Tree scattering amplitudes of the spin-$\frac{4}{3}$ fractional superstring. II. The twisted sectors

Philip C. Argyres*
School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540

S.-H. Henry Tye
Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853

(Received 22 November 1993)

The spin-$\frac{4}{3}$ fractional superstring is characterized by a world-sheet chiral algebra involving spin-$\frac{4}{3}$ currents. The discussion of the tree-level scattering amplitudes of this theory presented in the preceding paper is expanded to include amplitudes containing two twisted-sector states. These amplitudes are shown to satisfy spurious state decoupling. The restriction to only two external twisted-sector states is due to the absence of an appropriate dimension-one vertex describing the emission of a single twisted-sector state. This is analogous to the "old covariant" formalism of ordinary superstring amplitudes in which an appropriate dimension-one vertex for the emission of a Ramond-sector state is lacking. Examples of tree scattering amplitudes are calculated in a $c = 5$ model of the spin-$\frac{4}{3}$ chiral algebra realized in terms of free bosons on the string world sheet. The target space of this model is three-dimensional flat Minkowski space-time and the twisted-sector physical states are fermions in space-time. Since the critical central charge of the spin-4/3 fractional superstring theory is 10, this $c = 5$ model is not consistent at the string loop level.

PACS number(s): 11.25.Db

I. INTRODUCTION AND SUMMARY

Fractional superstrings [1] are string theories whose physical state conditions are generated by fractional-spin currents on the world sheet. The resulting critical central charges of these strings are found to be less than that of the superstring, suggesting the possibility of string theories with critical space-time dimensions less than 10. This paper is the second of two papers examining the properties of the simplest such fractional superstring, namely, the one with spin-4/3 currents on the world sheet. The first paper in this series [2] described tree-scattering amplitudes of this string for states in the untwisted sectors of the spin-4/3 fractional superconformal algebra, and showed that they obey duality and spurious state decoupling. Also, a specific $c = 5$ (noncritical) model of this string with a three-dimensional space-time interpretation was constructed, where it was shown that the untwisted-sector physical states correspond to space-time bosons including the graviton and Yang-Mills bosons. In Ref. [3] a no-ghost theorem for these $c = 5$ untwisted-sector states was presented.

This paper examines the twisted sectors of the spin-4/3 fractional superstring. In Sec. II we define the twisted sectors in terms of the monodromies of the fractional currents with states in those sectors. We derive general properties that models having twisted sectors must obey using the methods of Refs. [4,5]. It is found that there are two types of twisted sectors, which we denote by R and R', whose occurrence depends on how a $\mathbb{Z}_2$ automorphism of the spin-4/3 fractional superconformal algebra is realized in specific conformal field theory (CFT) models.

In Sec. III we show how scattering amplitudes containing one channel of either R or R' twisted-sector states and satisfying spurious state decoupling can be constructed. Various choices of physical state conditions in the R' sector are consistent with spurious state decoupling in the prescription for scattering amplitudes that we develop. Presumably, demanding duality of four-point amplitudes will further specify the set of R' physical state conditions. The restriction in our scattering prescription to only two external twisted-sector states is due to the absence of an appropriate dimension-one vertex describing the emission of a single twisted-sector state. This is closely analogous to the situation in the "old covariant" formalism of superstring amplitudes [6] in which, due to the absence of the Faddeev-Popov superghost fields, an appropriate dimension-one vertex for the emission of a Ramond-sector state is also lacking.

This analogy is made even closer upon consideration of the example of the $c = 5$ model of the spin-4/3 string where the R' sector is found to describe space-time fermions. In this model, discussed in Sec. IV, the twisted-sector states are realized by a $\mathbb{Z}_2$ orbifold twisted sector of the $c = 5$ CFT. In particular, the $c = 5$ CFT is a tensor product of three free coordinate boson fields X^μ on the world sheet which do not participate in the orbifolding, and an "internal" $\mathrm{so}(2,1)_2$ Wess-Zumino-Witten model described by two bosons compactified on a triangular lattice. The relevant twisted sector arises upon twisting by a reflection through the origin of that lattice. Some details of this construction are relegated to an Appendix. In Sec. IV we also calculate some of the

*Electronic mail address: argyres@guinness.ias.edu

low-lying physical states in the R' sector and compute a three-point coupling between two fermions and the massless vector boson of the untwisted sector. These computations place some restrictions on the choice of physical state conditions; however, the precise set of R'-sector physical state conditions consistent with duality of tree amplitudes remains an open question in this model.

There are many other ways of twisting the triangular $so(2,1)_2$ lattice, but Lorentz invariance is invariably broken in the resulting twisted sectors. This follows from the fact that under the action of all twists other than the $\mathbb{Z}_2$ twist used to define the R' sector, at least one of the $so(2,1)_2$ currents is not invariant. Currents not invariant under the twist cannot have zero modes in the twisted sector, implying that Lorentz invariance is broken in these sectors. Whether or not the states in various twisted sectors should be included in the string spectrum is determined by one loop modular invariance; however, since the $c = 5$ model described above is not a critical ($c = 10$) representation of the spin-4/3 fractional superstring, it is not expected to be modular invariant in any case.

In Sec. V we briefly describe a $c = 2$ model of the spin-4/3 fractional superstring in which the R (as opposed to R') twisted sector is realized. Unlike the R' sector, the R-sector physical state conditions are determined by the scattering amplitude prescription developed in Sec. III. Because the $c = 2$ model has only one flat dimension the resulting space-time physics is trivial. It remains an open question whether either the R or R' sector is actually realized in a critical ($c = 10$) model of the spin-4/3 fractional superstring.

II. TWISTED-SECTOR REPRESENTATION THEORY

We begin with a brief description of the spin-4/3 fractional superconformal (FSC) algebra. A more detailed discussion can be found in Ref. [2]. The fractional currents $G^{\pm}(z)$ and the energy-momentum tensor $T(z)$ together generate the FSC chiral algebra, encoded in the singular terms of the operator product expansions (OPE) [5]:

$$\begin{aligned} T(z)T(w) &= \frac{1}{(z-w)^4}\left\{\frac{c}{2} + 2(z-w)^2 T(w) + (z-w)^3 \partial T(w) + \cdots\right\}, \\ T(z)G^{\pm}(w) &= \frac{1}{(z-w)^2}\left\{\frac{4}{3}G^{\pm}(w) + (z-w)\partial G^{\pm}(w) + \cdots\right\}, \\ G^{\pm}(z)G^{\pm}(w) &= \frac{\lambda^{\pm}}{(z-w)^{4/3}}\left\{G^{\mp}(w) + \frac{1}{2}(z-w)\partial G^{\mp}(w) + \cdots\right\}, \\ G^{\pm}(z)G^{\mp}(w) &= \frac{1}{(z-w)^{8/3}}\left\{\frac{3c}{8} + (z-w)^2 T(w) + \cdots\right\}. \end{aligned} \tag{2.1}$$

The first OPE implies that $T(z)$ obeys the conformal algebra with central charge c, while the second implies that $G^{\pm}(z)$ are dimension-4/3 Virasoro primary fields. The constants $\lambda^{\pm}$ in the $G^{\pm}G^{\pm}$ OPE's are fixed by associativity to be

$$\begin{aligned} \lambda^+ = \lambda^- &= \sqrt{\frac{8-c}{6}} \quad \text{for } c < 8, \\ \lambda^+ = -\lambda^- &= \sqrt{\frac{c-8}{6}} \quad \text{for } c > 8. \end{aligned} \tag{2.2}$$

This algebra generates the physical state conditions for the spin-4/3 fractional superstring. Since there is only a single cut on the right-hand side of each OPE, the currents $G^{\pm}$ are Abelianly braided (or parafermionic). Under interchange of z and w (along a prescribed path, say a counterclockwise switch) the only consistent phase that G^+ or G^- can pick up with itself is $e^{2i\pi/3}$. The phase that develops upon interchange of G^+ with G^- can be taken to be $e^{-2i\pi/3}$.

The group of automorphisms of the FSC algebra organizes the representation theory of its highest-weight modules. The order-six automorphism group S_3 of the FSC algebra is generated by the transformations

$$G^{\pm} \to \omega^{\pm 1} G^{\pm}, \tag{2.3a}$$

$$G^{\pm} \to \delta G^{\mp}, \tag{2.3b}$$

where $\omega = e^{2\pi i/3}$, and $\delta = \mathrm{sgn}(8-c)$. Since the FSC algebra is supposed to be an organizing symmetry of the states of the spin-4/3 string, it is natural to assume that its automorphisms extend to automorphisms of the CFT representation of the FSC algebra. All states can then be assigned definite $\mathbb{Z}_3$ quantum numbers under the action of the $\mathbb{Z}_3$ group of automorphisms generated by the transformation (2.3a). The untwisted sectors of the FSC algebra consist of the set of states which obey the bypass relations

$$\chi_p(z) * \chi_q(w) = \omega^{2pq} \chi_p(z)\, \chi_q(w), \tag{2.4}$$

where χ_p is a state with $\mathbb{Z}_3$ charge p. The bypass relation $V(z) * W(w)$ denotes the analytic continuation of z along a closed path looping once around w in a counterclockwise sense as shown in Fig. 1(a). Clearly the currents themselves are untwisted-sector fields with $\mathbb{Z}_3$ charges ± 1 for $G^{\pm}$ and charge 0 for T.

We define the twisted sectors of FSC algebra representations in terms of the bypass relations the twist fields obey with the FSC currents. The basic property of the

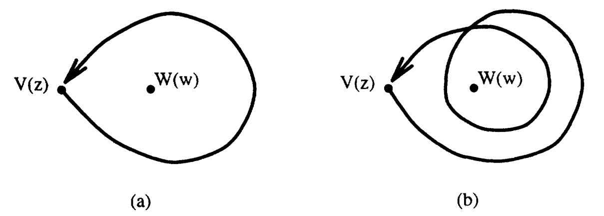

FIG. 1. Paths defining (a) the bypass relation $V * W$, and (b) the double bypass relation $V *^2 W$.

twisted sectors is that the twist fields are double valued with respect to the fractional currents $G^\pm(z)$. More precisely, with any twist field $\tau(w)$, the split algebra currents satisfy the bypass relations

$$T * \tau = T\,\tau, \quad G^\pm *^2 \tau = \omega G^\pm \tau. \tag{2.5}$$

The path defining the double-bypass relation $V(z) *^2 W(w)$ is shown in Fig. 1(b). The phase in the double-bypass relation is chosen so that the $G^\pm$ currents have zero modes in the twisted sectors. Indeed, the double-bypass relation implies that $z^{4/3}G^\pm(z)\tau(0)$ is a double-valued analytic function on the z plane with branch point at $z=0$ and thus defines the mode expansion

$$G^\pm(z)\tau(0) = \sum_{n\in\mathbb{Z}} z^{n/2-4/3} G^\pm_{n/2}\tau(0). \tag{2.6}$$

To further specify the twisted sectors we must also know the single bypass relations of the fractional currents $G^\pm$ with the twist fields. In general under a single bypass the $G^\pm$ currents go to some other pair of (dimension-4/3) currents in the CFT under consideration. Denoting these currents by $\widetilde{G}^\pm$ we have

$$G^\pm * \tau = \delta\omega^2 \widetilde{G}^\mp \tau, \quad \widetilde{G}^\pm * \tau = \delta\omega^2 G^\mp \tau. \tag{2.7}$$

We have included the factor of $\delta = \mathrm{sgn}(8-c)$ for later convenience. For these bypass relations to be consistent with associativity of the CFT operator product algebra, $\delta\widetilde{G}^\mp$ must be images of the $G^\pm$ currents under the action of some $\mathbb{Z}_2$ automorphism of the CFT. This implies, in particular, that $\widetilde{G}^\pm$ satisfy the same double-bypass relation (2.5) as $G^\pm$ do, implying that $\widetilde{G}^\pm$ have half-integral mode expansions as in (2.6). The single bypass relations (2.7) then imply the identification of modes

$$\widetilde{G}^\pm_{n/2} = \delta(-1)^n G^\mp_{n/2}. \tag{2.8}$$

There are some special choices for the $\widetilde{G}^\pm$ currents (or, equivalently, for the $\mathbb{Z}_2$ automorphism) which lead to particularly simple twisted-sector properties. The first is the choice $\widetilde{G}^\pm = G^\mp$, where the $\mathbb{Z}_2$ automorphism group is trivially realized by the identity transformation. In this case we simply recover the untwisted sectors. A second choice is $\widetilde{G}^\pm = \omega^{\mp p}G^\pm$, where $p \in \mathbb{Z}_3$. This choice gives rise to twisted sectors we refer to as the R sectors of the FSC algebra, since they are equivalent to the R sectors introduced in Ref. [5]. The general case where $\widetilde{G}^\pm \neq G^\pm$ gives rise to what we call R' twisted sectors.

The remainder of this section explores the representation theory of the R and R' sectors. We begin with the R sector since it is simpler, and derive the generalized commutation relations satisfied by the modes of $G^\pm$ when acting on an arbitrary twisted-sector state. We then turn to the R' sector which is complicated by the need to specify the operator product algebra of the $G^\pm$ currents with their images $\widetilde{G}^\pm$ under the $\mathbb{Z}_2$ automorphism. This depends, in general, on specific properties of the CFT representaton of the FSC algebra in question.

A. *R*-sector mode algebra

The R sectors arise from choosing the $\mathbb{Z}_2$ automorphism which relates $G^\pm$ and $\widetilde{G}^\pm$ to be an automorphism of the FSC algebra itself. There are three $\mathbb{Z}_2$ subgroups of the S_3 automorphism group generated by the transformations (2.3) of the spin-4/3 FSC algebra. They give rise to three twisted sectors with bypass relations

$$G^\pm * \tau_p = \delta\omega^{2\pm p} G^\mp \tau_p, \tag{2.9}$$

where $p \in \mathbb{Z}_3$. (The R-sector "C-disorder" fields φ_p introduced in Ref. [5] actually satisfy the bypass relations $G^\pm * \varphi_p = \delta\omega^{\pm p}G^\mp \varphi_{p\mp 1}$. With the change of basis $\tau_p = \omega^2\varphi_p + \varphi_{p-1} + \varphi_{p+1}$ these bypass relations become the bypass relations written above.) The single bypass relations (2.9) imply the mode identifications

$$G^-_{n/2} = \delta\omega^{-p}(-1)^n G^+_{n/2}, \tag{2.10}$$

when acting on τ_p. Thus in the R sector there is really only one independent fractional current, which we can take to be either G^+ or G^-.

The three R sectors labeled by $p \in \mathbb{Z}_3$ are related by the $\mathbb{Z}_3$ automorphism (2.3a) of the FSC algebra, and thus are isomorphic in models in which that automorphism extends to the whole CFT operator product algebra. Since we assume that this is always the case, we henceforth restrict the discussion to the $p=0$ R sectors, the other two sectors being identical.

In analogy to the superconformal gauge of the ordinary superstring, the physical states of the spin-4/3 fractional superstring are certain highest-weight states of the FSC algebra, that is, they are annihilated by all the positive modes of T and $G^\pm$. In order to show spurious state decoupling in scattering amplitudes, we will need to know the algebra satisfied by these modes. This algebra takes the form of generalized commutation relations (GCR's) for the modes of $G^\pm$ due to cuts in the FSC operator product algebra [4,5].

To derive the R-sector GCR's, consider the integral

$$\mathcal{I} = \frac{1}{4}\oint_\gamma \frac{dz}{2\pi i}\oint_\delta \frac{dw}{2\pi i}\left(\frac{\sqrt{z}+\sqrt{w}}{\sqrt{z}-\sqrt{w}}\right)^{\frac{1}{3}} z^{\frac{1}{3}+\frac{n}{2}} w^{\frac{1}{3}+\frac{m}{2}} \times G^+(z)G^-(w)\tau(0), \tag{2.11}$$

where the contours both wind twice around the origin with the δ contour inside the γ contour. The factors in the integrand have been chosen to make these contours closed: the whole integrand is a double-valued analytic function on both the z and w planes with branch points at $z = 0$ and $w = 0$ and possible poles at $z = w$ or $z = e^{2\pi i}w$ (i.e., at the point $z = w$ on both sheets). Evaluating this integral by shrinking the δ contour close to the origin and using the mode definition (2.6), which can be inverted as

$$G^{\pm}_{n/2}\tau(0) = \frac{1}{2}\oint_{\gamma}\frac{dz}{2\pi i}\, z^{1/3+n/2}G^{\pm}(z)\tau(0), \tag{2.12}$$

gives

$$\mathcal{I} = \sum_{\ell=0}^{\infty} D_{\ell}^{(-\frac{1}{3},\frac{1}{3})} G^{+}_{(n-\ell)/2} G^{-}_{(m+\ell)/2}, \tag{2.13}$$

where the $D_{\ell}^{(\alpha,\beta)}$ are binomial coefficients defined by the expansion

$$(1-x)^{\alpha}(1+x)^{\beta} = \sum_{\ell=0}^{\infty} D_{\ell}^{(\alpha,\beta)} x^{\ell}. \tag{2.14}$$

The integral $\mathcal{I}$ can also be evaluated in another way by deforming the γ contour to lie inside the δ contour. The result of this deformation is three contributions, as shown in Fig. 2. One contribution is just the same integral with γ and δ interchanged. The other two contributions pick up residues associated with the $G^{+}(z)G^{-}(w)\tau(0)$ OPE singularities at $z = w$ and $z = e^{2\pi i}w$. The former singularity can simply be read off from the $G^{+}G^{-}$ OPE; to evaluate the latter we must continue z once counterclockwise around the origin to $e^{2\pi i}z$ before letting it approach w (on the second sheet). By the single-bypass relation (2.9), this analytic continuation is

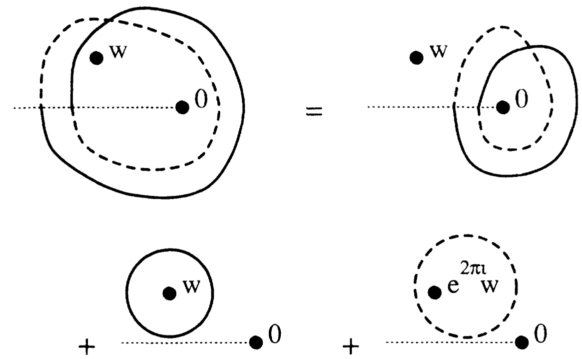

FIG. 2. Deformation of the γ contour in the cut z plane. The cut is chosen to lie along the negative real axis.

$$G^{+}(e^{2\pi i}z)G^{-}(w)\tau(0) = \delta\omega^{2}G^{-}(z)G^{-}(w)\tau(0). \tag{2.15}$$

Performing the same continuation on the other factors in the integrand, picking up the residues of poles from the FSC algebra, and combining all the contributions as shown in Fig. 2, gives

$$\delta\sum_{\ell=0}^{\infty} D_{\ell}^{(\frac{1}{3},-\frac{1}{3})}\left[G^{+}_{\frac{n-\ell}{2}}G^{+}_{\frac{m+\ell}{2}} + G^{+}_{\frac{m-\ell}{2}}G^{+}_{\frac{n+\ell}{2}}\right]$$

$$= 2^{-5/3}\lambda^{+}(-1)^{n+m}G^{+}_{\frac{n+m}{2}} + 2^{-4/3}\{(-1)^{n}+(-1)^{m}\}\left[L_{\frac{n+m}{2}} + \frac{c}{128}\left(6n^{2}-5\right)\delta_{n+m}\right], \tag{2.16}$$

where we have used (2.10) to write the GCR's in terms of G^{+} modes alone.

The commutation relations of the $G^{\pm}$ modes with the L_n modes of the stress-energy tensor follow in the standard way:

$$\left[L_n, G^{\pm}_{\frac{m}{2}}\right] = \left(\frac{n}{3}-\frac{m}{2}\right)G^{\pm}_{n+\frac{m}{2}}. \tag{2.17}$$

Acting on highest-weight states (states annihilated by all the positive modes of $G^{\pm}$ and T), one can show from (2.16) that the zero modes satisfy the relation

$$\delta\left(G_0^{+}\right)^{2} = 2^{-8/3}\lambda^{+}G_0^{+} + 2^{-4/3}\left[L_0 - \frac{5c}{128}\right]. \tag{2.18}$$

B. R'-sector mode algebra

We refer to a twisted sector obeying the bypass relations (2.7) with $G^{\pm} \neq \widetilde{G}^{\pm}$ as an R' sector. The R' sector of the FSC algebra is characterized by a $\mathbb{Z}_2$ automorphism of the particular CFT model in question which interchanges $G^{\pm} \leftrightarrow \delta\widetilde{G}^{\pm}$. This symmetry between $G^{\pm}$ and $\delta\widetilde{G}^{\pm}$ implies that the $\widetilde{G}\widetilde{G}$ OPE's form a second spin-4/3 FSC algebra by themselves (at the same central charge and with structure constants $\delta\lambda^{\pm}$), and that $\widetilde{G}^{\pm}$ have untwisted-sector $\mathbb{Z}_3$ charges ± 1, and so obey all the concomitant bypass relations with $G^{\pm}$. The $G\widetilde{G}$ OPE's, on the other hand, remain undetermined by this symmetry, and depend on the properties of the specific CFT representation of the FSC algebra under consideration. We will assume that the $G\widetilde{G}$ OPE's are of the form

$$\begin{aligned} G^{\pm}(z)\widetilde{G}^{\pm}(w) &= (z-w)^{-4/3}\left\{\mathcal{A}^{\mp}+\cdots\right\},\\ G^{\pm}(z)\widetilde{G}^{\mp}(w) &= (z-w)^{-8/3}\delta\left\{\frac{3c\mu}{8} + (z-w)^{2}[\mu T + \mathcal{B}^{\mp}] + \cdots\right\}. \end{aligned} \tag{2.19}$$

We will see that the R' sector of the $c = 5$ model to be considered in Sec. IV has currents obeying OPE's of this form.

The form of the $G\widetilde{G}$ OPE's (2.19) is actually more general, and follows for representations satisfying a few physical properties. Assume that these representations have a global D-dimensional Poincaré symmetry realized by the tensor product of D coordinate bosons X^μ with an "internal" CFT which has a positive definite spectrum of highest weights. $G^\pm$ and $\widetilde{G}^\pm$ then only involve derivatives of the X^μ fields with no vertex contributions of the form $e^{ik\cdot X}$, and are so$(D-1,1)$ singlets. This requirement is natural in representations which have a flat D-dimensional space-time interpretation in string theory. In addition, if no dimension-1/3 and no dimension-one so$(D-1,1)$ scalar fields exist, then (2.19) is the most general form the $G\widetilde{G}$ OPE's can take. Note that these last two conditions are not as strong as they might appear. For example, a pair of $\mathbb{Z}_3$ charge ± 1 dimension-1/3 scalar fields would obey braiding properties and operator product selection rules identical to those of $\mathbb{Z}_3$ parafermion currents; but associativity of the $\mathbb{Z}_3$ parafermion current algebra fixes the central charge to be $c = 4/5$ [4]. A similar argument applies to potential dimension-one currents, for example, a simple associativity argument shows that adding a single dimension-one current to the spin-4/3 FSC algebra fixes the central charge to be $c = 1$. These arguments are not proofs, though, since associativity constraints can be evaded by increasing the number of independent dimension-1/3 or dimension-one currents. In any case, however the reader judges the reasonableness of the above assumptions, there are at least two cases in which they are satisfied, namely, the $c = 5$ model to be discussed in detail in Sec. IV, and a $c = 7$ model described in Appendix C of Ref. [2].

It follows from (2.19) that the combinations of fields $\Omega^\pm \equiv \widetilde{G}^\pm - \delta\mu G^\pm$ are dimension-4/3, $\mathbb{Z}_3$ charge ± 1 untwisted-sector highest-weight operators with respect to the FSC algebra generated by $G^\pm$. The properties of the untwisted-sector highest-weight modules derived in Ref. [2], together with the assumed symmetry under interchanging $G^\pm$ and $\delta\widetilde{G}^\pm$, implies that

$$\mu = \frac{1}{3c}\left[c - 32 \pm 2\sqrt{(8-c)(32-c)}\right],$$
$$\mathcal{A}^\pm = \frac{\mu\lambda^\mp}{\mu+1}\left(G^\pm + \delta\widetilde{G}^\pm\right). \tag{2.20}$$

The dimension-two descendents of the $\Omega^\pm$ fields, $G^\mp_{-2/3}|w^\pm\rangle$, are the dimension-two $\mathcal{B}^\mp$ fields that enter in the $G^\pm\widetilde{G}^\mp$ OPE.

From the bypass relations (2.7) and the OPE's (2.19), and using the mode identifications (2.8), one deduces the GCR's (just as was done in the last subsection)

$$\sum_{\ell=0}^{\infty} D_\ell^{(-\frac{1}{3},\frac{1}{3})}\left[G^+_{\frac{n-\ell}{2}}G^-_{\frac{m+\ell}{2}} + G^-_{\frac{m-\ell}{2}}G^+_{\frac{n+\ell}{2}}\right] = \frac{2^{-5/3}\mu}{\mu+1}\left[(-1)^n\lambda^- G^+_{\frac{n+m}{2}} + (-1)^m\lambda^+ G^-_{\frac{n+m}{2}}\right]$$
$$+2^{-4/3}\left\{1+(-1)^{n+m}\right\}\left[L_{\frac{n+m}{2}} + \frac{c}{128}\left(6n^2-5\right)\delta_{n+m}\right], \tag{2.21}$$

and

$$\sum_{\ell=0}^{\infty} D_\ell^{(-\frac{2}{3},\frac{2}{3})}\left[G^\pm_{\frac{n-\ell}{2}}G^\pm_{\frac{m+\ell}{2}} - G^\pm_{\frac{m-\ell}{2}}G^\pm_{\frac{n+\ell}{2}}\right] = 2^{-5/3}\lambda^\pm(n-m)G^\mp_{\frac{n+m}{2}} + 2^{-4/3}\left\{(-1)^n + (-1)^m\right\}\frac{3\mu c}{32}n\delta_{n+m}. \tag{2.22}$$

The commutation relations (2.17) with the L_n modes follow in the standard way.

Note that, when acting on a highest-weight state, the second GCR gives no relation for the $G_0^\pm G_0^\pm$ zero mode product. One can obtain a relation for these zero modes by changing the power of the $(\sqrt{z}-\sqrt{w})/(\sqrt{z}+\sqrt{w})$ factor in the integrand of (2.11). One finds, in this way,

$$\sum_{\ell=0}^{\infty} D_\ell^{(\frac{1}{3},-\frac{1}{3})}\left[G^\pm_{\frac{n-\ell}{2}}G^\pm_{\frac{m+\ell}{2}} + G^\pm_{\frac{m-\ell}{2}}G^\pm_{\frac{n+\ell}{2}}\right] = 2^{-5/3}\lambda^\pm G^\mp_{\frac{n+m}{2}}$$
$$+\,2^{-4/3}\left\{(-1)^n + (-1)^m\right\}\left[\mu L_{\frac{n+m}{2}} + \mathcal{B}^\mp_{\frac{n+m}{2}} + \frac{\mu c}{128}\left(6n^2-5\right)\delta_{n+m}\right]. \tag{2.23}$$

Note, however, the appearance of the modes of the representation-dependent dimension-two operators $\mathcal{B}^\pm$ in (2.23).

III. SCATTERING AMPLITUDES

We now extend the prescription developed in Ref. [2] for dual N-point tree amplitudes of untwisted-sector states satisfying spurious state decoupling to include two twisted-sector states. In that prescription amplitudes could be written in either of the equivalent "pictures"

$$\mathcal{A}_N = 2\langle W^+|V(1)\frac{1}{L_0-\frac{1}{3}}V(1)\cdots\frac{1}{L_0-\frac{1}{3}}V(1)|W^-\rangle$$
$$= \langle V|V(1)\frac{1}{L_0-1}V(1)\cdots\frac{1}{L_0-1}V(1)|V\rangle, \tag{3.1}$$

where $W^\pm$ are untwisted-sector physical states of $\mathbb{Z}_3$ charge ± 1 and dimension 1/3, and V is a certain dimension-one descendent of these fields.

By analogy to the Ramond sector of the ordinary superstring in the "old covariant" formalism [6] we take the physical states $|\tau\rangle$ of the twisted sectors to be annihilated by the positive modes of the currents $G^{\pm}$ and T, and to be eigenstates of their zero modes with "intercepts" h_t and $\Lambda^{\pm}$:

$$L_0|\tau\rangle = h_t|\tau\rangle, \quad G_0^{\pm}|\tau\rangle = \Lambda^{\pm}|\tau\rangle. \tag{3.2}$$

The L_0 condition gives rise to the mass-shell condition for the physical states—a Klein-Gordon equation for each independent component of the physical state. The $G_0^{\pm}$ conditions, on the other hand, are linear in space-time derivatives (because the fractional currents have the form $G^{\pm} \sim \epsilon^{\pm} \cdot \partial X + \cdots$ for models with a flat space-time interpretation), and so should give rise to a Dirac equation for the components of the physical states. This will indeed turn out to be the case in the models discussed in Secs. IV and V. Of course, the Dirac equations must be consistent with the Klein-Gordon equations. This is automatically ensured in the R sector by the relation (2.18) between $G_0^{\pm}$ and L_0; however, it is not automatically satisfied in the R' sector, and must be imposed as a separate requirement.

We expect the form of tree amplitudes with one twisted-sector channel to consist of "in" and "out" twisted-sector physical states $|\tau\rangle$ sandwiching the vertex operators of physical untwisted-sector states W which are strung together with the Dirac propagator:

$$\mathcal{A}_N = \langle\tau_N|W_{N-1}^{\pm}(1)S^{\pm}\cdots S^{\pm}W_2^{\pm}(1)|\tau_1\rangle \ , \tag{3.3}$$

where

$$S^{\pm} = \frac{1}{G_0^{\pm} - \Lambda^{\pm}}. \tag{3.4}$$

The choice of $\mathbb{Z}_3$ charges of the vertices and propagators (i.e., their plus or minus superscripts) will be shown to be immaterial in the R sector, and will have to be further specified for the R' sector. To show spurious state decoupling for these amplitudes, we must be able to transform them to another "picture," similar to those in (3.1), in which the dimension-one V vertices appear instead of the dimension-1/3 $W^{\pm}$ vertices. We first describe this picture changing and the resulting spurious-state decoupling theorem in the R-sector case and then move onto the more complicated R'-sector case.

A. *R*-sector scattering amplitudes

We start with the ansatz for the N-point amplitude with one R-sector channel

$$\mathcal{A}_N = \langle\tau_N|W_{N-1}^{+}(1)S^{+}\cdots S^{+}W_2^{+}(1)|\tau_1\rangle. \tag{3.5}$$

This form for the amplitude can be rewritten in another "picture" using the commutator

$$[G_{n/2}^{\pm}, V(1)] = \left(L_0 + \frac{n}{2} - h_t\right) W^{\pm}(1) - W^{\pm}(1)\left(L_0 - h_t\right), \tag{3.6}$$

for any integer n, which follows from the representation theory of the untwisted-sector module that V belongs to. (See, for example, the derivation of Eq. (3.10) of Ref. [2].) In particular, rewrite a special case of (3.6) as

$$W^{+}(1)(L_0 - h_t) = (L_0 - h_t)W^{+}(1) - (G_0^{+} - \Lambda^{+})V(1) + V(1)(G_0^{+} - \Lambda^{+}). \tag{3.7}$$

Insert a factor of $1 = (L_0 - h_t)/(L_0 - h_t)$ before any S^{+} propagator in (3.5) and commute the $(L_0 - h_t)$ factor in the numerator to the left using (3.7). The first two terms on the right-hand side give vanishing contributions since the $(L_0 - h_t)$ factor can be continually commuted to the left using (3.7) until it annihilates the $\langle\tau_N|$ physical state, while the second term vanishes by the "canceled propagator" argument. Tree amplitudes with canceled propagators are holomorphic in the Mandelstam invariant of the canceled propagator channel, and thus vanish by analyticity if the amplitudes have sufficiently soft high-energy behavior, as string amplitudes do. The factor of $(G_0^{+} - \Lambda^{+})$ in the third term of (3.7) cancels the S^{+} propagator, leaving behind a Klein-Gordon propagator $\Delta = (L_0 - h_t)^{-1}$. Applying this argument repeatedly gives

$$\mathcal{A}_N = \langle\tau_N|V_{N-1}(1)\Delta\cdots\Delta V_3(1)\Delta W_2^{+}(1)|\tau_1\rangle. \tag{3.8}$$

The fact that one of the space-time boson vertices is still a highest-weight state W^{+}, and not one of the $\mathbb{Z}_3$-charge $q = 0$ descendents V, is familiar from tree amplitudes with one fermion line for the ordinary superstring in the old covariant formalism . The position of the lone W^{+}-vertex is arbitrary, as can be shown by manipulations similar to those used to derive (3.8). Furthermore, the above manipulations can be reversed using Eq. (3.7) with W^{-}, G_0^{-}, and Λ^{-} to show that any choice of $\pm$ superscripts in (3.3) is equivalent to (3.5). It is important to note in this connection that due to the equivalence of G^{+} and G^{-} modes (2.10) in the R sector, that we must have $\Lambda^{+} = \delta\Lambda^{-}$.

The following trick allows us to reexpress the amplitude (3.8) completely in terms of V vertices. Insert $1 = (L_0 - h_t)/(L_0 - h_t)$ between $W_2^{+}(1)$ and $|\tau_1\rangle$ and use (3.7) once more to finally obtain the amplitude in the form

$$\mathcal{A}_N = \left\langle \tau_N \left| V_{N-1}(1)\Delta\cdots\Delta V_2(1)\frac{G_0^{+} - \Lambda^{+}}{L_0 - h_t}\right| \tau_1 \right\rangle. \tag{3.9}$$

This form for the amplitude is slightly delicate because both the numerator and denominator in the factor before $|\tau_1\rangle$ annihilate it. However, this potential ambiguity can be resolved by noting that the $L_0 - h_t$ operator, when acting on the spinor wave function $u(k)$ associated with τ_1, is the Klein-Gordon operator $k^2 + m^2$, while $G_0^{+} - \Lambda^{+}$ is the associated Dirac operator $ik \cdot \gamma + m$. For massive states the simple zeros of the numerator and denominator cancel, and for massless states it can be defined as the limit as k goes on mass shell.

If we start with physical states in the amplitude (3.9), will they scatter only to other physical states? We can reformulate this question in terms of spurious state decoupling. A state $\langle s|$ obeying the zero-mode conditions in Eq. (3.2) is called a spurious state if it is orthogonal to all physical states. Since the physical state conditions are the only restriction on a generic physical state, it follows that $\langle s|$ can be written as

$$\langle s| = \sum_{m>0} \langle \chi_m | L_m + \sum_{n>0} \langle \psi^{\pm}_{\frac{n}{2}} | G^{\pm}_{\frac{n}{2}} \tag{3.10}$$

in terms of some other states χ_m and $\psi^{\pm}_{\frac{n}{2}}$. All states not satisfying the physical state conditions must have a spurious component. A physical state can itself be spurious, in which case it is a null state (since it is orthogonal to itself), and should also decouple from all scattering amplitudes. Thus, the decoupling of all spurious states from scattering amplitudes of physical states is a prerequisite for a sensible interpretation of those amplitudes. For this decoupling to be true, no spurious states should contribute to residues of poles in amplitudes when an internal propagator goes on shell.

Suppose we fix the external momenta k_i of the vertices V_i such that some state $|s\rangle$ in the string Fock space at momentum $\kappa = k_{M+1} + \cdots + k_N$ is on shell: $(L_0 - h_t)|s\rangle = (G^{\pm} - \Lambda^{\pm})|s\rangle = 0$. Factorize the amplitude in Eq. (3.9) by inserting a sum over a complete set of states σ_i of momentum κ at the propagator between V_{M+1} and V_M,

$$\mathcal{A}_N = \sum_i \langle \tau_N | V_{N-1}(1) \cdots \Delta V_{M+1}(1) \frac{G_0^+ - \Lambda^+}{L_0 - h_t} |\sigma_i\rangle \frac{1}{i\kappa\cdot\gamma + m_i} \langle \sigma_i | V_M(1) \cdots \Delta V_2(1) \frac{G_0^+ - \Lambda^+}{L_0 - h_t} |\tau_1\rangle . \tag{3.11}$$

Then the $|s\rangle\langle s|$ term in the sum contributes a pole in momentum space. The requirement of spurious state decoupling is that if $|s\rangle$ is spurious, its contribution to the residue of the pole should vanish:

$$\langle s | V_M(1) \Delta \cdots \Delta V_2(1) \frac{G_0^+ - \Lambda^+}{L_0 - h_t} |\tau_1\rangle = 0 . \tag{3.12}$$

To prove this, consider one term, say $\langle \psi | G^+_{n/2}$ with $n \geq 0$, in the presentation of $\langle s|$ as a sum of descendent states, Eq. (3.10), where ψ must satisfy $(L_0 + \frac{n}{2} - h_t)|\psi\rangle = 0$. [The $G^-_{n/2}$ descendent pieces can be shown to decouple by the same argument using the R-sector equivalence of G^+ and G^- (2.10). The L_n pieces decouple by a simpler argument.] The $G^+_{n/2}$ mode can be commuted to the right in Eq. (3.12) using Eq. (3.6) and the identity

$$G^{\pm}_{\frac{n}{2}} \frac{1}{L_0 - h_t} = \frac{1}{L_0 - h_t - \frac{n}{2}} G^{\pm}_{\frac{n}{2}} , \tag{3.13}$$

which follows from (2.17). The insertions coming from the right-hand side of Eq. (3.6) again vanish by a canceled propagator argument. Finally, the $G^+_{n/2}$ mode can be seen to pass through the $G_0^+ - \Lambda^+$ factor and annihilate $|\tau_1\rangle$ by the physical state conditions using the R-sector GCR (2.16). This proves spurious state decoupling.

Since there is no appropriate dimension-one commuting vertex in the R sector, we cannot extend our scattering amplitude prescription (3.5) or (3.8) to include more than two R-sector vertices. By the same token we cannot prove cyclic symmetry (duality) of these amplitudes in the present formalism. This situation is closely analogous to what happens in the old covariant formalism in the ordinary superstring. There, dual amplitudes with spurious state decoupling can be formulated for scattering of Neveu-Schwarz sector states, and can only be extended to include two Ramond-sector states as the "in" and "out" states in the correlator, thus losing manifest cyclic symmetry. So presumably, just as in the Ramond sector of the superstring, our inability to incorporate more than two R-module physical states in our scattering prescription means that there is a nontrivial contribution to R-module scattering amplitudes coming from the "fractional superghost" fields on the world sheet.

B. R'-sector scattering amplitudes

The main difference between the R and R' sector scattering amplitudes is that in the R' sector there is no identification of G^+ with G^- modes analogous to (2.10). This means that the zero-mode conditions (3.2) generate two *a priori* independent Dirac equations for the physical state wave functions. This will lead to overly restrictive physical state conditions unless the $G^{\pm}$ intercepts are appropriately tuned, or only a subset of all the $G^{\pm}_{n/2}$ modes with $n \geq 0$ are used as R'-sector physical state conditions. Since, as was emphasized at the end of Sec. II, the mode algebra obeyed by the fractional currents in the R' sector depends on the details of the particular CFT model under consideration, we can only hope to determine the appropriate R'-sector physical state conditions in the context of a specific model.

However, we can still demonstrate spurious state decoupling in amplitudes with one R'-sector channel. Indeed, all the manipulations of the last subsection for the R sector scattering go through unchanged in the R' sector as long as all the untwisted vertices $W^{\pm}$ and Dirac propagators $S^{\pm}$ in (3.3) have the same $\mathbb{Z}_3$ charge (i.e., all their superscripts are the same). The point is simply that since the G_0^+ and G_0^- modes are not related in any general way in the R' sector, the associated Dirac propagators are not equivalent—in particular $(G_0^+ - \Lambda^+)/(G_0^- - \Lambda^-)$ is not proportional to the identity.

In fact, one can show that the ansatz for the one channel R'-sector scattering amplitude

$$\mathcal{A}_N = \langle \tau_N | W^{\alpha}_{N-1}(1) S^{\alpha} \cdots S^{\alpha} W^{\alpha}_2(1) |\tau_1\rangle \tag{3.14}$$

obeys spurious state decoupling, where α is any fixed complex number and we define

$$\begin{aligned} G^{\alpha}_{n/2} &= G^{+}_{n/2} + \alpha G^{-}_{n/2}, \\ \Lambda^{\alpha} &= \Lambda^{+} + \alpha\Lambda^{-}, \\ S^{\alpha} &= (G^{\alpha}_{0} - \Lambda^{\alpha})^{-1}, \\ W^{\alpha} &= W^{+} + \alpha W^{-}. \end{aligned} \tag{3.15}$$

This follows from the fact that the analogue of the commutation relation (3.6) is satisfied for the above combinations:

$$\begin{aligned} [G^{\alpha}_{n/2}, V(1)] = \left(L_0 + \frac{n}{2} - h_t\right) W^{\alpha}(1) \\ - W^{\alpha}(1)\,(L_0 - h_t)\,, \end{aligned} \tag{3.16}$$

Thus, we have a one-parameter family of satisfactory scattering amplitudes in the R' sector.

Tree-level duality may restrict the value of α and the correct set of physical state conditions in the R' sector. In particular, by duality, if one factorizes the amplitude (3.14) in a channel other than the R'-sector channel displayed, one should find an infinite tower of poles corresponding to untwisted-sector physical states. Presumably this occurs only if the R'-sector states τ obey the correct set of physical state conditions. We will discuss the choice of physical state conditions in the context of a concrete example in the next section, though to the level we compute we will not be able to put many restrictions on the possible choices. As an example of the kind of choices that could make sense, note that associated with each value of α is a natural choice of physical state conditions for R'-sector states:

$$\begin{aligned} (L_n - h_t\delta_{n,0})\,|\tau\rangle = 0, \\ \left(G^{\alpha}_{n/2} - \Lambda^{\alpha}\delta_{n,0}\right)|\tau\rangle = 0, \end{aligned} \tag{3.17}$$

for $n \geq 0$. Note also that by the commutation relation (2.17) all the possible physical state conditions are generated by L_0, L_1, $G^{\pm}_{0}$, and $G^{\pm}_{1/2}$. In particular, the commutator of L_1 with $G^{\pm}_{0}$ generates all the integer-moded $G^{\pm}_{m}$ conditions. Thus, it may also be consistent to restrict the physical state conditions to only the integer-moded T and G^{α} annihilation operators. These, however, are only the simplest guesses. The fact that R'-sector states are $\mathbb{Z}_2$-twist fields obeying nontrivial monodromies with each other may indicate that the physical state conditions and scattering amplitudes for R'-sector "matter" fields (i.e., after integrating out the fractional superghost pieces of the full physical vertices) may be more complicated than what we have presented here.

IV. SPACE-TIME FERMIONS IN THE $c = 5$ MODEL

Strings propagating in D flat space-time dimensions are described by a world-sheet CFT which includes D massless scalar fields $X^{\mu}(z)$. The spin-4/3 fractional superstring CFT also includes a set of fields $\epsilon^{\pm}_{\mu}(z)$ of conformal dimension 1/3, transforming as vectors under space-time Lorentz transformations. The simplest nontrivial such CFT which is also a representation of the spin-4/3 FSC algebra can be constructed from five free massless scalar fields on the world sheet, and hence has central charge $c = 5$. (A list of known representations of the spin-4/3 FSC algebra is given in Appendix C of Ref. [2].) Three of the scalars are just coordinate boson fields $X^{\mu}(z)$, $\mu = 0, 1, 2$, with the standard operator products $X^{\mu}(z)X^{\nu}(w) = -g^{\mu\nu}\ln(z-w)$, where $g^{\mu\nu}$ is the three-dimensional Minkowski metric with signature $(-++)$. The remaining two fields $\varphi^{i}(z)$, $i = 1, 2$, are compactified on a triangular lattice: $\varphi^{i}(z) = \varphi^{i}(z) + 2\pi$, with $\varphi^{i}(z)\varphi^{j}(w) = -g^{ij}\ln(z-w)$, where

$$g^{ij} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.1}$$

These two bosons form a representation of the $so(2,1)_2$ Wess-Zumino-Witten model. Thus the $c = 5$ model has a global three-dimensional Poincaré symmetry.

Vertex operators in the $so(2,1)_2$ CFT, $V_{\mathbf{m}} = \exp\{im_1\varphi^1 + im_2\varphi^2\}$ for integer m_i, describe Virasoro-primary operators transforming as integer-spin fields under the $so(2,1)$ symmetry. For example, some of the simplest $so(2,1)_2$ Virasoro-primary fields are the dimension-1/3 $so(2,1)$-vector fields $\epsilon^{\pm}_{\mu}$, the dimension-4/3 $so(2,1)$-scalars $s^{\pm}$, and the dimension-one U_{μ}'s, given by

$$\begin{aligned} \epsilon^{\pm}_{\mu} &= \left(V_{(\mp1,\mp1)}\,,\, V_{(\pm1,0)}\,,\, V_{(0,\pm1)}\right), \\ s^{\pm} &= \tfrac{1}{3}\left(V_{(\pm2,\pm2)} + V_{(\mp2,0)} + V_{(0,\mp2)}\right), \\ U_{\mu} &= \Big(V_{(1,-1)} + V_{(-1,1)}\,,\, V_{(1,2)} + V_{(-1,-2)}\,, \\ &\qquad V_{(2,1)} + V_{(-2,-1)}\Big). \end{aligned} \tag{4.2}$$

The properties of these and other fields in the $c = 5$ model are discussed in more detail in Ref. [2].

The fractional supercurrents $G^{\pm}$ satisfying the spin-4/3 FSC algebra (2.1) are given in terms of these fields by

$$G^{\pm} = \frac{1}{\sqrt{2}}\left(\pm\epsilon^{\pm}\cdot\partial X - \frac{3}{2}s^{\pm}\right). \tag{4.3}$$

There are actually six solutions for the supercurrents, which can be obtained from the above solution by making the transformations $G^{\pm} \to \omega^{\pm q}G^{\pm}$ or $G^{\pm} \to \omega^{\pm q}\widetilde{G}^{\pm}$, where $q \in \mathbb{Z}_3$ and $\widetilde{G}^{\pm}$ are given by

$$\widetilde{G}^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\epsilon^{\pm}\cdot\partial X - \frac{3}{2}s^{\pm}\right), \tag{4.4}$$

which differs from (4.3) by a sign change in the $\epsilon^{\pm}\cdot\partial X$ terms. The existence of these six solutions is a consequence of the $\mathbb{Z}_2 \times S_3$ automorphism group of the $c = 5$ CFT generated by $X^{\mu} \to -X^{\mu}$, $V_{\mathbf{m}} \to \omega^{m_1+m_2}V_{\mathbf{m}}$, and $V_{\mathbf{m}} \to V_{-\mathbf{m}}$, which leaves the $so(2,1)$ generators invariant.

In what follows, we construct the R' twisted sector corresponding to the $\mathbb{Z}_2$ automorphism $V_{\mathbf{m}} \to V_{-\mathbf{m}}$ which is generated by the transformation $\varphi^{j} \to -\varphi^{j}$. Recalling the discussion of Sec. II, the fields of this twisted sector are characterized by the property that under single-bypass around a twisted-sector field, untwisted-sector fields transform according to this automorphism. Thus the defining property of the R' twisted sector is that with any twist field, $\tau(w)$, the free boson fields $\varphi^j(z)$ satisfy the basic bypass relation

$$\varphi^j * \tau = -\varphi^j \tau. \tag{4.5}$$

This is also the definition of the twisted sector of a $\mathbb{Z}_2$ orbifold [7] of the $\mathrm{so}(2,1)_2$ CFT under the action of the symmetry which reflects the triangular φ^j-boson lattice through the origin. Note that this $\mathbb{Z}_2$ transformation maps the $G^\pm$ currents into the $\widetilde{G}^\pm$ currents. It is straightforward to show that these currents obey OPE's of the form (2.19) with

$$\begin{aligned}\mu &= -3/5,\\ \mathcal{A}^\pm &= -\frac{3}{2\sqrt{2}}(G^\pm + \widetilde{G}^\pm),\\ \mathcal{B}^\pm &= \tfrac{3}{5}T_\varphi - \tfrac{2}{5}T_X \pm \tfrac{1}{2}U\cdot\partial X,\end{aligned} \tag{4.6}$$

where T_X and T_φ are the stress-energy tensors for the X^μ and $\mathrm{so}(2,1)_2$ CFTs, respectively. Thus we are indeed constructing precisely the R' sector discussed in Sec. II.

There is a physical reason for expecting this sector to appear in the spin-4/3 fractional superstring: it includes the space-time fermionic states of the $c=5$ model. The untwisted-sector states described so far are all space-time bosonic states, corresponding to integer-spin representations of $\mathrm{so}(2,1)$, in the $c=5$ FSC algebra representation. In general, the $\mathrm{so}(N)_2$ Wess-Zumino-Witten model can be realized as the $\mathbb{Z}_2$ orbifold of the $\mathrm{su}(N)_1$ model, with the spinor representations of $\mathrm{so}(N)$ appearing as the $\mathbb{Z}_2$-twisted fields [8]. Thus we expect to find the space-time fermionic physical states of the $c=5$ model of the spin-4/3 fractional superstring in the $\mathbb{Z}_2$-twisted sector of $\mathrm{so}(2,1)_2$. In the next subsection we construct this twisted sector in some detail, and show along the way that the R'-sector bypass relations (2.7) are indeed satisfied. In the subsequent subsections we compute some low-lying R'-sector physical states and discuss their scattering amplitudes using the results of Sec. III.

A. The twisted sector of $\mathrm{so}(2,1)_2$

From the bypass relation (4.5), it follows that the $\varphi^j(z)$ bosons have the mode expansion

$$\varphi^j(z) = \phi^j + i\sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r}\beta^j_r z^{-r}, \tag{4.7}$$

with modes satisfying the commutation relations

$$[\beta^i_r, \beta^j_s] = r g^{ij}\delta_{r+s}. \tag{4.8}$$

It is easy to show that

$$\left[i\sum_{r>0}\frac{1}{r}\beta^i_r z^{-r}\,,\, i\sum_{s<0}\frac{1}{s}\beta^j_s w^{-s}\right] = -g^{ij}\ln\left(\frac{\sqrt{z}-\sqrt{w}}{\sqrt{z}+\sqrt{w}}\right), \tag{4.9}$$

implying the basic OPE $\varphi^i(z)\varphi^j(w) = -g^{ij}\ln(z-w) + \cdots$. The quantization of the zero mode ϕ^j is crucial for correctly reproducing the operator products of untwisted sector operators when acting on states in the twisted sector. As shown in Ref. [8], the correct quantization of the ϕ^j zero modes of $\varphi^j(z)$ is

$$[\phi^i, \phi^j] = i\pi\varepsilon^{ij}, \tag{4.10}$$

where ε^{ij} is the antisymmetric two-index tensor normalized by $\varepsilon^{12}=1$. These commutation relations result from the proper quantization of free boson zero modes in the presence of the constraints arising from identifications under lattice translations and the inclusion of a constant antisymmetric background field [8].

In the twisted sector (i.e., with a $\mathbb{Z}_2$ twist field inserted at the origin) the basic vertex operators $V_{\mathbf{m}} =: e^{i\mathbf{m}\cdot\varphi}:$ have the normal ordered expression

$$\begin{aligned}V_{\mathbf{m}}(z) = {}&\exp\left\{-\frac{i\pi}{2}m_1m_2\right\}2^{-\mathbf{m}\cdot\mathbf{m}}z^{-\frac{1}{2}\mathbf{m}\cdot\mathbf{m}}\exp\{i\mathbf{m}\cdot\boldsymbol{\phi}\}\\ &\times\exp\left\{-\sum_{r<0}\frac{1}{r}\mathbf{m}\cdot\boldsymbol{\beta}_r z^{-r}\right\}\exp\left\{-\sum_{r>0}\frac{1}{r}\mathbf{m}\cdot\boldsymbol{\beta}_r z^{-r}\right\}.\end{aligned} \tag{4.11}$$

The first factor provides the signs needed to "Wick rotate" from $\mathrm{so}(3)$ to $\mathrm{so}(2,1)$. No further cocycles are needed, since the $\mathrm{so}(3)$ signs are automatically taken care of by the commutation relations of the ϕ^j zero modes. The normal ordering factors $(2\sqrt{z})^{-\mathbf{m}\cdot\mathbf{m}}$ are required to ensure the factorizability of amplitudes (associativity of the operator product). Using this explicit form (4.11) for the vertex and the mode commutation relations (4.8) and (4.10), it is easy to check that the vertex operators indeed obey the correct untwisted-sector operator product expansion

$$V_{\mathbf{m}}(z)V_{\mathbf{n}}(w) = (-1)^{m_2n_1}(z-w)^{\mathbf{m}\cdot\mathbf{n}}V_{\mathbf{m}+\mathbf{n}}(w) + \cdots \tag{4.12}$$

as they should since the operator products encode only local information and do not depend on whether any twist fields are located elsewhere on the world sheet.

The representations of the zero mode algebra (4.10) determine the properties of the twisted-sector ground state. It is shown in the Appendix that there are only three representations of the zero mode algebra which preserve the global so(2,1) symmetry of the $\mathrm{so}(2,1)_2$ CFT. This implies that there are three twisted sectors $\mathcal{T}_p$, labeled by $p \in \mathbb{Z}_3$, and that if τ_p is an arbitrary twist field in $\mathcal{T}_p$ and $V_{\mathbf{m}}$ is an untwisted-sector vertex operator, then $V_{\mathbf{m}}\tau_p \in \mathcal{T}_p$. Thus the three twisted sectors are disjoint; in fact, they are just copies of one another under the action of a $\mathbb{Z}_3$ symmetry of the $\mathrm{so}(2,1)_2$ CFT. Indeed, from the expression (4.11) for the $V_{\mathbf{m}}$ vertex acting on an arbitrary twisted-sector state $\tau_p(0)$ one obtains the bypass relation

$$\begin{aligned} V_{\mathbf{m}}(z) * \tau_p(0) &= e^{-i\pi\mathbf{m}\cdot\mathbf{m}} e^{2i\mathbf{m}\cdot\phi} V_{-\mathbf{m}}(z)\,\tau_p(0) \\ &= e^{-i\pi\mathbf{m}\cdot\mathbf{m}} \omega^{p(m_1+m_2)} V_{-\mathbf{m}}(z)\,\tau_p(0), \end{aligned} \quad (4.13)$$

where in the second line we have used a property of the zero-mode representations derived in the Appendix. This bypass relation implies the double-bypass relation $V_{\mathbf{m}} *^2 \tau_p = e^{-2i\pi\mathbf{m}\cdot\mathbf{m}} V_{\mathbf{m}}\,\tau_p$, which in turn implies the existence of the mode expansion

$$V_{\mathbf{m}}(z)\tau_p(0) = \sum_{n\in\mathbb{Z}} z^{-\frac{1}{2}\mathbf{m}\cdot\mathbf{m}-\frac{n}{2}} (V_{\mathbf{m}})_{\frac{n}{2}}\,\tau_p(0). \quad (4.14)$$

It is clear from (4.13) that the $\mathcal{T}_p$ twisted sectors can be "rotated" into each other by the $\mathbb{Z}_3$ automorphism of the $\mathrm{so}(2,1)_2$ CFT: $V_{\mathbf{m}} \to \omega^{m_1+m_2} V_{\mathbf{m}}$. Thus the three twisted sectors are equivalent and we henceforth restrict ourselves (without loss of generality) to the $p = 0$ sector. In this sector the single-bypass relation (4.13) for the basic fields are

$$\begin{aligned} \epsilon^{\mu\pm} * \tau &= \omega^2 \epsilon^{\mu\mp}\tau, \\ s^{\pm} * \tau &= \omega^2 s^{\mp}\tau, \\ G^{\pm} * \tau &= \omega^2 \widetilde{G}^{\mp}\tau, \end{aligned} \quad (4.15)$$

the last of which is precisely the defining bypass relation (2.7) of the R' sector. These bypass relations along with the mode expansion (4.14) imply that $\mathbb{Z}_2$-even fields have only integer moding, while the $\mathbb{Z}_2$-odd fields have only half-odd integer modings. For example,

$$\begin{aligned} \epsilon^{\mu+}_{n/2} &= (-1)^n \epsilon^{\mu-}_{n/2}, \\ s^{+}_{n/2} &= (-1)^n s^{-}_{n/2}, \\ G^{\pm}_{n/2} &= (-1)^n \widetilde{G}^{\mp}_{n/2}. \end{aligned} \quad (4.16)$$

We now discuss the properties of the twist fields. In order to characterize the twisted-sector states we must take into account the action of the zero modes ϕ^j on the twist states. In particular, the twisted-sector ground state forms a representation of the nontrivial zero-mode algebra (4.10). As a result, it is shown in the Appendix, the ground states in the twisted sectors are doubly degenerate. We label the corresponding twist fields by $\sigma^a(z)$ and the ground states by $|a\rangle = \sigma^a(0)|0\rangle$, where $a = 0$ or 1.

The action of the zero mode of an arbitrary untwisted-sector operator on the twisted-sector ground states can be worked out using the methods of the Appendix. In particular, one finds that the zero modes of the so(2,1) scalar and vector fields act on the ground state as

$$\begin{aligned} s_0^{\pm}|a\rangle &= 2^{-8/3}|a\rangle, \\ \epsilon_0^{\mu\pm}|a\rangle &= 2^{-2/3}(\gamma^\mu)^a_b|b\rangle, \\ U_0^{\mu}|a\rangle &= 2^{-1}(\gamma^\mu)^a_b|b\rangle, \end{aligned} \quad (4.17)$$

where γ_μ is a Dirac Γ matrix obeying

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - \varepsilon^{\mu\nu\rho}\gamma_\rho, \quad (4.18)$$

where $\varepsilon^{\mu\nu\rho}$ is the antisymmetric tensor in three dimensions normalized by $\varepsilon^{012} = 1$. These zero-mode actions imply that the twisted sector ground states transform as spinors under the global so(2,1) symmetry. Thus twisted sector physical states describe space-time fermionic excitations of the spin-4/3 fractional string.

Since the ground states satisfy $\beta^j_r|a\rangle = 0$ for $r > 0$, a simple computation reveals

$$\begin{aligned} &\langle a|T(z)|b\rangle \\ &= \lim_{w\to z} \langle a| \left\{ -\frac{1}{2}\partial\varphi^i(w) g_{ij}\partial\varphi^j(z) - \frac{1}{(w-z)^2} \right\} |b\rangle \\ &= C^b_a \frac{1}{8} z^{-2}, \end{aligned} \quad (4.19)$$

implying that the conformal dimension of the twisted-sector ground state is $h(\sigma^a) = 1/8$. C^b_a is the spinor metric which can be taken to be $(\gamma^0)^b_a$. All other twist fields are created from σ^a by the repeated action of the φ^j creation modes β^j_r with $0 > r \in \mathbb{Z} + \frac{1}{2}$. Thus the twist states have the spectrum of conformal dimensions $h = \frac{1}{8} + \frac{n}{2}$ for $n \geq 0$ an integer.

B. Mode algebra in the $\mathrm{so}(2,1)_2$ twisted sector

Acting on a twisted-sector state, the FSC currents $G^{\pm}$ have the mode expansion

$$G^{\pm}(z)\tau(0) = \sum_{n\in\mathbb{Z}} z^{-\frac{4}{3}-\frac{n}{2}} G^{\pm}_{n/2} z^{-n/2}\tau(0), \quad (4.20)$$

following from (4.14). Since the space-time coordinate boson fields X^μ of the $c = 5$ representation of the FSC algebra are unaffected by the orbifolding procedure, their mode expansion is the usual one,

$$X^\mu(z) = x^\mu - i\alpha_0^\mu \ln(z) + i\sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n z^{-n}, \quad (4.21)$$

satisfying the standard commutation relations $[x^\mu, \alpha^\nu_0] = ig^{\mu\nu}$ and $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n} g^{\mu\nu}$. Combining this with the mode expansion of the $\mathrm{so}(2,1)_2$ fields acting on twisted-sector states, and recalling the form of the currents (4.3), we obtain

$$G^{\pm}_{n/2} = \frac{1}{\sqrt{2}} \left\{ \mp i \sum_{m\in\mathbb{Z}} \alpha_m \cdot \epsilon^{\pm}_{\frac{n}{2}-m} - \tfrac{3}{2} s^{\pm}_{\frac{n}{2}} \right\}. \quad (4.22)$$

Although we could, in principle, use the mode expansion (4.22) to calculate the action of the $G^{\pm}$ modes on arbitrary twisted-sector states using expression (4.11) for the vertex operators in terms of the modes of φ^j, this is in practice a complicated way of computing. A more efficient way which preserves Lorentz invariance in intermediate steps is to express all the twisted-sector states in terms of products of modes of $\epsilon^{\pm}_{\mu}$ acting on the twisted-sector ground-state spinor. The set of all products of $\epsilon^{\pm}_{\mu}$ modes is not a linearly independent basis, however, so we need to compute the algebra satisfied by the $\epsilon^{\pm}_{\mu}$ modes. Also, in order to compute the action of the $G^{\pm}$ and T modes on twisted-sector states, we have to express them in terms of the $\epsilon^{\pm}_{\mu}$ modes.

The algebra of the $\epsilon^{\pm}_{\mu}$ modes can be written as a set of generalized commutation relations (GCR's) derived from the $\epsilon^{\pm}_{\mu}$ OPE's in the same way that the $G^{\pm}$ GCR's were derived in Sec. II. Noting from (4.16) that the ϵ^{+}_{μ} and ϵ^{-}_{μ} modes are related, we define

$$\epsilon^{\mu}_{n/2}=\epsilon^{\mu-}_{n/2}=(-1)^n\epsilon^{\mu+}_{n/2}. \tag{4.23}$$

Using the OPE's $\epsilon^{\pm}_{\mu}\epsilon^{\mp}_{\nu} = z^{-2/3}g_{\mu\nu}+\cdots$ and $\epsilon^{\pm}_{\mu}\epsilon^{\pm}_{\nu} = z^{-1/3}\varepsilon_{\mu\nu}{}^{\rho}\epsilon^{\mp}_{\rho}+\cdots$, the mode algebra for the ϵ^{μ}'s is found to be

$$\sum_{\ell=0}^{\infty}D_{\ell}^{(-\frac{2}{3},-\frac{1}{3})}\left\{\epsilon^{\mu}_{\frac{n-\ell}{2}}\epsilon^{\nu}_{\frac{m+\ell}{2}}-\epsilon^{\nu}_{\frac{m-\ell-1}{2}}\epsilon^{\mu}_{\frac{n+\ell+1}{2}}\right\}$$
$$=2^{-4/3}(-1)^n g^{\mu\nu}\delta_{n+m}+2^{-2/3}(-1)^{n+m}\varepsilon^{\mu\nu}{}_{\rho}\epsilon^{\rho}_{\frac{n+m}{2}}, \tag{4.24}$$

where the coefficients $D_{\ell}^{(\alpha,\beta)}$ are given in Eq. (2.14). Any $\mathrm{so}(2,1)_2$ twisted-sector state can be written as a polynomial in the ϵ^{μ} creation modes acting on the twisted-sector ground state. The GCR (4.24) plus the identity

$$\epsilon_{-1/2}\cdot\epsilon_0|a\rangle=0, \tag{4.25}$$

which is easily verified from the expression for vertex operators in terms of the φ^j modes (4.11), are sufficient to reduce any set of such states to a linearly independent basis.

The current modes can be expressed in terms of ϵ^{μ} modes as follows. Since the s^+ and s^- modes are related by the identifications (4.16), we can define

$$s_{n/2}=s^{-}_{n/2}=(-1)^n s^{+}_{n/2}, \tag{4.26}$$

and from the OPE $\epsilon^{\pm}\cdot\epsilon^{\pm}=3z^{2/3}s^{\mp}+\cdots$ one then derives

$$s_{m+\frac{1}{2}}=-\frac{2^{-1/3}}{3}\sum_{\ell=0}^{\infty}D_{\ell}^{(-\frac{5}{3},\frac{2}{3})}\epsilon_{\frac{m-\ell}{2}}\cdot\epsilon_{\frac{m+\ell+1}{2}},$$
$$s_m=\frac{2^{-4/3}}{3}\sum_{\ell=0}^{\infty}D_{\ell}^{(-\frac{5}{3},\frac{2}{3})}\left\{\epsilon_{\frac{m-\ell}{2}}\cdot\epsilon_{\frac{m+\ell}{2}}+\epsilon_{\frac{m-\ell-1}{2}}\cdot\epsilon_{\frac{m+\ell+1}{2}}\right\}. \tag{4.27}$$

Similarly, since $\epsilon^{\pm}\cdot\epsilon^{\mp}=3z^{-2/3}+z^{4/3}T_{\varphi}+\cdots$,

$$L^{\varphi}_m=\frac{1}{8}\delta_{m,0}-(-1)^m2^{-2/3}\sum_{\ell=0}^{\infty}D_{\ell}^{(\frac{1}{3},-\frac{7}{3})}\epsilon_{\frac{m-\ell-1}{2}}\cdot\epsilon_{\frac{m+\ell+1}{2}}, \tag{4.28}$$

where L^{φ}_m are the modes of T_{φ}. The mode expansion of the full stress-energy tensor is then

$$L_n=\frac{1}{2}\sum_{\ell=-\infty}^{\infty}\alpha_{m-\ell}\cdot\alpha_{\ell}+L^{\varphi}_m. \tag{4.29}$$

Using (4.27) and (4.28) in (4.29) and (4.22) gives the current modes solely in terms of ϵ and α modes.

C. Simple physical states and scattering amplitudes

As discussed in Sec. III B, the correct set of physical state conditions in the R' sector is not known. So, for the moment, we assume the maximal set:

$$(L_n-h_t\delta_{n,0})|\tau\rangle=0,$$
$$(G^{\pm}_{n/2}-\Lambda^{\pm}\delta_{n,0})|\tau\rangle=0, \tag{4.30}$$

for $n\geq 0$ an integer. The h_t intercept determines the conformal dimension of physical state vertex operators in the twisted sector. In terms of the polarization spinors of the twisted sector states, the L_0 condition gives the usual Klein-Gordon equation fixing the mass of the state. The $G^{\pm}_0$ conditions, likewise, give Dirac equations for the spinor wave functions, which also fix the mass of the state. The consistency of these three zero-mode conditions determines the $G^{\pm}_0$ intercepts $\Lambda^{\pm}$ in terms of the L_0 intercept h_t. We now solve these physical state conditions for the lowest two levels of twisted-sector states.

The lowest twisted-sector state is

$$|\tau_0\rangle=u_a|a;k\rangle, \tag{4.31}$$

where $|a;k\rangle=e^{ik\cdot X}|a\rangle$ and u_a is a spinor wave function. The only nontrivial physical state conditions come from the L_0 and $G^{\pm}_0$ modes:

$$(L_0-h_t)|\tau_0\rangle=0\Rightarrow\left(k^2-2h_t+\tfrac{1}{4}\right)u=0, \tag{4.32a}$$

$$(G^{\pm}_0-\Lambda^{\pm})|\tau_0\rangle=0\Rightarrow\left(\pm ik\cdot\gamma+\tfrac{3}{8}+2^{2/3}\sqrt{2}\Lambda^{\pm}\right)u=0. \tag{4.32b}$$

Consistency of (4.32aa) with (4.32ab) implies $\Lambda^{\pm}$ and h_t are related by

$$2^{2/3}\sqrt{2}\Lambda^{\pm}=-\tfrac{3}{8}\pm\sqrt{\tfrac{1}{4}-2h_t}. \tag{4.33}$$

Now we examine the first excited state in the twisted sector. These are the four dimension-5/8 states $\beta^j_{-1/2}|a\rangle$. Written in this manner their so(2,1) properties are not apparent. These can be made more manifest by introducing the combinations

$$|\mu,a\rangle = \epsilon^\mu_{-1/2}|a\rangle. \tag{4.34}$$

From the identity (4.25) it follows that

$$\gamma_\mu|\mu,a\rangle = 0. \tag{4.35}$$

Thus, $|\mu,a\rangle$ describes a spin-3/2 so(2,1) representation. If desired, they could be written in terms of φ^j modes as $|\mu,a\rangle = \mathbf{m}_\mu\cdot\boldsymbol{\beta}_{-1/2}(\gamma_\mu)^a_b|b\rangle$ (no sum on μ implied), where $\mathbf{m}_0 = \{-1,-1\}$, $\mathbf{m}_1 = \{1,0\}$, and $\mathbf{m}_2 = \{0,1\}$. The general first excited twisted-sector state is then

$$|\tau_1\rangle = u^\mu_a|\mu,a;k\rangle, \tag{4.36}$$

where $|\mu,a;k\rangle = e^{ik\cdot X}|\mu,a\rangle$. The constraint (4.35) implies that we are free to redefine the spin-3/2 polarization by $u^\mu \to u^\mu + \gamma^\mu\tilde{u}$, where $\tilde{u}$ is an arbitrary spinor. We fix this freedom by taking

$$\gamma_\mu u^\mu = 0. \tag{4.37}$$

The nontrivial physical state conditions come from the L_0, $G^\pm_0$, and $G^\pm_{1/2}$ modes:

$$(L_0 - h_t)|\tau_1\rangle = 0 \quad\Rightarrow\quad \left(k^2 - 2h_t + \tfrac{5}{4}\right)u^\mu = 0, \tag{4.38a}$$

$$(G^\pm_0 - \Lambda^\pm)|\tau_1\rangle$$

$$= 0 \quad\Rightarrow\quad \left(\pm ik\cdot\gamma + \tfrac{5}{8} - 2^{2/3}\sqrt{2}\Lambda^\pm\right)u^\mu = 0, \tag{4.38b}$$

$$G^\pm_{1/2}|\tau_1\rangle = 0 \quad\Rightarrow\quad k\cdot u = 0. \tag{4.38c}$$

Together with (4.37) these are the correct equations of motion for a spin-3/2 particle. Consistency of the Dirac equations (4.38b) with the L_0 condition (4.38a) implies the intercepts are related by

$$2^{2/3}\sqrt{2}\Lambda^\pm = \tfrac{5}{8} \mp \sqrt{\tfrac{5}{4} - 2h_t}. \tag{4.39}$$

Note that the $G^\pm_{1/2}$ condition (4.38c) is redundant, since it follows from the Dirac equations (4.38b) along with the constraint (4.37).

From these two lowest R'-sector levels, it already follows that the maximal set of physical state conditions (4.30) must be modified. In particular, the relations (4.33) and (4.39) between the $\Lambda^\pm$ and h_t intercepts are different and have no common solution. If, however, we restrict ourselves to the set of physical state conditions (3.17) discussed in Sec. III B and parametrized by α, we find that there is then a common solution for the intercepts only for $\alpha = 0$, $h_t = 1/8$, and $\Lambda^+ = -3\cdot 2^{-25/6}$. In other words, a consistent set of R'-sector physical state conditions may be to impose only the vanishing of the G^+ annihilation operators and the $G^+_0 - \Lambda^+ = 0$ condition:

$$\begin{aligned}(L_n - h_t\delta_{n,0})|\tau\rangle &= 0,\\ (G^+_{n/2} - \Lambda^+\delta_{n,0})|\tau\rangle &= 0,\end{aligned} \tag{4.40}$$

for $n \geq 0$ an integer. The action of the G^- operators, and in particular the value of the G^-_0 intercept, would then be free to vary from state to state as determined by the GCR's for the $c = 5$ model. With this choice of physical state conditons, the lowest-level R'-sector state τ_0 is a massless Majorana spinor, and the next level describes a massive spin-3/2 particle. Note also that since the $G^+_{1/2}$ condition (4.38c) is redundant, it might also be consistent to discard the half-odd-integral modings of G^+ as physical state conditions. A potentially useful exercise would be to compute R'-sector physical states at higher levels to check these conjectures.

The simplest nontrivial scattering amplitude we can write using the prescription (3.8) developed in Sec. III is a three-point coupling for two R'-sector ground states τ_0 and the massless vector state from the untwisted sector. This latter state was worked out for the $c = 5$ model in Ref. [2], and is described (in the $\mathbb{Z}_3$ charge $+1$ sector) by the vertex

$$W^+(z) = \xi^\mu\left[\epsilon^+_\mu(z) + i\varepsilon_\mu{}^{\nu\rho}k_\nu\epsilon^+_\rho(z)\right]e^{ik\cdot X(z)}, \tag{4.41}$$

where the momentum and polarization satisfy $k\cdot k = k\cdot\xi = 0$. The three-point coupling is then easily worked out:

$$\begin{aligned}\mathcal{A}_{tvt} &= \langle\tau_0;v,k_1|W^+(\xi,k_2;1)|\tau_0;u,k_3\rangle\\ &= 2^{-2/3}\bar{v}\xi_\mu\left(\gamma^\mu + i\varepsilon^\mu{}_{\nu\rho}k_2^\nu\gamma^\rho\right)u\,\delta^3(k_1+k_2+k_3).\end{aligned} \tag{4.42}$$

The first term is just the expected minimal coupling of the fermions to the gauge field. The second term represents a derivative coupling which is higher-order in the string tension, and therefore is suppressed at energies below the Planck scale. This string correction to minimal coupling does not occur in the corresponding ordinary superstring amplitude, although string correction terms do appear in higher-point functions.

Higher-point tree amplitudes can be calculated similarly using W^+ vertices and $S^+ = (G^+_0 - \Lambda^+)^{-1}$ propagators, in accordance with the prescription developed in Sec. III B. As mentioned in that section, even without an understanding of the world-sheet "fractional superghost" system, the correctness of our twisted-sector scattering prescription can still be tested at the tree level by computing four- or higher-point scattering amplitudes and checking whether duality is satisfied. In particular, it would be interesting to work out some four-point amplitudes with two twisted-sector states and two untwisted-sector states. One could then check for duality by looking at the spectrum of poles in the t channel to see if it matched the spectrum of the untwisted sector for some values of the twisted-sector intercepts, while factorizing in the s channel will give information on the physical twisted-sector spectrum and may help clarify the correct R'-sector physical state conditions.

V. THE R SECTOR IN A $c = 2$ MODEL

The issue of the correct set of physical state conditions in the R sector is simpler than in the R' sector because of the mode identifications (2.10) which imply that there is really only one independent fractional supercurrent in the R sector. In particular, since $G_0^- = \delta G_0^+$, the $G_0^\pm$ intercepts must be related to each other by $\Lambda^- = \delta\Lambda^+$. [Recall that $\delta = \mathrm{sgn}(8-c)$.] Also, by (2.18), the G_0^+ intercept is related in a model-independent way to the L_0 intercept:

$$\Lambda^+\left(\Lambda^+ - \frac{\delta\lambda^+}{2^{8/3}}\right) = \frac{\delta}{2^{4/3}}\left(h_t - \frac{5c}{128}\right). \tag{5.1}$$

This implies that

$$\left(G_0^+ + \Lambda^+ - \frac{\delta\lambda^+}{2^{8/3}}\right)\left(G_0^+ - \Lambda^+\right) = \frac{\delta}{2^{4/3}}\left(L_0 - h_t\right), \tag{5.2}$$

thus ensuring that the Dirac equation resulting from the $G_0^+ - \Lambda^+ = 0$ physical state condition is automatically consistent with the mass-shell constraint coming from the $L_0 - h_t = 0$ physical state condition.

It is natural to ask whether the R sector can be realized in the $c = 5$ model described in the last section. From the discussion of Sec. II, the R sector is characterized by the automorphism of the FSC algebra which interchanges $G^+ \leftrightarrow \delta G^-$. This automorphism is extended to the whole $c = 5$ CFT by the *simultaneous* transformations $X^\mu \to -X^\mu$ and $\varphi^j \to -\varphi^j$. Thus the R sector can be realized in the $c = 5$ model, but as the twisted-sector of a $\mathbb{Z}_2$ orbifold of all five boson fields on the world sheet. This has unfortunate consequences for the physical interpretation of states in this sector since orbifolding the coordinate boson fields X^μ does not leave the generators of space-time translations ∂X^μ invariant. Thus there are no translation-invariant states in the R sector of the $c = 5$ model.

However, one should not conclude from this that the R sector is in general badly behaved from a space-time point of view, rather, this behavior is only a property of specific CFT models of the spin-4/3 FSC algebra. As an example in support of this statement, we briefly describe in this section a $c = 2$ model in which the R sector appears without orbifolding the coordinate boson fields. Unfortunately, this model has only one space-time dimension, so the resulting space-time physics is trivial; however, we can still demonstrate the existence of R-sector physical states which are translationally invariant in the one space-time dimension.

The $c = 2$ model is written in terms of two free bosons, X and φ, satisfying $X(z)X(w) = -\ln(z-w)$ and $\varphi(z)\varphi(w) = -\frac{2}{3}\ln(z-w)$, with φ compactified on the unit circle $\varphi = \varphi + 2\pi$ [9]. The vertex operators in the φ-CFT, $V_m = e^{im\varphi}$ for $m \in \mathbb{Z}$, have conformal dimensions $h(V_m) = m^2/3$, and carry the (untwisted-sector) $\mathbb{Z}_3$ charge $q = m \bmod 3$. Denoting the dimension-1/3 and 4/3 operators by

$$\epsilon^\pm = V_{\pm 1}, \quad s^\pm = V_{\mp 2}, \tag{5.3}$$

it is easy to check that the spin-4/3 FSC algebra currents are given by

$$G^\pm = \frac{1}{\sqrt{2}}\left(i\epsilon^\pm\partial X + \frac{1}{\sqrt{2}}s^\pm\right). \tag{5.4}$$

Comparing to the expression (4.3) for the currents in the $c = 5$ model, the important difference for our purposes is the absence of $\pm$ signs in front of the $\epsilon^\pm\partial X$ term in (5.4). This means that the automorphism interchanging $G^+ \leftrightarrow G^-$ is realized in the $c = 2$ CFT by the transformation $\varphi \to -\varphi$ *without* any accompanying reflection of the X coordinate boson.

Thus, the R sector states of the $c = 2$ model are realized as states in the twisted sector of the $\mathbb{Z}_2$ orbifold of the single φ boson. Acting on this sector, φ has its mode expansion shifted in the standard way,

$$\varphi(z) = \phi + i\sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r}\beta_r z^{-r}, \tag{5.5}$$

with modes satisfying the commutation relations $[\beta_r, \beta_s] = (2r/3)\delta_{r+s}$. The ϕ zero mode commutes with everything, and so can be taken to be a constant, which we set to zero. The basic vertex operators V_m have the normal ordered expansion

$$V_m(z) = 2^{-2m^2/3}z^{-m^2/3}\exp\left\{-\sum_{r<0}\frac{1}{r}m\beta_r z^{-r}\right\} \times\exp\left\{-\sum_{r>0}\frac{1}{r}m\beta_r z^{-r}\right\}. \tag{5.6}$$

The normal ordering factors $(4z)^{-m^2/3}$ are required to ensure the factorizability of amplitudes (associativity of the operator product). From this expression acting on an arbitrary twisted-sector state τ one obtains the bypass relation $V_m * \tau = \omega^{-m^2}V_{-m}\tau$, where $\omega = e^{2\pi i/3}$. In particular, the single-bypass relation for the basic fields are

$$\begin{aligned} \epsilon^\pm * \tau &= \omega^2\epsilon^\mp\tau, \\ s^\pm * \tau &= \omega^2 s^\mp\tau, \\ G^\pm * \tau &= \omega^2 G^\mp\tau, \end{aligned} \tag{5.7}$$

the last of which is precisely the defining bypass relation (2.9) of the R sector. (More precisely, this is the bypass relation of the $p = 0$ R sector; the $p = \pm 1$ R sectors can be realized by letting the φ zero mode take the values $\phi = 2\pi/3$ and $4\pi/3$.) In general, these bypass relations imply that $\mathbb{Z}_2$-even fields have only integer moding, while the $\mathbb{Z}_2$-odd fields have only half-odd integer modings.

The twisted-sector ground state is nondegenerate. We denote the corresponding twist field by $\sigma(z)$ and the twist ground state by $|\Omega\rangle = \sigma(0)|0\rangle$. The action of the zero mode of an arbitrary untwisted-sector operator on the twisted-sector ground state is simply $(V_m)_0|\Omega\rangle = 2^{-2m^2/3}|\Omega\rangle$. Since the ground states satisfy $\beta_r|\Omega\rangle = 0$

for $r > 0$, a simple computation analogous to (4.19) reveals that the conformal dimension of the twisted-sector ground state is $h(\sigma) = 1/16$. All other twist fields are created from σ by the repeated action of the φ creation modes β_r with $0 > r \in \mathbb{Z} + \frac{1}{2}$. Thus the twist states have the spectrum of conformal dimensions $h = \frac{1}{16} + \frac{n}{2}$ for $n \geq 0$ an integer.

Acting on a twisted-sector state, the FSC currents $G^{\pm}$ have the mode expansion

$$G^{\pm}(z)\tau(0) = \sum_{n\in\mathbb{Z}} z^{-\frac{4}{3}-\frac{n}{2}} G^{\pm}_{n/2} z^{-n/2}\tau(0). \tag{5.8}$$

Since the coordinate boson field X is unaffected by the orbifolding procedure, its mode expansion is the usual one, as in Eq. (4.21). Recalling the form of the currents (5.4), we obtain

$$G^{\pm}_{n/2} = \frac{1}{\sqrt{2}}\left\{\sum_{m\in\mathbb{Z}} \alpha_m \epsilon^{\pm}_{\frac{n}{2}-m} + \frac{1}{\sqrt{2}} s^{\pm}_{\frac{n}{2}}\right\}. \tag{5.9}$$

The lowest-level state in the R sector is simply $|\tau_0\rangle = e^{ikX}|\Omega\rangle$. The nontrivial physical state conditions are

$$(L_0 - h_t)|\tau_0\rangle = 0 \quad\Rightarrow\quad k^2 - 2h_t + \tfrac{1}{8} = 0, \tag{5.10a}$$

$$(G^+_0 - \Lambda^+)|\tau_0\rangle = 0 \quad\Rightarrow\quad k + 2^{-5/2} - 2^{2/3}\sqrt{2}\Lambda^+ = 0. \tag{5.10b}$$

Consistency of (5.10a) with (5.10b) implies Λ^+ and h_t are related by

$$\Lambda^+ = 2^{-11/3} \pm 2^{-2/3}\sqrt{h_t - \tfrac{1}{16}}, \tag{5.11}$$

which is equivalent to the relation (5.1) derived from the R-sector mode algebra.

The first excited state in the twisted sector is $|\tau_1\rangle = \beta_{-1/2}e^{ikX}|\Omega\rangle$. The nontrivial physical state conditions are

$$(L_0 - h_t)|\tau_1\rangle = 0 \quad\Rightarrow\quad k^2 - 2h_t + \tfrac{9}{8} = 0, \tag{5.12a}$$

$$(G^+_0 - \Lambda^+)|\tau_1\rangle = 0 \quad\Rightarrow\quad k + 13\times 2^{-5/2} + 3\times 2^{2/3}\sqrt{2}\Lambda^+ = 0, \tag{5.12b}$$

$$G^+_{1/2}|\tau_1\rangle = 0 \quad\Rightarrow\quad k - 2^{-3/2} = 0. \tag{5.12c}$$

Consistency of (5.12b) with (5.12a) implies the intercepts are related by

$$\Lambda^+ = -\tfrac{1}{3}\left(13\times 2^{-11/3} \pm 2^{-2/3}\sqrt{h_t - \tfrac{9}{16}}\right). \tag{5.13}$$

Although this is a different functional relation between Λ^+ and h_t than appears in (5.1), they have the common solution $h_t = 5/8$ and $\Lambda^+ = -5\times 2^{-11/3}$, which is precisely realized when the condition (5.12c) is satisfied.

At higher levels in the R sector similar physical states will be found, all with specific values of the intercepts satisfying the relation (5.1). *A priori*, there is no reason to expect all these states, or even an infinite subset of them, to have the same value of the intercept. Of course, in one space-time dimension duality does not require the existence of infinite towers of states. We have thus shown that, to the same level of consistency, both the R and R' sectors can be realized in models of the spin-4/3 fractional superstring. It remains an open question whether either of these sectors is actually realized in a critical ($c = 10$) model of the fractional superstring.

ACKNOWLEDGMENTS

It is a pleasure to thank K. Dienes, J. Grochocinski, Z. Kakushadze, A. LeClair, and E. Witten for useful discussions and comments. P.C.A. would also like to thank the Center for Theoretical Physics at M.I.T., and B. Zwiebach in particular, for their hospitality. The work of P.C.A. was supported by NSF Grant No. PHY92-45317 and by the Ambrose Monell Foundation, and the work of S.-H.H.T. was supported in part by the National Science Foundation.

APPENDIX: TWISTED-SECTOR ZERO MODES OF so(2,1)$_2$

We take into account the action of the φ^j zero modes ϕ^j on the twist states. In particular, the twist ground state forms a representation of the nontrivial zero-mode algebra $[\phi^1, \phi^2] = i\pi$, and so is degenerate. We denote this degeneracy by an index a on the twist field $\sigma^a(z)$ and write for the twist ground state simply

$$|\widetilde{a}\rangle = \sigma^a(0)|0\rangle. \tag{A1}$$

We choose these states to be eigenstates of ϕ^2 (the second component of ϕ^j):

$$\phi^2|\widetilde{a}\rangle = \pi a|\widetilde{a}\rangle, \tag{A2}$$

where, for the moment, a can be any real number. The zero-mode algebra $[\phi^1, \phi^2] = i\pi$ implies that ϕ^1 and ϕ^2 are conjugate variables, and that

$$e^{im\phi^1}|\widetilde{a}\rangle = |\widetilde{a+m}\rangle. \tag{A3}$$

Recall that the classical boson fields φ^j take values on the torus defined by the lattice identifications $\phi^j = \phi^j + 2\pi$. This implies, first of all, that only exponentials $\exp(im_j\phi^j)$ with $m_j \in \mathbb{Z}$ should be considered (since they are single-valued on the torus), and secondly by (A2) that we should identify

$$|\widetilde{a+2}\rangle = e^{2i\phi^1}|\widetilde{a}\rangle = \beta|\widetilde{a}\rangle, \tag{A4}$$

where β will be determined momentarily. Note that since $[e^{2i\phi^1}, e^{i\mathbf{m}\cdot\phi}] = 0$ for integer m_j, $\beta = e^{2i\phi^1}$ is a constant.

The fact that the m_j are constrained to be integers means that for each $0 \leq a < 1$ and every choice of β there is a separate, inequivalent two-dimensional representation of the zero-mode algebra, consisting of the states $|\widetilde{a}\rangle$

and $|a+1\rangle$. Note that in terms of their ϕ^2 eigenvalues, these two states differ by a half-lattice translation, as do the fixed points of the $\mathbb{Z}_2$ orbifold. In this way we match up with the familiar result that the number of (ground state) twist fields in an asymmetric (chiral) orbifold is the square-root of the number of fixed points of the orbifold action [7,10]. Note also that the existence of these infinite number of representations of the zero-mode algebra is a reflection of the symmetry of the vertex operator algebra which takes $V_{\mathbf{m}} \to \beta^{m_1/2}\gamma^{m_2/2}V_{\mathbf{m}}$, where $\beta = e^{2i\phi^1}$ and $\gamma = e^{2i\phi^2} = e^{2i\pi a}$.

We now show that only three of these infinite number of inequivalent representations of the zero-mode algebra are consistent with the so(2,1) symmetry of the CFT. In particular, the zero modes $(U_\mu)_0$ of the generators of the so$(2,1)_2$ current algebra symmetry must obey the so(2,1) algebra

$$[(U_\mu)_0, (U_\nu)_0] = \varepsilon_{\mu\nu\rho}(U^\rho)_0. \tag{A5}$$

Using the definitions of the U_μ fields in terms of vertices (4.2), and the identification of the zero mode of a vertex acting on the ground state as

$$(V_{(m_1,m_2)})_0 = e^{-\frac{i\pi}{2}m_1m_2}2^{-m_ig^{ij}m_j}Z_{(m_1,m_2)}, \tag{A6}$$

from (4.11) where

$$Z_{(m_1,m_2)} = \exp(im_j\phi^j), \tag{A7}$$

one can show, using the Hausdorff formula $e^Ae^B = e^{A+B}e^{\frac{1}{2}[A,B]}$, that the so(2,1) algebra (A5) is satisfied only if

$$\begin{aligned} Z_{(2,1)} + Z_{(-2,-1)} + Z_{(0,3)} + Z_{(0,-3)} &= 0,\\ Z_{(1,2)} + Z_{(-1,-2)} + Z_{(3,0)} + Z_{(-3,0)} &= 0,\\ Z_{(1,-1)} + Z_{(-1,1)} + Z_{(3,3)} + Z_{(-3,-3)} &= 0. \end{aligned} \tag{A8}$$

Acting on the ground states $|\widetilde{a}\rangle$, using (A4) and the Hausdorff formula one can show that these equations reduce to

$$\begin{aligned} \gamma + \gamma^{-2} - \beta - \beta^{-1}\gamma^{-1} &= 0,\\ \beta + \beta^{-2} - \gamma - \gamma^{-1}\beta^{-1} &= 0,\\ \beta\gamma + \beta^{-2}\gamma^{-2} - \beta^{-1} - \gamma^{-1} &= 0, \end{aligned} \tag{A9}$$

where we have defined $\gamma = e^{2i\pi a}$. The only solutions to these equations are $\beta = \gamma = \omega^p$ where $p \in \mathbb{Z}_3$ and $\omega = e^{2\pi i/3}$. These three twisted-sector representations give rise to three inequivalent highest-weight modules of the FSC algebra.

Introducing the new notation $|a\rangle_p$ for the twisted sector ground states to remove the fractional part of the index of the $|\widetilde{a}\rangle$ states,

$$|a\rangle_p \equiv |\widetilde{a+\tfrac{p}{3}}\rangle, \tag{A10}$$

where $a = 0$ or 1, we can write

$$\begin{aligned} e^{i\mathbf{m}\cdot\phi}|a\rangle_p &= \exp\left\{\frac{i\pi}{2}m_2\left(m_1 + 2a + \frac{2p}{3}\right)\right\}|a+m_1\rangle_p,\\ |a+2\rangle_p &= \omega^p|a\rangle_p. \end{aligned} \tag{A11}$$

Using the explicit form of the zero-mode representations given in (A11) and the expression (A6) for the zero mode of a vertex acting on the twisted-sector ground state, it is easy deduce the zero mode actions (4.17) and (4.18).

[1] P.C. Argyres, A. LeClair, and S.-H.H. Tye, Phys. Lett. B **253**, 306 (1991).

[2] P.C. Argyres and S.-H.H. Tye, preceding paper, Phys. Rev. D **49**, 5326 (1994).

[3] Z. Kakushadze and S.-H.H. Tye, Phys. Rev. D **49**, 4122 (1994).

[4] A.B. Zamolodchikov and V.A. Fateev, Zh. Eksp. Teor. Fiz. **89**, 380 (1985) [Sov. Phys. JETP **62**, 215 (1985)]; **90**, 1553 (1986) [**63**, 913 (1986)].

[5] A.B. Zamolodchikov and V.A. Fateev, Theor. Math. Phys. **71**, 451 (1987).

[6] A. Neveu and J.H. Schwarz, Phys. Rev. D **4**, 1109 (1971); for a detailed explanation of this formalism, see, e.g., Chap. 7 of M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).

[7] L. Dixon, J.A. Harvey, C.Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985); **B274**, 285 (1986); S. Hamidi and C. Vafa, *ibid.* **B279**, 465 (1987); L. Dixon, D. Friedan, E. Martinec, and S. Shenker, *ibid.*, **B282**, 13 (1987).

[8] K. Itoh, M. Kato, H. Kunimoto, and M. Sakamoto, Nucl. Phys. **B306**, 362 (1988); M. Sakamoto, Phys. Lett. B **231**, 258 (1989).

[9] P.C. Argyres, E. Lyman, and S.-H.H. Tye, Phys. Rev. **D46**, 4533 (1992).

[10] K.S. Narain, M.H. Sarmadi, and C. Vafa, Nucl. Phys. **B288**, 551 (1987).