N}{2gK(k)}\sum_{i=1}^{4}\frac{d\gamma_i}{\prod_{j=1, j\neq i}(\gamma_i - \gamma_j)} \ ,\end{aligned} \quad (69)$$

where $r(\theta, k') = \sqrt{1 - k'^2\sin^2\theta}$ and where we have used Eq. (A25) in the Appendix. We have also used the fact that the derivative of $\pi$ in Eq. (64) is a constant, and so

$$0 = \sum_{i=1}^{4}\alpha_i \frac{E(k) - k'^2\sin^2\beta_i K(k)}{r(\beta_i, k')} d\beta_i \ . \quad (70)$$

This explicit conformal map allows us to calculate the four-point amplitude. We first write the amplitude in the $\rho$ plane, and then make a conformal map to the $z$ plane. Let the modular parameter be $\hat{\tau} = \tau + i\theta$, where $\tau$ is the distance between the splitting strings, and $\theta$ is the relative rotation. Then, with a fair amount of work, one can find the Jacobian from $\hat{\tau}$ to $\hat{x}$.

Let us define $\hat{x}$ as

$$\hat{x} = \frac{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_4)}{(\gamma_2 - \gamma_4)(\gamma_3 - \gamma_1)} \ , \quad (71)$$

so that

$$d\hat{x} = \hat{x}(1-\hat{x})\frac{(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_4)} d\gamma_1 \ . \quad (72)$$

Putting everything together, we now find

$$\frac{d\hat{\tau}}{d\hat{x}} = -\frac{\pi N}{2K(k)g\hat{x}(1-\hat{x})(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)} \ . \quad (73)$$

If we take only the tachyon component of $|\Phi\rangle$, then the four-point amplitude can be written as

$$\begin{aligned}A_4 &= \langle V_3|\frac{b_0\bar{b}_0}{L_0 + \bar{L}_0 - 2}|V_3\rangle \\ &= \int d\tau d\theta \left\langle V(\infty)V(1)\left(\int_C dz\frac{dz}{dw}b_{zz}\right)\left(\int_C \frac{d\bar{z}}{d\bar{w}}b_{\bar{z}\bar{z}}\right)V(\hat{x})V(0)\right\rangle \\ &= \int d^2\hat{\tau}\left|\exp\left(\sum_i [ip_i\cdot\phi(i) + \epsilon_i\phi(i)]\right)A_G\right|^2 \ ,\end{aligned} \quad (74)$$

where we must sum over all permutations so that we integrate over the entire complex plane, where $b_0$ defined in the $\rho$ plane transforms into $\int_C dz(dz/dw)b_{zz}$ in the $z$ plane, where $C$ is the image in the $z$ plane of a circle in the $\rho$ plane which slices the intermediate closed string, where $V(z) = c(z)\tilde{c}(z)V_0(z)$, where $V_0$ is the tachyon vertex without ghosts, and where the ghost part $A_G$ equals

$$\begin{aligned}A_G &= \int_C \frac{dz}{2\pi i}\frac{dz}{dw}\exp\left\{-\sum_{i\leq j}\langle\sigma_i\sigma_j\rangle + \sum_j\langle\sigma_j\sigma_+(z)\rangle\right\} \\ &= \int_C \frac{dz}{2\pi i}\frac{dz}{dw}\frac{\prod_{i<j}(\gamma_i - \gamma_j)}{\prod_{j=1}^{4}(z - \gamma_j)}\end{aligned} \quad (75)$$

$$= 2\frac{g}{\pi c}\hat{x}(1-\hat{x})K(k)(\gamma_1 - \gamma_3)^3(\gamma_2 - \gamma_4)^3 \ . \quad (76)$$

(Notice that we have made a conformal transformation from the $\rho$ world sheet to the $z$ complex plane. In general, we pick up a determinant factor, proportional to the determinant of the Laplacian defined on the world sheet. However, after making the conformal transformation, we find that the determinant of the Laplacian on the flat $z$ plane reduces to a constant. Thus, we can in general ignore this determinant factor.)

Putting the Jacobian, the ghost integrand, and the string integrand together, we finally find

$$A_4 = \int d^2\hat{x}\left|\hat{x}^{2p_1\cdot p_2}(1-\hat{x})^{2p_2\cdot p_3}\right|^2 \ . \quad (77)$$

In two dimensions, we have $p_i\cdot p_j = p_i p_j - \epsilon_i\epsilon_j$ where $\epsilon_i =$

$\sqrt{2}+\chi_i p_i$, where $\chi$ is the "chirality" of the tachyon state, and so we reproduce the integral found in matrix models and Liouville theory. (The amplitude is nonzero only if the chiralities are all the same except for one external line.)

However, so far the region of integration does *not* cover the entire complex $z$ plane. This is because we have implicitly assumed in the constraints $\hat{\tau}_{ij} = \rho(z_i) - \rho(z_j)$ that there is no four-string interaction. However, as we have shown in [22], the complete region of integration contains a "missing region" which is precisely filled by the four-string interaction. This calculation carries over, without any change, to the $D < 26$ case.

With the missing region filled by the four-string tetrahedron graph, we finally have the complete shifted Shapiro-Virasoro amplitude, as expected.

Last, we would like to mention the direction for possible future work. Two problems come to mind. The most glaring deficiency of this approach is that we have set the cosmological constant to zero. However, the theory becomes quite nonlinear for a nonzero cosmological constant, and so the calculations become much more difficult.

The second problem is that we have not shown the equivalence of this approach to the Das-Jevicki action [28,29], which is the second-quantized field theory of matrix models. This action is based strictly on the tachyon, and so we speculate that, once we gauge away the BRST trivial states and integrate out the discrete states, our action should reduce down to the Das-Jevicki action (for $\mu = 0$). This problem is still being investigated.

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## APPENDIX

We will find it convenient to define the holomorphic expressions for the operators as follows. (It is understood that we must double the operators in order to describe the closed string.) If we define $\phi^M = \{X^i, \phi, \sigma\}$, then

$$\partial_z \phi^M = \sum_{n=-\infty}^{\infty} \{-i\alpha_n^i, -i\phi_n, \sigma_n\} z^{-n-1} , \tag{A1}$$

where

$$[\phi_n^M, \phi_m^N] = n\delta^{MN}\delta_{n,-m} , \tag{A2}$$

where $\delta^{MN} = \mathrm{diag}\,\{\delta^{ij}, 1, 1\}$.

Physical states without ghost indices are defined via the conditions

$$\begin{aligned} L_n\, |\Phi\rangle &= \bar{L}_n|\Phi\rangle = 0 , \\ (L_0 - 1)|\Phi\rangle &= (\bar{L}_0 - 1)|\Phi\rangle = 0 , \\ (L_0 - \bar{L}_0)|\Phi\rangle &= 0 . \end{aligned} \tag{A3}$$

The tachyon state is defined as

$$|p^\mu\rangle = |p^i, \epsilon\rangle = e^{ip\cdot X + \epsilon\phi}(0)|0\rangle , \tag{A4}$$

where $\alpha_0^\mu|p\rangle = p^\mu|p\rangle$.

To solve for $\epsilon$ and the mass of the tachyon, we must solve the on-shell condition

$$L_0|p,\epsilon\rangle = \bar{L}_0|p,\epsilon\rangle = \left(\frac{1}{2}p_i^2 - \frac{1}{2}\epsilon(\epsilon + Q)\right)|p,\epsilon\rangle , \tag{A5}$$

so that

$$p_i^2 - \epsilon(\epsilon + Q) - 2 = 0 . \tag{A6}$$

To put this in more familiar mass-shell form, let us define $E = \epsilon + (1/2)Q$. Thus, the mass-shell condition can be written as

$$p_i^2 - E^2 = -\left(\frac{1}{4}Q^2 - 1\right)^2 = -m^2 , \tag{A7}$$

which defines the tachyon mass. This means that the tachyon mass obeys the relation

$$m^2 = \left(\frac{1-D}{12}\right)^2 . \tag{A8}$$

As a check, we find that this simply reproduces the usual relationship between the tachyon mass and dimension. So therefore the tachyon is massless in $D = 1$ (or in two dimensions, if we consider the Liouville field to be a dimension).

On the other hand, we can solve the mass-shell condition for $\epsilon$ directly, yielding

$$\epsilon = \frac{-Q \pm \sqrt{Q^2 - 8 + 4p_i^2}}{2} . \tag{A9}$$

We shall be mainly interested in the case of two dimensions, or $D = 1$, and so we find $Q = 2\sqrt{2}$ and

$$\epsilon = -\sqrt{2} + \chi p , \tag{A10}$$

where $\chi = \pm 1$ is called the "chirality" of the tachyon state. The ground state, with arbitrary ghost number $\lambda$, can therefore be written as

$$|p,\epsilon,\lambda\rangle = e^{ipX + \epsilon\phi + \lambda\sigma}(0)|0\rangle , \tag{A11}$$

where $\sigma_0|p,\lambda\rangle = \lambda|p,\lambda\rangle$. We will choose $\lambda = 1$ for the ghost vacuum.

Our tachyon state is then defined as $|p^M\rangle = |p,\epsilon,\lambda\rangle$.

In addition, we also have the $b$-$c$ ghost system. We define the SL$(2,R)$ vacuum in the usual way:

$$\langle 0|c_{-1}c_0c_1|0\rangle = 0 \ , \tag{A12}$$

and so the ghost system has background charge $-3$. Then the ghost part of the tachyon field is given by $c_1\bar{c}_1|0\rangle$.

If we let $c_1|0\rangle = |-\rangle$, with ghost number $-1/2$, then the open string wave function is based on the vacua $|-\rangle$ and $c_0|-\rangle = |+\rangle$. For the closed string case, the string wave function $|\Phi\rangle$ is based on four possible vacua, so that

$$|\Phi\rangle = \varphi_{--}|-\rangle|-\rangle + \varphi_{-+}|-\rangle|+\rangle + \varphi_{+-}|+\rangle|-\rangle + \varphi_{++}|+\rangle|+\rangle \ . \tag{A13}$$

With this ground state, we can then construct the vertex functions, once we know the Neumann functions. These can be defined via the Green's function on the string world sheet in the usual way:

$$\begin{aligned} N(\rho_r,\tilde{\rho}_s) &= -\delta_{rs}\left\{\sum_{n\geq 1}\frac{2}{n}e^{-n|\xi_r-\tilde{\xi}_s|}\cos(n\sigma_r)\cos(n\tilde{\sigma}_s) - 2\max(\xi_r,\tilde{\xi}_s)\right\} + 2\sum_{n,m\geq 0}N^{rs}_{nm}e^{n\xi_r+m\tilde{\xi}_s}\cos(n\sigma_r)\cos(n\tilde{\sigma}_s) \\ &= \ln|z-\tilde{z}| + \ln|z-\tilde{z}^*| \ . \end{aligned} \tag{A14}$$

By taking the Fourier transform of the previous equation, one can invert the relation and find an expression for $N^{rs}_{nm}$:

$$\begin{aligned} N^{rs}_{nm} &= \frac{1}{nm}\oint_{z_r}\frac{dz}{2\pi i}\oint_{z_s}\frac{d\tilde{z}}{2\pi i}\frac{1}{(z-\tilde{z})^2}e^{-n\rho_r(z)-m\tilde{\rho}_s(\tilde{z})} \ , \\ N^{rs}_{n0} &= \frac{1}{n}\oint_{z_r}\frac{dz}{2\pi i}\frac{1}{z-z_s}e^{-n\rho_r(z)} \ . \end{aligned} \tag{A15}$$

In addition to these Neumann functions, we must also define the $B_N$ line integrals, which are found in the calculation of any $N$-point tree graph and hence must appear in the vertex function as well.

We have

$$B_N = \prod_{j=1}^{N}(b_0-\bar{b}_0)_j\prod_{k=1}^{2N-6}b_{\mu_k}d\tau_k \ , \tag{A16}$$

where

$$b_{\mu_k} = \int\frac{d^2\xi}{2\pi}\left[\mu_k b(z) + \text{c.c.}\right] \ , \tag{A17}$$

where $\tau_k$ are the modular parameters which specify the polyhedra and where $\mu_k$ are the $2N-6$ Beltrami differentials which correspond to the $2N-6$ quasiconformal deformations which typify how the polyhedral vertex function changes as the moduli parameters $\tau_i$ vary. These $\tau_i$, in turn, are functions of the angles $\theta_{ij}$.

With these Neumann functions, we can construct the four-point scattering amplitude. However, the Jacobian from the world sheet to the complex $z$ plane requires elliptic integrals.

Our conventions are those of Ref. [27]. First elliptic integrals are defined as

$$\begin{aligned} F(\phi,k) &= \int_0^y\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ &= \int_0^\phi\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \\ &= \text{sn}^{-1}(y,k) \ , \end{aligned} \tag{A18}$$

where $y=\sin\phi$ and $\phi = \text{am}\ u_1$.

Second elliptic integrals are defined as

$$\begin{aligned} E(\phi,k) &= \int_0^y\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt \\ &= \int_0^\phi\sqrt{1-k^2\sin^2\theta}d\theta \ . \end{aligned} \tag{A19}$$

Third elliptic integrals are defined as

$$\begin{aligned} \Pi(\phi,\alpha^2,k) &= \int_0^y\frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}} \\ &= \int_0^\phi\frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^{u_1}\frac{du}{1-\alpha^2\text{sn}^2u} \ . \end{aligned} \tag{A20}$$

Complete first elliptic integrals are defined as

$$K(K) = K = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\pi/2,k) \ . \tag{A21}$$

Complete second elliptic integrals are defined as

$$E(\pi/2,k) = E = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta \ . \tag{A22}$$

Heuman's $\lambda$ function is defined as

$$\Lambda_0(\phi,k) = \frac{2}{\pi}[EF(\phi,k') + KE(\phi,k') - KF(\pi,k')] \ . \tag{A23}$$

The Jacobi $\zeta$ function is defined as

$$Z(\phi,k) = E(\phi,k) - \frac{E}{K}F(\phi,k) \ . \tag{A24}$$

In the text, we have used the following differential equations:

$$\begin{aligned} \frac{d}{dk}\left[K(k')Z(\beta_i,k')\right] &= \frac{k'E(K')}{k^2}\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')} \ , \\ \frac{d}{dk}\Lambda_0 &= \frac{2}{\pi k}\left[E(k)-K(k)\right]\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')} \ , \\ \frac{d}{d\beta_i}\left[K(k')Z(\beta_i,k')\right] &= \frac{r^2(\beta_i,k')K(k')-E(k')}{r(\beta_i,k')} \ , \\ \frac{d}{d\beta_i}\Lambda_0(\beta_i,k) &= \frac{2}{\pi r(\beta_i,k')}\left[E(k)-k^{2'}\sin^2\beta_iK(k)\right] \ . \end{aligned} \tag{A25}$$

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