Tree scattering amplitudes of the spin-$\frac{4}{3}$ fractional superstring. I. The untwisted sectors

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Scattering amplitudes of the spin-$\frac{4}{3}$ fractional superstring are shown to satisfy spurious state decoupling and cyclic symmetry (duality) at the tree level in the string perturbation expansion. This fractional superstring is characterized by the spin-$\frac{4}{3}$ fractional superconformal algebra—a parafermionic algebra studied by Zamolodchikov and Fateev involving chiral spin-$\frac{4}{3}$ currents on the world sheet in addition to the stress-energy tensor. Examples of tree scattering amplitudes are calculated in an explicit $c = 5$ representation of this fractional superconformal algebra realized in terms of free bosons on the string world sheet. The target space of this model is three-dimensional flat Minkowski space-time with a level-2 Kač-Moody so(2,1) internal symmetry, and has bosons and fermions in its spectrum. Its closed string version contains a graviton in its spectrum. Tree-level unitarity (i.e., the no-ghost theorem for space-time bosonic physical states) can be shown for this model. Since the critical central charge of the spin-4/3 fractional superstring theory is 10, this $c = 5$ representation cannot be consistent at the string loop level. The existence of a critical fractional superstring containing a four-dimensional space-time remains an open question.

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I. INTRODUCTION

String theories are characterized by the local symmetries of two-dimensional field theories on the string world sheet. The bosonic string is invariant under diffeomorphisms and local Weyl rescalings on the world sheet, and the superstring is characterized by a locally supersymmetric version of these symmetries. It is natural to ask whether other symmetries on the world sheet can give rise to consistent string theories. Since fractional-spin fields exist in two-dimensional theories, one can imagine new local symmetries on the world sheet involving fractional-spin currents (replacing the spin-3/2 supercurrent of the superstring). A proposal for a large class of new string theories, called fractional superstrings, based on these fractional symmetries was advanced in Ref. [1]. The critical central charges of the fractional superstrings are smaller than that of the ordinary superstring. Evidence has been presented for the existence of fractional superstrings with potentially realistic phenomenologies in space-times of dimensions four and six [2,3]. This paper presents a spin-4/3 fractional superstring model that is consistent at the tree level in string perturbation theory, and has a low-energy spectrum and scattering amplitudes describing gravity, Yang-Mills theory, and fermions.

The basic idea behind the fractional superstring is to replace the world-sheet supersymmetry of ordinary superstring theory with a world-sheet "fractional supersymmetry." Such a fractional supersymmetry relates world-sheet coordinate boson fields X^μ not to fermions but rather to fields ϵ^i_μ of fractional world-sheet spin h. The fractional supersymmetry is generated by a generalization of the supercurrent, a set of new chiral currents G^i [4–6] whose conformal dimensions are $1+h$. The computationally simplest case after the ordinary superstring is the spin-4/3 fractional superstring where $h = 1/3$, and is the subject of this paper. The dimension-4/3 fractional supercurrents $G^\pm$ are of the form

$$G^\pm(z) \sim \epsilon^\pm_\mu \partial X^\mu + \cdots \ , \tag{1.1}$$

and generate, along with the stress-energy tensor T, the spin-4/3 fractional superconformal algebra. Classically, this spin-4/3 algebra is the constraint algebra arising from gauge fixing the local world-sheet symmetry. Quantum mechanically, the constraints generate physical state conditions which pick out the propagating degrees of freedom from the larger string state space. Although the classical world-sheet gauge symmetry giving rise to a spin-4/3 constraint algebra is not understood at present, we can make progress by taking the constraint algebra itself as a starting point, and checking the consistency of the resulting string theory by constructing unitary scattering amplitudes for the physical states. This approach mimics the original construction of the superstring.

By analogy with the superconformal gauge of the superstring, the stress-energy tensor and fractional supercurrents are assumed to generate the physical state conditions. In particular, physical states are taken to be annihilated by all the positive modes of T and $G^\pm$. The physical states are thus highest-weight states of the frac-

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tional superconformal algebra. In Sec. II we derive the properties of a class of highest-weight modules of this algebra, the *untwisted* modules, using the techniques developed by Zamolodchikov and Fateev [4,5] for studying parafermionic algebras. These modules are organized by a $\mathbb{Z}_3$ symmetry of the fractional superconformal algebra. Highest-weight states with $\mathbb{Z}_3$ charge ± 1 are said to belong to D modules, while those with $\mathbb{Z}_3$ charge 0 are in S modules. The results we derive for these modules are independent of the choice of particular conformal field theory representations of the spin-4/3 fractional superconformal algebra.

We then take a first step toward showing the consistency of fractional superstrings by defining tree scattering amplitudes and showing that they are consistent with the assumed physical state conditions following from the spin-4/3 fractional superconformal algebra. In other words, in tree scattering, physical states never scatter to unphysical states, and null states can also be consistently decoupled from scattering of other physical states. This property is commonly referred to as *spurious state decoupling*, and is shown in Sec. III in a representation-independent way. The argument for spurious state decoupling follows closely that used in the "old covariant formalism" [7] for ordinary superstring amplitudes; however, due to the nonlinearity of the spin-4/3 fractional superconformal algebra, an extra independent cancellation is required for the argument to succeed compared to the ordinary superstring case. The fact that this cancellation does occur is not trivial, and is additional evidence for the basic consistency of fractional superstrings.

Scattering of D-module states can be written in three physically equivalent "pictures," reflecting the $\mathbb{Z}_3$ symmetry of the fractional superconformal algebra, in which the vertex operators for scattering can be one of $W^{\pm}$ of conformal dimension 1/3 and $\mathbb{Z}_3$ charge ± 1 or $V^{(+)}$ of conformal dimension 1 and $\mathbb{Z}_3$ charge 0. This is closely analogous to the two different pictures for scattering of Neveu-Schwarz sector states in the old covariant formalism for the ordinary superstring, in which vertex operators can be either G-parity even dimension-1/2 operators or G-parity odd dimension-1 operators. Scattering of S-module states is more problematic due to the absence of an appropriate dimension-1 vertex operator in that sector. In this respect the S-module states are analogous to the Ramond sector states of the ordinary superstring. However, unlike the Ramond sector, the S-module sector includes the scalar ground state of the spin-4/3 string. From this point of view the S-module states are analogous to the Gliozzi-Scherk-Olive- (GSO-) projected states of the Neveu-Schwarz sector, which include a tachyon state. Indeed, S-module states can also be shown to decouple from tree scattering amplitudes with D-module states by a $\mathbb{Z}_3$ analogue of the GSO projection [8].

A separate issue that can be addressed at the tree level is the unitarity of scattering amplitudes. In particular, spurious state decoupling implies unitarity only if one can prove that the space of physical states has non-negative norm. This latter property is called the *no-ghost theorem*. We will not prove a no-ghost theorem in this paper; however, such a theorem is discussed in Ref. [9] for the three-dimensional model presented in Sec. IV of this paper.

The model presented in Sec. IV is a particular conformal field theory representation of the spin-4/3 fractional superconformal algebra with central charge $c = 5$. It is made up of three free coordinate boson fields X^{μ} on the world sheet and a two-boson representation of the $\mathrm{so}(2,1)_2$ Wess-Zumino-Witten model. This model thus has a global three-dimensional Poincaré invariance. The nonlinear nature of the spin-4/3 fractional superconformal algebra makes the existence of such a representation nontrivial. Also, the states in the model are found to be space-time bosons or fermions, showing that the existence of fractional-spin constraints on the world-sheet need not imply fractional spins in space-time. The untwisted sectors of the fractional superconformal algebra describe space-time bosonic physical states in this representation. Some simple states and their scattering amplitudes are discussed in Sec. IV. In particular, the lowest-mass D-module states describe massless gauge fields for the open string and a graviton for closed or heterotic-type fractional superstrings. Appendix A collects some useful details of the free boson construction of the $\mathrm{so}(2,1)_2$ conformal field theory. Appendix B briefly describes other known representations of the spin-4/3 fractional superconformal algebra.

Fields transforming in $\mathrm{so}(2,1)_2$ spinor representations in the $c = 5$ representation (i.e, as space-time fermions) appear in the twist sector of the $\mathbb{Z}_2$ orbifold of the two-boson theory describing the $\mathrm{so}(2,1)_2$ current algebra. The resulting physical states are highest-weight states of *twisted* modules of the fractional superconformal (FSC) algebra. A companion paper [10] discusses the properties of these modules and the spurious state decoupling argument for scattering amplitudes involving the twist sector highest-weight states.

The structure of the highest-weight modules of the spin-4/3 fractional superconformal algebra (i.e., its Kač determinant formula) can be used to place restrictions on the values of the central charge and the intercepts in various sectors consistent with unitarity. In the bosonic and superstrings, for representations with one timelike (space-time) dimension, a non-negative physical state space occurs up to a maximum value of the central charge. As one passes through this critical value of the central charge the norm of some physical states change sign, implying that at the critical central charge there are extra null states. Thus, one can check for the existence of a critical central charge in a representation-independent way by searching for the occurrence of extra sets of zero-norm physical states. For the spin-4/3 fractional superstring, the critical value of the central charge is found to be $c = 10$ [1].

One immediate consequence of this value of the critical central charge is that the three-dimensional spin-4/3 fractional superstring model that we present as an example in Sec. IV is *not* a critical string since its central charge is $c = 5$. This fact, however, has no significance at the level of tree scattering amplitudes. It is only for loop amplitudes that one expects the condition $c = 10$ to manifest itself, since it comes from an anomaly cancellation condition. Indeed, this is precisely what occurs in the bosonic and superstrings, where unitary tree amplitudes exist for $c \leq 26$ and $c \leq 15$, respectively, but loop amplitudes are only sensible at the upper bounds of these ranges, i.e., at the critical central charges. Since three-dimensional Minkowski space-time is too small to decribe nature, it is encouraging that the central charge of our three-dimensional model is less than 10, allowing the possibility of a critical spin-4/3 fractional superstring containing four-dimensional Minkowski space-time.

This paper is organized so that the technical matter appears in Sec. II. Since some readers may be unfamiliar with the considerations involved in analyzing the properties of highest-weight modules of parafermionic algebras, we have tried to make the other parts of the paper intelligible without reading that section. In particular, if the reader reads only the first two paragraphs of Sec. II and is willing to accept the results summarized in Eqs. (2.24) and (2.25) and (2.27)–(2.31), the discussion of spurious state decoupling in tree scattering amplitudes in Sec. III should be self-contained. However, we have tried to provide sufficient detail to make the arguments of Sec. II intelligible to any reader familiar with the basics of two-dimensional conformal field theory. Also, the three-dimensional model given in Sec. IV provides a concrete example of the abstract considerations of Sec. II, in which all the computations are easy to carry through since only free scalar fields are involved.

II. THE SPIN-4/3 FRACTIONAL SUPERCONFORMAL ALGEBRA

The fractional currents $G^{\pm}(z)$ and the energy-momentum tensor $T(z)$ together generate the fractional superconformal (FSC) chiral algebra, encoded in the singular terms of their operator product expansions (OPE):

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{(z-w)^4}\left\{\frac{c}{2} + 2(z-w)^2 T(w) + (z-w)^3 \partial T(w) + \cdots\right\} ,\\
T(z)G^{\pm}(w) &= \frac{1}{(z-w)^2}\left\{\frac{4}{3}G^{\pm}(w) + (z-w)\partial G^{\pm}(w) + \cdots\right\} ,\\
G^{+}(z)G^{+}(w) &= \frac{\lambda^{+}}{(z-w)^{4/3}}\left\{G^{-}(w) + \frac{1}{2}(z-w)\partial G^{-}(w) + \cdots\right\},\\
G^{-}(z)G^{-}(w) &= \frac{\lambda^{-}}{(z-w)^{4/3}}\left\{G^{+}(w) + \frac{1}{2}(z-w)\partial G^{+}(w) + \cdots\right\},\\
G^{+}(z)G^{-}(w) &= \frac{1}{(z-w)^{8/3}}\left\{\frac{3c}{8} + (z-w)^2 T(w) + \cdots\right\}.
\end{aligned}
\tag{2.1}
$$

The first OPE implies that $T(z)$ obeys the conformal algebra with central charge c, while the second implies that $G^{\pm}(z)$ are dimension-4/3 Virasoro primary fields. The FSC algebra was first studied by Zamolodchikov and Fateev [4,5]. The constants $\lambda^{\pm}$ in the $G^{\pm}G^{\pm}$ OPE's are real parameters which are definite functions of c. We will show below that associativity fixes $\lambda^{+}\lambda^{-} = (8-c)/6$. Using a remaining freedom to rescale the $G^{\pm}$ currents, we choose the conventional values of $\lambda^{\pm}$ to be

$$
\begin{aligned}
\lambda^{+} = \lambda^{-} &= \sqrt{\frac{8-c}{6}} \quad \text{for } c < 8,\\
\lambda^{+} = -\lambda^{-} &= \sqrt{\frac{c-8}{6}} \quad \text{for } c > 8.
\end{aligned}
\tag{2.2}
$$

Conformal invariance fixes all the other coefficients in (2.1). This algebra generates the physical state conditions for the spin-4/3 fractional string. The holomorphic chiral algebra (2.1) is suitable for describing open string states and scattering. For closed strings one must include an antiholomorphic copy of this algebra. We will focus almost exclusively on the open string case, since the generalization to closed string states and tree-scattering amplitudes is straightforward.

An important property of the FSC algebra is its group of automorphisms, which organizes the representation theory of its highest-weight modules. The order-six automorphism group S_3 of the FSC algebra is generated by the transformations

$$G^{\pm} \to \omega^{\pm 1} G^{\pm}, \quad G^{\pm} \to \mathrm{sgn}(8-c) G^{\mp}, \tag{2.3}$$

where $\omega = e^{2\pi i/3}$ is a cube root of unity. We will exploit the $\mathbb{Z}_3$ subgroup of automorphisms generated by the first transformation in (2.3) in the remainder of this section to analyze the properties of the untwisted modules of the FSC algebras using conformal field theory (CFT) techniques developed for parafermionic algebras [4,5]. A basis of states in the untwisted modules can be taken to have definite $\mathbb{Z}_3$ charges q. Highest-weight states with $q = 0$ are said to be in an S module, while D modules have pairs of highest-weight states with $q = \pm 1$. The reader interested in getting to the prescription for spin-4/3 fractional superstring scattering amplitudes can skip to Sec. III which only uses the operator product expansions summarized at the end of this section. In a companion paper [10] we will exploit the $\mathbb{Z}_2$ subgroup of automorphisms generated by the second transformation in (2.3) in order to understand the twisted modules of the spin-4/3 algebra. The next six paragraphs comment on some general features of the spin-4/3 algebra, after which we begin the detailed analysis of its untwisted modules.

Since the choice of the algebra (2.1) essentially defines the spin-4/3 fractional superstring, it is worth briefly mentioning the reasons why one might expect it to give rise both to a sensible and a computationally manageable string theory. The representation theory of the FSC algebra (and related nonlocal algebras) is well studied [4–6,11,1,12,9]. It is known to have a representation theory similar to that of the conformal and superconformal algebras. In particular, it has a series of unitary minimal representations realized by the coset models $\mathrm{su}(2)_4\otimes\mathrm{su}(2)_L/\mathrm{su}(2)_{L+4}$ with central charges which accumulate at a particularly simple $c = 2$ model as $L \to \infty$, analogous to the free field representation of the conformal algebra with central charge 1, or the superconformal algebra with central charge 3/2. Presumably a continuum of representations exists for $c \geq 2$; some simple examples will be given later in this paper.

An important feature of the FSC algebra is the appearance of cuts in the GG OPE's. Since there is only a single cut on the right-hand side of each OPE, upon continuation of a correlation function involving, say, $G^\pm(z)G^\pm(w)$ along a contour interchanging z and w it is consistent for the correlator to pick up a definite phase. This situation is described by saying that the currents $G^\pm$ are *Abelianly braided* (or *parafermionic*). Under interchange of z and w (along a prescribed path, say a counterclockwise switch) the only consistent phase that G^+ or G^- can pick up with itself is $e^{2i\pi/3}$. The phase that develops upon interchange of G^+ with G^- can be taken to be $e^{-2i\pi/3}$.

Because the operator algebra (2.1) is Abelianly braided, one can derive a Ward identity relating correlators with a G^+G^- pair to ones with the pair removed [4,5]. One can then solve for the structure constants $\lambda^\pm$ by imposing the associativity condition—independence of which G^+G^- pair we apply the Ward identity to—on, say, the four-point function $\langle G^+(z_1)G^+(z_2)G^-(z_3)G^-(z_4)\rangle$, giving (2.2). The crucial fact that enables us to integrate the Ward identity is the absence of fractional cuts not allowed by Abelian braiding property on the right-hand side of the OPE's (2.1), even among the "regular" terms. This argument is described in more detail in Appendix C of Ref. [13]; however, we will not pursue it further here since we will be able to derive (2.2) from the generalized commutation relations satisfied by the modes of the currents, to be discussed below.

The Abelian braiding of the currents tightly constrains the form of the FSC algebra. For example, the appearance of a new primary dimension-7/3 field on the right-hand side of the G^+G^+ or G^-G^- OPE would not be consistent with Abelian braiding. On the other hand, a primary dimension-1 field (and its Virasoro descendents) could appear in the G^+G^- OPE consistent with Abelian braiding as long as it appeared with opposite sign in the G^-G^+ OPE, similar to the way the spin-1 current enters in the $N = 2$ superconformal algebra. However, we exclude such an operator from (2.1) because it can be shown that such an algebra is only associative for $c = 1$, making it unsuitable for constructing a string theory.

An important consequence of the associativity condition (2.2) is that representations of the FSC algebra can*not* be tensored together to form new representations of the FSC algebra. Given two representations of the FSC algebra with the same central charge c_0, and therefore the same structure constants $\lambda^\pm(c_0)$, and currents $G_i^\pm$, T_i for $i = 1, 2$, it may seem that one could form a new representation by tensoring them together. The tensor-product algebra would have currents $G^\pm = G_1^\pm + G_2^\pm$, $T = T_1 + T_2$, central charge $c = 2c_0$, and structure constant $\lambda^\pm(c_0) = \lambda^\pm(c/2)$; however, the new central charge and structure constant are no longer related by (2.2), indicating that the tensor-product representation is not really a representation of the FSC algebra (2.1). The problem is not that associativity somehow breaks down for the tensor-product representation, but rather that taking tensor products introduces new fractional powers among the regular terms of the OPE's, implying that the braid relations of $G^\pm$ in the tensor-product CFT are different from those in the FSC algebra. For example, the first regular term that would appear in the G^+G^+ OPE in the tensor-product CFT is $G^+(z)G^+(w) \sim (z-w)^0 :G_1^+G_2^+:(w)$. This term and its descendents all appear with integer powers of $(z-w)$. Although these terms do not introduce "cuts," they do nevertheless involve powers of $(z-w)$ that do not appear (mod integers) among the leading terms of the FSC algebra OPE's. The basic lesson is that it is not the OPE's alone that define a chiral algebra; they must be supplemented by the braid relations satisfied by the currents.

Thus, the nonlinearity of the FSC algebra, indicated by the dependence of the structure constants $\lambda^\pm$ on c, implies the absence of tensor-product representations of the algebra. We will see that it is this nonlinearity, rather than the fractional dimension of the currents $G^\pm$, that raises the main obstacles to the existence and tractability of the spin-4/3 fractional superstring. Indeed, the existence of sensible tree-scattering amplitudes despite the nonlinearity of the FSC algebra will appear to occur, in our formulation, due to an "accidental" algebraic cancellation which has no counterpart in the analogous formulation of bosonic or superstring scattering amplitudes.

A. The FSC mode algebra

The physical states of the spin-4/3 fractional superstring are annihilated by the positive modes of T and $G^\pm$, in analogy to the "old covariant" formulation of the bosonic string and superstring in (super)conformal gauge. In this section we will define what we mean by the modes of the $G^\pm$ current, and will derive the algebra that these modes satisfy. This discussion will actually only be valid for certain "untwisted" sectors of states analogous to the Neveu-Schwarz sector of the superstring. The analysis for the analogues of the Ramond sector appears in Ref. [10].

It will be important in the sequel to know the *monodromies* of the currents $G^\pm$. The monodromies are the phases picked up when the insertion point of one current is continued along a closed path around the insertion point of another current. We choose this path to be a simple counterclockwise closed loop, and denote the analytic continuation of a field $V(z)$ around $W(w)$ by a *bypass relation* [4], denoted $V(z) * W(w)$, and illustrated in Fig. 1. The monodromies are thus simply the phases acquired upon braiding a pair of fields twice. The bypass relations satisfied by the fractional currents can be read off from the FSC algebra OPE's (2.1):

$$\begin{aligned} T(z) * T(w) &= T(z)\, T(w), \\ T(z) * G^{\pm}(w) &= T(z)\, G^{\pm}(w), \\ G^{\pm}(z) * G^{\pm}(w) &= e^{-2i\pi/3} G^{\pm}(z)\, G^{\pm}(w), \\ G^{\pm}(z) * G^{\mp}(w) &= e^{+2i\pi/3} G^{\pm}(z)\, G^{\mp}(w). \end{aligned} \tag{2.4}$$

As was pointed out in Ref. [5], the FSC algebra (2.1) has a $\mathbb{Z}_3$ symmetry that is useful in organizing its representation theory. In particular, the currents G^+ and G^- can be assigned $\mathbb{Z}_3$ charges $q = 1$ and -1, respectively, while the energy-momentum tensor T (as well as the identity) has charge $q = 0$. It is natural to assume that, since the FSC algebra is supposed to be an organizing symmetry of our theory, all the fields in a representation will have definite $\mathbb{Z}_3$ charges, and that these fields will have the same monodromies with the FSC currents as the currents have with themselves. These conditions define the class of untwisted representations of the FSC algebra. Along with the parafermionic nature of the FSC algebra, these properties enable one to learn much about the structure of these representations [4].

So, assume the state space of FSC algebra representations falls into sectors $\mathcal{U}_q$ labeled by their $\mathbb{Z}_3$ charge. The currents G^+ and G^- have $\mathbb{Z}_3$ charges $q = +1$ and $q = -1$, respectively, and so act on the Fock space sectors as $G^{\pm} : \mathcal{U}_q \to \mathcal{U}_{q\pm 1}$. A state $\chi_q \in \mathcal{U}_q$ obeys the bypass relations

$$T * \chi_q = T\, \chi_q, \quad G^{\pm} * \chi_q = e^{\mp 2i\pi q/3} G^{\pm}\, \chi_q. \tag{2.5}$$

Note that these monodromies are consistent with (2.4) and the $\mathbb{Z}_3$ charge assignments of T and $G^{\pm}$. These bypass relations imply that we can define the mode expansions of G^+ and G^- acting on any state χ_q by

$$\begin{aligned} G^+(z)\chi_q(0) &= \sum_{n\in\mathbb{Z}} z^{n-q/3} G^+_{-1-n-(1-q)/3}\chi_q(0)\ , \\ G^-(z)\chi_q(0) &= \sum_{n\in\mathbb{Z}} z^{n+q/3} G^-_{-1-n-(1+q)/3}\chi_q(0)\ . \end{aligned} \tag{2.6}$$

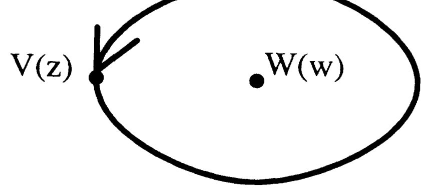

FIG. 1. Path defining the bypass relation $V * W$.

The point is simply that the powers appearing on the right-hand side are the only ones which pick up phases consistent with (2.5) upon a counterclockwise continuation of z around 0. The moding of the currents labels the (operator) coefficients of these terms with the convention that the value of the mode number is the negative of the dimension of the mode operator. The mode expansions (2.6) can be inverted to give

$$\begin{aligned} G^+_{n-(1-q)/3}\chi_q(0) &= \oint_\gamma \frac{dz}{2\pi i} z^{n+q/3} G^+(z)\chi_q(0)\ , \\ G^-_{n-(1+q)/3}\chi_q(0) &= \oint_\gamma \frac{dz}{2\pi i} z^{n-q/3} G^-(z)\chi_q(0)\ . \end{aligned} \tag{2.7}$$

Here, γ is a contour encircling the origin once, where $\chi_q(0)$ is inserted. The allowed modings of the currents in the different $\mathbb{Z}_3$ sectors are summarized in Fig. 2. Note that the action of a $G^{\pm}$ mode on a state in a given sector will map it to a different sector, where, in general, different modings are allowed.

Following the arguments of Ref. [4], the generalized commutation relations (GCR's) satisfied by the current modes of the FSC algebra (2.1) can be derived. We briefly review this argument by deriving the GCR's of the modes of G^+ with G^-. The general procedure for deriving GCR's for the modes of any Abelianly braided operators should be clear from this example.

Consider the integral

$$\mathcal{I} = \oint_\gamma \frac{dz}{2\pi i} \oint_\delta \frac{dw}{2\pi i} z^{m+q/3} w^{n-q/3} \times (z-w)^{p+2/3} G^+(z) G^-(w)\chi_q(0), \tag{2.8}$$

where m, n, and p are arbitrary integers. The contours γ and δ encircle the origin, with δ inside γ. The fractional parts of the exponents in the integrand are chosen so that the whole integrand is single valued in both the z and w planes. This is possible only because of the Abelian nature of the G^+G^- OPE. Evaluate $\mathcal{I}$ by letting δ shrink down to a small circle near to the origin. In this limit, expand the $(z-w)^\alpha$ factor as $(z-w)^\alpha = \sum_{\ell=0}^{\infty} C_\ell^{(\alpha)} z^{\alpha-\ell} w^\ell$, where $C_\ell^{(\alpha)}$ are the appropriate fractional binomial coefficients:

$$C_\ell^{(\alpha)} = (-1)^\ell \binom{\alpha}{\ell}. \tag{2.9}$$

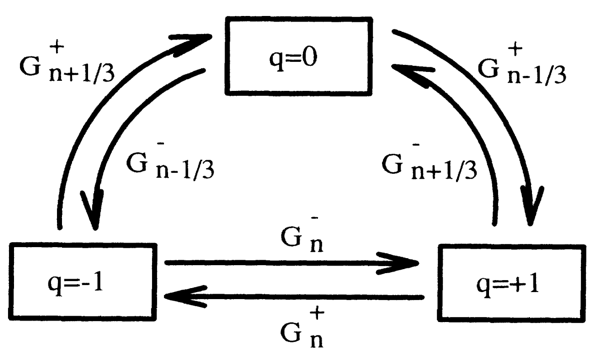

FIG. 2. Modings of $G^{\pm}$ acting on $\mathbb{Z}_3$ sectors of charge q.

Inserting this expansion into (2.8) and using the mode definitions (2.7) gives

$$\mathcal{I} = \sum_{\ell=0}^{\infty} C_\ell^{(p+2/3)} G^+_{m+p-\ell+(1+q)/3} G^-_{n+\ell-(1+q)/3} \chi_q(0). \tag{2.10}$$

$\mathcal{I}$ can also be evaluated in another way, by first deforming the γ contour so that it lies inside δ. Upon performing this deformation, one picks up in the usual way two contributions corresponding to the same integral with γ and δ interchanged, and a contribution where the γ contour encircles the G^- insertion at the point w on the z plane; see Fig. 3. The contribution with the γ and δ contours interchanged is evaluated in the same way as outlined above after interchanging $G^+(z)$ and $G^-(w)$ as well as z and w in the $(z-w)^{p+2/3}$ factor. Taking care to perform these interchanges along equivalent paths in the complex plane gives an overall phase $e^{i\pi(-2/3)} \times e^{i\pi(p+2/3)} = (-1)^p$ (where the Abelian braiding of G^+ with G^- has been used). The second contribution, where γ only encircles the point w in the z plane, is evaluated by letting this contour shrink to a small circle around w and replacing $G^+(z)G^-(w)$ by their OPE. The value of the integer p in the integrand controls the number of terms in the OPE that contribute. For example, taking $p = -1$ and assembling the three contributions shown in Fig. 3 results in the generalized commutation relation for the G^+ and G^- modes given below in Eq. (2.11).

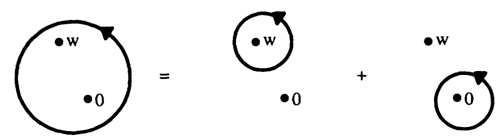

FIG. 3. Deformation of the γ contour in the z plane.

Alternatively we could have chosen another value of p, which would pick up different contributions from the G^+G^- OPE. It is clear that by letting p take more negative values, more complicated GCR's involving more terms from the G^+G^- OPE can be obtained. By conformal invariance, this tower of GCR's is consistent. Indeed, the GCR obtained with $p = p_0$ can be derived from the GCR with $p = p_0 - 1$ using the binomial coefficient identity $C_\ell^{(\alpha)} - C_{\ell-1}^{(\alpha)} = C_\ell^{(\alpha+1)}$. So, there are many GCR's that can be derived from a single OPE, depending on how many terms on the right-hand side of the OPE one wishes to include. We will include only the singular terms, shown in Eq. (2.1).

With this choice, the FSC algebra GCR's become [5]

$$\begin{aligned}
\sum_{\ell=0}^{\infty} C_\ell^{(-2/3)} \left[G^+_{\frac{q}{3}+n-\ell} G^+_{\frac{2+q}{3}+m+\ell} - G^+_{\frac{q}{3}+m-\ell} G^+_{\frac{2+q}{3}+n+\ell} \right] &= \frac{\lambda^+}{2}(n-m) G^-_{\frac{2+2q}{3}+n+m}, \\
\sum_{\ell=0}^{\infty} C_\ell^{(-2/3)} \left[G^-_{-\frac{q}{3}+n-\ell} G^-_{\frac{2-q}{3}+m+\ell} - G^-_{-\frac{q}{3}+m-\ell} G^-_{\frac{2-q}{3}+n+\ell} \right] &= \frac{\lambda^-}{2}(n-m) G^+_{\frac{2-2q}{3}+n+m}, \\
\sum_{\ell=0}^{\infty} C_\ell^{(-1/3)} \left[G^+_{\frac{1+q}{3}+n-\ell} G^-_{-\frac{1+q}{3}+m+\ell} + G^-_{-\frac{2+q}{3}+m-\ell} G^+_{\frac{2+q}{3}+n+\ell} \right] &= L_{n+m} + \frac{3c}{16}\left(n+1+\frac{q}{3}\right)\left(n+\frac{q}{3}\right)\delta_{n+m},
\end{aligned} \tag{2.11}$$

where these expressions are understood to be acting on a state in $\mathcal{U}_q$. Because of the infinite sum on the left-hand sides, the mode algebra in Eq. (2.11) is not a graded Lie algebra, but a new algebraic structure on the string world sheet. This infinite sum is a reflection of the fractional dimension of the current G and the resulting cuts in its OPE's. Although the GCR's look complicated, they are as useful as the familiar (anti)commutators of the (super)Virasoro algebra. The reason for this is that the integer ℓ appearing in the infinite sum is bounded from below. Acting on any state of fixed conformal dimension, the left-hand sides of the GCR's will have only a finite number of nonzero terms since for large enough ℓ the $G^\pm$ modes will annihilate the state. Examples of the use of the GCR's will be given later in this section.

For completeness, we also write down the standard commutators following from the conformal algebra and the fact that $G^\pm$ are dimension-4/3 Virasoro primary fields:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}\ , \\
[L_m, G_r^\pm] &= \left(\frac{m}{3} - r\right) G^\pm_{m+r}\ ,
\end{aligned} \tag{2.12}$$

where the moding r is the one appropriate to whichever $\mathbb{Z}_3$ sector the $G^\pm$ currents are acting on, and the L_n are the standard modes of the stress-energy tensor defined by

$$T(z)\chi_q(0) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \chi_q(0), \tag{2.13}$$

independent of q.

B. FSC highest-weight modules

A *highest-weight state* (or *primary state*) $|\chi\rangle$ of the FSC algebra is a state which is annihilated by all the positive modes of T and $G^\pm$:

$$L_n|\chi\rangle = G^\pm_{n/3}|\chi\rangle = 0, \qquad n \in \mathbb{Z} > 0. \tag{2.14}$$

A *highest-weight module* of the FSC algebra is a highest-weight state $|\chi\rangle$ along with all its *descendent* states formed from $|\chi\rangle$ by the action of creation (or zero) modes of T and $G^\pm$. It is easy to see from the $LG^\pm$ commutator (2.12) that any sequence of L and $G^\pm$ creation modes can be reordered (to a different set) so that the L's are to the right of the $G^{\pm}$'s. Thus the general descendent of $|\chi\rangle$ can be written

$$G^{\pm}_{r_1}\cdots G^{\pm}_{r_p}L_{n_1}\cdots L_{n_q}|\chi\rangle, \quad r_i\in\mathbb{Z}/3\leq 0, n_j\in\mathbb{Z}\leq 0. \tag{2.15}$$

States annihilated by the positive L_n modes are *Virasoro primaries*, while states created from a primary state by the action of the L_{-n} modes alone are *Virasoro descendents*. In general, we will use the term "descendent" without modifier to mean descendents with respect to the FSC algebra. Thus, descendent states can be Virasoro primary or Virasoro descendent.

Before describing the properties of the FSC modules in detail, let us first outline how the FSC mode algebra (2.11) is used in practice. The basic problem that the mode algebra should answer is how any sequence of $G^{\pm}$ and L creation or annihilation modes in which the sum of all the modings is nonpositive, can be written as a descendent state as in (2.15). For the L's alone, this follows from the Virasoro algebra (2.12) by repeatedly commuting the positively moded L's to the right until they annihilate the highest-weight state χ. The analogous operation for the $G^{\pm}$ modes is less clear due to the infinite sums in their GCR's (2.11).

To explain how this can be done, we first need to show a basic property of the GCR algebra (2.11). For any highest-weight state χ, and any $r_i\in\mathbb{Z}/3$,

$$G^{\pm}_{r_1}\cdots G^{\pm}_{r_p}|\chi\rangle = 0, \quad \text{if } \sum_{i=1}^{p} r_i > 0. \tag{2.16}$$

Consider the $p=2$ case first, where we want to show that $G^{\alpha}_{r}G^{\beta}_{s}|\chi\rangle=0$, if $r+s>0$, where α and β are $\pm$. If $s>0$ the expression vanishes because χ is highest weight, so we only need to examine the case $r>-s\geq 0$. From the form of the GCR's (2.11) it follows that

$$\sum_{\ell=0}^{\infty} C^{(a)}_{\ell}\left[G^{\alpha}_{r-\ell}G^{\beta}_{s+\ell}\pm G^{\beta}_{s'-\ell}G^{\alpha}_{r'-\ell}\right]|\chi\rangle \sim \left(G^{\gamma}_{r+s} \text{ or } L_{r+s}\right)|\chi\rangle, \tag{2.17}$$

where $s'=s-1/3$ or $s-2/3$ and $r'=r+1/3$ or $r+2/3$. Since χ is highest weight, and using $r>0$, $r+s>0$, and $C^{(a)}_0=1$, the above expression reduces to the finite sum

$$G^{\alpha}_{r}G^{\beta}_{s}|\chi\rangle = \sum_{\ell=1}^{-s} C^{(a)}_{\ell}G^{\alpha}_{r-\ell}G^{\beta}_{s+\ell}|\chi\rangle. \tag{2.18}$$

Note that all the terms in the sum on the right-hand side are of the same form as the original term on the left-hand side, except that the modings of G^{β} are less negative. Repeatedly applying the same argument to these terms, one can eventually show that they are zero. For the general case $p>2$ perform the same argument on $G^{\alpha}_{r_1}G^{\beta}_{r_2}$ in (2.16) using the $p=2$ result and proceed by induction on p.

This shows how to perform the operation analogous to "commuting a mode to the right" using the GCR's (2.11). Namely, the GCR's relate the product of the mode in question and its neighbor to the right to an infinte sum of products of modes as in (2.17). However, by (2.16) all but a finite number of these terms vanish, and repeated applications of the GCR's on the remaining terms will eventually convert them all to creation operators.

As an example of the use of the GCR's, we derive the associativity constraint (2.2) on the structure constants $\lambda^{\pm}$. Consider a highest-weight state χ with $\mathbb{Z}_3$-charge $q=0$ satisfying $L_0|\chi\rangle=h|\chi\rangle$ and its descendent state

$$|\chi'\rangle = G^{-}_{0}G^{+}_{0}G^{+}_{-1/3}|\chi\rangle. \tag{2.19}$$

We can simplify χ' by using the G^+G^+ GCR of Eq. (2.11) with $q=0$, $m=-1$, $n=0$ to find (since χ is highest weight)

$$|\chi'\rangle = \frac{\lambda^+}{2}G^{-}_{0}G^{-}_{-1/3}|\chi\rangle. \tag{2.20}$$

Now using the G^-G^- GCR with the same values of q, m, and n gives

$$|\chi'\rangle = \frac{\lambda^+\lambda^-}{4}G^{+}_{-1/3}|\chi\rangle. \tag{2.21}$$

Alternatively, we can try to simplify (2.19) with the G^-G^+ GCR. Acting on the state $G^{+}_{-1/3}|\chi\rangle$ (which is in the $q=+1$ sector) with $q=1$, $m=1$, $n=-1$, this GCR gives

$$\begin{aligned}\left(G^{+}_{-1/3}G^{-}_{1/3}+G^{-}_{0}G^{+}_{0}\right)G^{+}_{-1/3}|\chi\rangle &= \left(L_0-\frac{c}{24}\right)G^{+}_{-1/3}|\chi\rangle \\ &= \left(h+\frac{1}{3}-\frac{c}{24}\right)G^{+}_{-1/3}|\chi\rangle,\end{aligned} \tag{2.22}$$

where we have used (2.16) to remove all but two terms from the infinite sum. Using the G^-G^+ GCR again with $q=0$, $m=1$, $n=-1$ on χ shows $G^{-}_{1/3}G^{+}_{-1/3}|\chi\rangle = L_0|\chi\rangle = h|\chi\rangle$, which, when substituted in (2.22), shows that

$$|\chi'\rangle = \left(\frac{1}{3}-\frac{c}{24}\right)G^{+}_{-1/3}|\chi\rangle. \tag{2.23}$$

Comparing (2.21) and (2.23) determines $\lambda^+\lambda^-$.

1. *S modules*

The basic properties of the FSC modules built on highest-weight states with $\mathbb{Z}_3$ charge $q=0$ follow from the nonvanishing modings of $G^{\pm}$ and their GCR's. The modules based on these states are called "S modules" [5]. On states of $\mathbb{Z}_3$ charge zero, the only nonvanishing modings $G^{\pm}_{r}$ have $r\in\mathbb{Z}-1/3$. In particular, there is no (nonvanishing) $G^{\pm}$ zero mode. The only zero mode is thus L_0, and let us choose highest-weight states W_s for these modules to be L_0 eigenstates of eigenvalue h_s. A given S module will be completely determined by h_s and c, the value of the central charge in whatever representation of the FSC algebra we are considering.

The first descendents of $|W_s\rangle$ are $G^{\pm}_{-1/3}|W_s\rangle$, which are Virasoro primary states of dimension $h_s+\frac{1}{3}$ with $\mathbb{Z}_3$ charges $q=\pm 1$. To create further descendents from these states by the action of $G^{\mp}_r$ modes requires either $r\in\mathbb{Z}$ or $r\in\mathbb{Z}-\frac{2}{3}$ (see Fig. 2). Proceeding in this way, it is easy to see that the S module will consist of $q=0$ states of dimension h_s+n, and $q=\pm 1$ states of dimension $h_s+\frac{1}{3}+n$, where $n\geq 0$ is an integer.

One may try to build an infinite series of S-module descendent states of given conformal dimension by the action of the $G^{\pm}_0$ modes. For example, at dimension $h_s+\frac{1}{3}$ we have $(G^-_0G^+_0)^pG^+_{-1/3}|W_s\rangle$, for any non-negative integer p. However, the GCR's (2.11) show that these states are not independent: they are equal to $(\lambda^+\lambda^-/4)^pG^+_{-1/3}|W_s\rangle$. In general, the problem of finding a basis of independent states at each level can be complicated. The number of independent states at each level has been determined in Ref. [9].

The structure of the S-module descendents can be summarized by the operator product expansions of the FSC current $G^{\pm}$ with the W_s primary state and its descendents. They are

$$\begin{aligned}
&G^{\pm}(z)W_s=\frac{V_s^{\pm}}{z}+\cdots,\quad G^{\pm}(z)V_s^{\pm}=\left(\frac{\lambda^{\pm}}{2}\right)\frac{1}{z^{4/3}}\left\{V_s^{\mp}+\frac{2z}{3h_s+1}\partial V_s^{\mp}\right\}+\frac{\widetilde{V}_s^{\mp}}{z^{1/3}}+\cdots,\\
&G^{\pm}(z)V_s^{\mp}=\frac{h_s}{z^{5/3}}\left\{W_s+\frac{z}{2h_s}\partial W_s\right\}\pm\frac{\widetilde{W}_s}{z^{2/3}}+\cdots,\\
&G^{\pm}(z)\widetilde{W}_s=\pm\left(\frac{2-8h_s-c}{6}\right)\frac{1}{z^2}\left\{V_s^{\pm}+\frac{z}{3h_s+1}\partial V_s^{\pm}\right\}\pm\lambda^{\pm}\frac{\widetilde{V}_s^{\pm}}{z}+\frac{h_s}{3}\frac{\widetilde{\widetilde{V}}_s^{\pm}}{z}+\cdots.
\end{aligned}\tag{2.24}$$

For ease of writing, we have inserted the S-module vertex operators W_s, etc., at the origin of the complex plane and have dropped their arguments. $V_s^{\pm}$, $\widetilde{V}_s^{\pm}$, and $\widetilde{\widetilde{V}}_s^{\pm}$ are new Virasoro (though not FSC) primaries of conformal dimension $h_s+\frac{1}{3}$, $h_s+\frac{4}{3}$, and $h_s+\frac{4}{3}$, respectively, while $\widetilde{W}_s$ is a dimension h_s+1 Virasoro primary. They are defined by

$$\begin{aligned}
&|V_s^{\pm}\rangle=G^{\pm}_{-1/3}|W_s\rangle,\\
&|\widetilde{V}_s^{\pm}\rangle=G^{\mp}_{-1}|V_s^{\mp}\rangle-\frac{\lambda^{\mp}}{3h_s+1}L_{-1}|V_s^{\pm}\rangle,\\
&|\widetilde{\widetilde{V}}_s^{\pm}\rangle=2G^{\pm}_{-4/3}|W_s\rangle-\frac{5}{3h_s+1}L_{-1}|V_s^{\pm}\rangle,\\
&|\widetilde{W}_s\rangle=\left(G^+_{-2/3}G^-_{-1/3}-G^-_{-2/3}G^+_{-1/3}\right)|W_s\rangle.
\end{aligned}\tag{2.25}$$

The various coefficients appearing in (2.24) were determined by the FSC mode algebra (2.11). For example,

$$\begin{aligned}
G^+_{1/3}|V_s^-\rangle&=G^+_{1/3}G^-_{-1/3}|W_s\rangle\\
&=L_0|W_s\rangle=h_s|W_s\rangle,
\end{aligned}\tag{2.26}$$

giving the first coefficient in the $G^+V_s^-$ OPE in (2.24). Here the G^+G^- GCR (2.11) was used in the second equality.

We should think of $V_s^{\pm}$ as forming a "fractional supermultiplet" with W_s. Note that W_s is single valued with respect to the currents T and $G^{\pm}$, while $V_s^{\pm}$ have cuts with the fractional current, reflecting the "fractional statistics" of $V_s^{\pm}$ on the world sheet. Summarizing, S-module are characterized by the fields $(W_s,V_s^{\pm})$ belonging to a fractional superconformal multiplet with conformal dimensions $(h_s,h_s+\frac{1}{3})$ and with world-sheet statistics (*bosonic, fractional*).

2. D modules

FSC modules built on highest-weight states with $\mathbb{Z}_3$ charge $q=\pm 1$ are called "D modules" [5]. The nonvanishing modings $G^{\pm}_r$ on $q=\pm 1$ states have $r\in\mathbb{Z}$ or $r\in\mathbb{Z}-2/3$. If we choose the highest-weight states of the D module to have conformal dimension (L_0 eigenvalue) h_d, it follows that descendents will have dimension h_d+n in the $q=\pm 1$ sectors and dimension $h_d+\frac{2}{3}+n$ in the $q=0$ sector, for n a non-negative integer. However, the structure of D modules is more complicated than that of S modules because of the action of the $G^{\pm}_0$ modes on the highest-weight state.

Consider a highest-weight state W^+ with $q=+1$, and conformal dimension h_d. In general, this state will be degenerate with another state $|W^-\rangle=G^+_0|W^+\rangle$ which also has dimension h_d, but has charge $q=-1$. The G^+G^- GCR in (2.11) implies that $G^-_0|W^-\rangle=(h_d-\frac{c}{24})|W^+\rangle$, so fractional highest-weight states are doubly degenerate. It is convenient to normalize these states to satisfy

$$G^{\pm}_0|W_d^{\pm}\rangle=\Lambda^{\pm}|W_d^{\mp}\rangle,\tag{2.27}$$

where

$$\begin{aligned}
&\Lambda^+=\Lambda^-=\sqrt{h_d-\frac{c}{24}}\qquad\text{for}\quad c<24h_d,\\
&\Lambda^+=-\Lambda^-=\sqrt{\frac{c}{24}-h_d}\qquad\text{for}\quad c>24h_d.
\end{aligned}\tag{2.28}$$

The main properties of D modules are summarized by the OPE's of the currents $G^{\pm}$ with the highest weight vertex operators $W_d^{\pm}(z)$ of dimension h_d and their first descendent operators $V_d^{(\pm)}(z)$ of dimension $h_d+\frac{2}{3}$:

$$G^{\pm}(z)W_d^{\pm} = \frac{\Lambda^{\pm}}{z^{4/3}}\left\{W_d^{\mp} + \frac{2z}{3h_d}\partial W_d^{\mp}\right\} + \frac{\widetilde{W}_d^{\mp}}{z^{1/3}} + \cdots, \quad G^{\mp}(z)W_d^{\pm} = \frac{1/2}{z^{2/3}}\left\{V_d^{(+)} \mp V_d^{(-)}\right\} + \cdots,$$
$$G^{\pm}(z)V_d^{(+)} = \left(h_d + \frac{c}{12} + \lambda^{\pm}\Lambda^{\mp}\right)\frac{1}{z^2}\left\{W_d^{\pm} + \frac{z}{3h_d}\partial W_d^{\pm}\right\} - \left(\Lambda^{\pm} - \frac{1}{2}\lambda^{\pm}\right)\frac{\widetilde{W}_d^{\pm}}{z} + \cdots, \tag{2.29}$$
$$G^{\pm}(z)V_d^{(-)} = \mp\left(h_d + \frac{c}{12} - \lambda^{\pm}\Lambda^{\mp}\right)\frac{1}{z^2}\left\{W_d^{\pm} + \frac{z}{3h_d}\partial W_d^{\pm}\right\} \pm \left(\Lambda^{\pm} + \frac{1}{2}\lambda^{\pm}\right)\frac{\widetilde{W}_d^{\pm}}{z} + \cdots.$$

The first OPE defines the two (Virasoro primary) descendent operators of conformal dimension $h_d + 1$ and $\mathbb{Z}_3$ charge $q = \pm 1$,

$$|\widetilde{W}_d^{\pm}\rangle = G_{-1}^{\mp}|W_d^{\mp}\rangle - \frac{2\Lambda^{\mp}}{3h_d}L_{-1}|W_d^{\pm}\rangle, \tag{2.30}$$

while the second OPE defines the $\mathbb{Z}_3$ charge $q = 0$ Virasoro primary descendents of conformal dimension $h_d + \frac{2}{3}$:

$$|V_d^{(\pm)}\rangle = G_{-2/3}^{+}|W_d^{-}\rangle \pm G_{-2/3}^{-}|W_d^{+}\rangle. \tag{2.31}$$

We have put the $\pm$ superscript on the V_d descendent states in parentheses to emphasize that they do *not* refer to the $\mathbb{Z}_3$ charge of these states. We have chosen the particular definition (2.31) of $V_d^{(\pm)}$ for later convenience. Just as in the S-module case, the coefficients of the OPE's (2.29) are determined from the FSC mode algebra (2.11).

We think of $V_d^{(\pm)}$ as forming a fractional supermultiplet with $W_d^{\pm}$. The fractional currents $G^{\pm}$ are single valued with respect to the $q = 0$ descendent $V_d^{(\pm)}$ but has cuts with the $q = \pm 1$ highest-weight states $W_d^{\pm}$. In summary, D modules are characterized by the central charge c and the conformal dimension h_d of their two highest-weight fields. The D module fractional supermultiplets are always of the form of a set of fields $(W_d^{\pm}, V_d^{(\pm)})$ with conformal dimensions $(h_d, h_d + \frac{2}{3})$ and with world-sheet statistics (*fractional* , *bosonic*).

III. TREE SCATTERING AMPLITUDES

In this section we formulate tree-level scattering amplitudes of physical states in the spin-4/3 fractional superstring, and show that these amplitudes obey spurious state decoupling and duality properties. In what follows, we construct open string scattering amplitudes. Closed string scattering amplitudes at the tree level are easily formed by combining two open string amplitudes using a level-matching condition for left and right movers [14]. The construction we use is closely analogous to that of open Ramond-Neveu-Schwarz superstring tree amplitudes in the "old covariant" formalism [7].

We take the physical states of the fractional superstring to be highest-weight states of the FSC algebra. Thus the physical state conditions are the requirement that the positive (annihilation) modes of the FSC currents vanish when acting on physical states. This definition of physical states can be motivated as follows. In the usual superstring, the physical state conditions are constraints following from gauge fixing the local world-sheet symmetry. Classically these constraints in the superconformal gauge are given by the vanishing of the energy-momentum tensor and superconformal current. The full local world-sheet symmetry of the spin-4/3 fractional superstring is unknown, though it should include invariances under reparametrizations and Weyl rescalings of the world sheet. Assume that some analogue of the superconformal gauge exists in the fractional superstring, giving rise to an algebra of constraints generated by the vanishing of $T(z)$ and the fractional superconformal currents $G^{\pm}(z)$. In other words, assume that the fractional superconformal algebra is the quantum version of some classical constraint algebra. Thus, although we do not know of any classical local symmetry on the world sheet that gives rise to a spin-4/3 current as a constraint upon gauge fixing, we nevertheless assume the weak physical state conditions

$$\langle\psi|T(z)|\phi\rangle = \langle\psi|G^{\pm}(z)|\phi\rangle = 0 \ , \tag{3.1}$$

for any physical states $|\phi\rangle$ and $|\psi\rangle$. Just as the stress-energy constraint is satisfied if the physical states are defined to be those states annihilated by the stress-energy modes L_n with $n > 0$, we can factorize the fractional current constraint by demanding that all physical states are annihilated by non-negative modes of $G^{\pm}$. Thus, a physical state $|\phi\rangle$ should satisfy

$$(L_n - h\delta_{n,0})|\phi\rangle = 0 \ , \quad 0 \le n \in \mathbb{Z} \ ,$$
$$G_r^{\pm}|\phi\rangle = 0 \ , \quad 0 < r \in \mathbb{Z}/3 \ , \tag{3.2}$$

where r is the appropriate moding depending on the $\mathbb{Z}_3$ charge of the state, as in Eq. (2.6). Note that, by (2.12), all the positively moded constraints can be generated from those of the set $\{L_1, L_2, G_{1/3}^{\pm}, G_{2/3}^{\pm}, G_1^{\pm}, G_{4/3}^{\pm}\}$.

From the physical state conditions (3.2) it is clear that physical states are highest-weight states of the FSC algebra. If the state has $\mathbb{Z}_3$ charge $q = 0$ it is the highest weight state of an S module with conformal dimension $h_s = h$. If the state has $\mathbb{Z}_3$ charge $q = \pm 1$ it is the highest-weight state of a D module with $h_d = h$. Here h is the "intercept," a normal ordering constant in the definition of T. The value of this intercept should be determined by demanding consistency (unitarity, anomaly cancellation) of the string scattering amplitudes.

The above argument suggests that there should also be a $G_0^{\pm}$ physical state condition for D-module states (integer moding is not allowed on S-module highest-weight states, see Fig. 2). However, since the FSC algebra essentially determines the action of the $G_0^{\pm}$ modes on the two highest-weight states of a D module in terms of their common L_0 intercept h_d as described in Sec. II B 2, we find that we do not need to impose any extra zero-mode physical state condition to those of Eq. (3.2).

The standard properties of spurious and null states follow from the physical state conditions. A state $|s\rangle$ obeying the zero-mode conditions in Eq. (3.2) is called a spurious state if it is orthogonal to all physical states. Such a state can be written as

$$\langle s| = \sum_{n>0} \langle \chi_n | L_n + \sum_{r>0} \langle \psi_r^{\pm} | G_r^{\pm} \ , \tag{3.3}$$

in terms of some other states $|\chi_n\rangle$ and $|\psi_r^{\pm}\rangle$. Since $|s\rangle$ is orthogonal to all physical states, the operator $|s\rangle\langle s|$ must annihilate all physical states. Since the physical state conditions (3.2) are the only restriction on a generic physical state, it follows that $|s\rangle\langle s| = \sum_{n>0} X_n L_n + \sum_{r>0} \Psi_r^{\pm} G_r^{\pm}$ for some operators X_n and $\Psi_r^{\pm}$. Equation (3.3) follows with $\langle \chi_n| = \langle s| X_n$ and $\langle \psi_r^{\pm}| = \langle s|\Psi_r^{\pm}$. All states not satisfying the physical state conditions must have a spurious component. A physical state can itself be spurious, in which case it is a null state (since it is orthogonal to itself), and should decouple from all scattering amplitudes. Thus, the decoupling of all spurious states from scattering amplitudes of physical states is a prerequisite for a sensible interpretation of those amplitudes.

We formulate scattering amplitudes of physical states by first satisfying the requirement of conformal invariance on the string world sheet. This essentially ensures decoupling of spurious states which are created solely by modes of T in (3.3). This consideration and the resulting description of scattering amplitudes is identical to that encountered in the "old covariant" formulation of bosonic string amplitudes [7]. We briefly describe heuristic arguments that lead to a prescription for fractional superstring scattering; however, this presription is only really justified by the spurious state decoupling argument which then follows.

The world sheet in an open string tree scattering process is conformally equivalent to a unit disc with vertex operators $V(x)$ representing the asymptotic scattering states inserted at points on the boundary. Since we must be able to integrate these vertex operators over their insertion positions, they must be dimension-one operators in the two-dimensional world-sheet theory. Furthermore, as in the bosonic string, they must be Virasoro primary operators. We can conformally map the disk to the complex upper half-plane, fixing the positions of three of the vertex insertions at ∞, 1, and 0 on the real axis, with the remaining insertions at points $1 < x_i < \infty$. The boundary conditions at the ends of the string (the real axis) can be implemented by the standard trick of extending the amplitude to the full complex plane so that holomorphic functions on the upper half-plane correspond to left-moving excitations of the open string and holomorphic functions on the lower half-plane correspond to right-moving modes. The boundary conditions then imply the continuity of these functions across the real axis. This picture is suitable for writing the string amplitude as a correlator of holomorphic operators only with a radial-ordering prescription:

$$\mathcal{A}_N = \int \frac{dx_3 \cdots dx_{N-1}}{x_3 \cdots x_{N-1}} \langle V_N | V_{N-1}(x_{N-1}) \cdots V_2(1) | V_1 \rangle \ , \tag{3.4}$$

where the "in" and "out" states are the insertions at $x_1 = 0$ and $x_N = \infty$, and the integration is over all x_i preserving the order $1 < x_3 < \cdots < x_{N-1} < \infty$. A vertex insertion at x can be rewritten as $V(x) = x^{L_0} V(1) x^{-L_0}$, and the positions of the insertions explicitly integrated over to give the amplitude in the form

$$\mathcal{A}_N = \langle V_N | V_{N-1}(1) \widetilde{\Delta} \dots \widetilde{\Delta} V_2(1) | V_1 \rangle, \tag{3.5}$$

where the propagator is $\widetilde{\Delta} = (L_0 - 1)^{-1}$.

From the presentation on the disk, it is clear that $\mathcal{A}_N$ should be invariant under cyclic permutations of the vertex ordering. The cyclic symmetry of open string amplitudes is known as "duality." It can be formulated in the picture corresponding to (3.4) as the requirement that after passing the V_N vertex to the right through all the other vertices the value of $\mathcal{A}_N$ must be unchanged. Now, suppose the string describes particles in some flat space-time with coordinate fields $X^{\mu}(z)$. Then, by translation invariance, the general vertex in (3.4) will be of the form

$$V_i(x) = V_0(k_i, x) e^{ik_i \cdot X(x)}, \tag{3.6}$$

where V_0 depends only on derivatives of X (as well as any other conformal fields on the world sheet). Upon commuting the e^{ikX} factors of two vertices, one picks up the phase $\exp[i\pi k_i \cdot k_j \epsilon(x_i - x_j)]$, where $\epsilon(x) = +1$ if $x > 0$ and -1 if $x < 0$. Commuting this exponential part of the V_N vertex to the right past all the other vertices gives the factor $\exp(-i\pi k_N^2)$, where we have used momentum conservation. Since this factor is independent of the number of vertices V_N was commuted through, whereas the phase that $V_0(k_N, x)$ picks up will depend on how many other V_0's it commutes with, the only requirement consistent with having nonzero scattering of arbitrary numbers of particles is that $k_N^2 \in 2\mathbb{Z}$ and the $V_0(k_i, x)$ commute with each other.

Now, from the representation theory of the FSC algebra, only world-sheet fields with $\mathbb{Z}_3$ charge zero can be commuting operators. Combined with the condition that the V_i vertices have conformal dimension one, this implies tight restrictions on the possible candidate states appearing in the scattering amplitudes. In particular, if the physical state we want to scatter is a D-module highest-weight state $W_d^{\pm}$, the appropriate operators appearing in (3.4) would have to be the $q = 0$ Virasoro primary descendents of $W_d^{\pm}$ of conformal dimension $h_d + n + \frac{2}{3}$. The lowest level such states are $V_d^{(\pm)}$, with dimension $h_d + \frac{2}{3}$. This choice for the V vertex in (3.5) implies the L_0 intercept for D-module highest-weight states to be $h_d = \frac{1}{3}$ in order for the total dimension of the vertex to be 1. We will examine this possibility below, and return to the case of S-module physical state scattering later.

Consider scattering of D-module highest-weight states, where the vertices in Eq. (3.5) may correspond to the $V_d^{(\pm)}$ descendent states in a FSC D module. We can convert Eq. (3.5) to a different "picture" involving the highest-weight states $W_d^{\pm}$ using the general properties of D modules. Evaluate the commutator

$$[G_r^{\pm}, V_d^{(+)}(w)] \equiv \oint_w \frac{dz}{2\pi i} z^{r+1/3} G^{\pm}(z) V_d^{(+)}(w) \quad (3.7)$$

(where the integration contour is around the point w) by inserting the $G^{\pm}(z)V_d^{(+)}(w)$ OPE in (2.29) on the right-hand side, since it involves only integer powers of $z-w$. Setting $w=1$, one finds

$$[G_r^{\pm}, V_d^{(+)}(1)] = \left(h_d + \frac{c}{12} + \lambda^{\pm}\Lambda^{\mp}\right)\left\{(r+\tfrac{1}{3})W_d^{\pm}(1) + \frac{1}{3h_d}\partial W_d^{\pm}(1)\right\} - (\Lambda^{\pm} - \tfrac{1}{2}\lambda^{\pm})\widetilde{W}_d^{\pm}(1). \quad (3.8)$$

In deriving this commutator, we have only used general properties of D modules. For string scattering, though, we should set the dimension of $W_d^{\pm}$ to $h_d = \frac{1}{3}$, since at this value of the intercept $V_d^{(+)}$ has dimension one, as required by conformal invariance. The commutator (3.8) dramatically simplifies at this value of h_d:

$$[G_r^{\pm}, V_d^{(+)}(1)] = (r+\tfrac{1}{3})W_d^{\alpha}(1) + \partial W_d^{\alpha}(1). \quad (3.9)$$

Since $W_d^{\pm}$ are Virsoro primaries of dimension h_d, $[L_0, W_d^{\pm}(1)] = h_d W_d^{\pm}(1) + \partial W_d^{\pm}(1)$, and using this in (3.9) with $h_d = 1/3$ then gives

$$[G_r^{\pm}, V_d^{(+)}(1)] = (L_0 + r - \tfrac{1}{3})W_d^{\pm}(1) - W_d^{\pm}(1)(L_0 - \tfrac{1}{3}) \quad (3.10)$$

for all $r \in \mathbb{Z}/3$.

The crucial point for what follows is that at $h_d = 1/3$ the dimension-$(1+h_d)$ descendents $\widetilde{W}_d^{\pm}$ decoupled from the commutator (3.8). It is this unexpected decoupling at precisely the physical value of the intercept which will allow us to construct sensible tree-scattering amplitudes from Eq. (3.10). Note that the decoupling of $\widetilde{W}_d^{\pm}$ has no counterpart in the analogous formulation of ordinary superstring scattering amplitudes. The occurrence of this operator in the first place is due to the nonlinear structure of the FSC algebra, and its decoupling appears as the result of an algebraic "accident" in this formulation of fractional superstring scattering amplitudes. Note also that the decoupling does not occur for the $V_d^{(-)}$ descendent state. As a result, only $V_d^{(+)}$ will be a consistent choice for the vertices appearing in the amplitude (3.5).

With this understanding, we interpret all the vertices in (3.5) as $V_d^{(+)}$ descendent states of D module physical states $W_d^{\pm}$, and we drop the d subscript. We can now replace the "in" state $|V_1^{(+)}\rangle$ in (3.5) with a physical state using $|V^{(+)}\rangle = G_{-2/3}^{-}|W^{+}\rangle + G_{-2/3}^{+}|W^{-}\rangle$ which follows from Eq. (2.31). The $G_{-2/3}^{\pm}$ modes can be commuted to the left using Eq. (3.10) as well as the relation

$$G_r^{\pm}(L_0 - a - r)^{-1} = (L_0 - a)^{-1} G_r^{\pm}, \quad (3.11)$$

following from Eq. (2.12). Acting on the "out" state, $\langle V^{(+)}|G_{-2/3}^{\pm} = \alpha\langle W^{\pm}|$, which is a consequence of the third OPE in Eq. (2.29) with $h_d = 1/3$. The extra insertions coming from the right-hand side of Eq. (3.10) vanish by a "canceled propagator" argument, since setting $r = -2/3$ in Eq. (3.10) gives factors of $L_0 - 1$ and $L_0 - 1/3$ which cancel the propagators to the left and right, respectively. Tree amplitudes with canceled propagators are holomorphic in the Mandelstam invariant of the canceled propagator channel, and thus, by analyticity, vanish if the amplitudes have Regge asymptotic behavior. We will see in the next section that they do have this soft high energy behavior. The resulting form for the scattering amplitude is

$$\begin{aligned}\mathcal{A}_N = &\langle W_N^{+}|V_{N-1}^{(+)}(1)\Delta\cdots\Delta V_2^{(+)}(1)|W_1^{-}\rangle \\ &+\langle W_N^{-}|V_{N-1}^{(+)}(1)\Delta\cdots\Delta V_2^{(+)}(1)|W_1^{+}\rangle,\end{aligned} \quad (3.12)$$

where the propagator in this picture is $\Delta = (L_0 - \frac{1}{3})^{-1}$. The two terms appearing in (3.12) are actually equal, as is easy to see using the normalization of the $W_d^{\pm}$ states given in (2.27). In particular, one can rewrite $\langle W_N^{+}|$ as $(\Lambda^{-})^{-1}\langle W_N^{-}|G_0^{-}$ and then commute the G_0^{-} to the right using (3.10), (3.11), and the canceled propagator argument until it acts on $|W_1^{-}\rangle$ to give $\Lambda^{-}|W_1^{+}\rangle$. Thus, the final form we find for the scattering amplitude of D-module physical states is

$$\begin{aligned}\mathcal{A}_N &= 2\langle W_N^{+}|V_{N-1}^{(+)}(1)\Delta\cdots\Delta V_2^{(+)}(1)|W_1^{-}\rangle \\ &= 2\langle W_N^{-}|V_{N-1}^{(+)}(1)\Delta\cdots\Delta V_2^{(+)}(1)|W_1^{+}\rangle.\end{aligned} \quad (3.13)$$

These two forms for $\mathcal{A}_N$ along with the expression (3.5) in terms of $q=0$ vertices comprise three physically equivalent "pictures" for computing scattering amplitudes. They are clearly closely related to the $\mathbb{Z}_3$ symmetry of the spin-4/3 FSC algebra.

Now we can investigate the crucial issue of spurious state decoupling in our amplitudes. If we start with physical states defined as highest-weight vectors of FSC modules, will they scatter only to other physical states? For this to be true, only physical states must contribute to residues of poles in amplitudes when an internal propagator goes on-shell. Suppose we fix the external momenta such that some state $|s\rangle$ in the string Fock space at momentum $\kappa = k_{M+1} + \cdots + k_N$ is on-shell: $(L_0 - 1/3)|s\rangle = 0$. If we factorize the amplitude in Eq. (3.13) by inserting a sum over a complete set of states of momentum κ at the propagator between $V_{M+1}^{(+)}$

and $V_M^{(+)}$, then the $|s\rangle\langle s|$ term in the sum will contribute a pole in momentum space. The requirement of spurious state decoupling is that if $|s\rangle$ is spurious, its contribution to the residue of the pole should vanish:

$$\langle s|V_M^{(+)}(1)\Delta\cdots\Delta V_2^{(+)}(1)|W_1^-\rangle = 0. \quad (3.14)$$

To prove this, consider one term, say $\langle\psi^-|G_r^-$ with $r>0$, in the presentation of $\langle s|$ as a sum of descendent states, Eq. (3.3), where $|\psi^-\rangle$ must satisfy $(L_0+r-\frac{1}{3})|\psi^-\rangle = 0$. (The G_r^+ descendent pieces can be shown to decouple by the same argument, and the L_n pieces by a similar, simpler argument.) The G_r^- mode can be commuted to the right in Eq. (3.14) using Eqs. (3.10) and (3.11). The insertions coming from the right-hand side of Eq. (3.10) again vanish by a canceled propagator argument. Finally, the G_r^- mode acting on the "in" state $|W_1\rangle$ vanishes by the physical state conditions Eq. (3.2), thus proving spurious state decoupling.

To examine whether similar considerations can give sensible scattering amplitudes for S-module physical states, first of all note that the intercept in this sector does not have to be the same as that of the D-module physical states. As discussed above, the conditions coming from conformal invariance for an operator to represent a scattering vertex in (3.5) are that it have $\mathbb{Z}_3$ charge $q=0$ and be Virasoro primary of conformal dimension 1. In an S module, appropriate operators would have to be W_s itself (of conformal dimension h_s) or one of its Virasoro primary descendents of dimension h_s+n in order to satisfy the $q=0$ condition. Choosing the intercept $h_s=1$ implies that the V_i in (3.5) be identified with S-module highest-weight states W_s. However, from the S-module OPE's (2.24) we can derive the relation analogous to (3.10):

$$[G_r^\pm, W_s(1)] = V_s^\pm(1) \qquad \text{for all } r\in\mathbb{Z}/3. \quad (3.15)$$

Because this commutator does not have the factors of L_0 on the right-hand side similar to those that appeared in (3.10), there can be no canceled propagator argument to remove the right-hand side of (3.15) when used in evaluating the residue of a spurious state pole, and thus the spurious state decoupling proof fails. One might wish to consider instead the S-module descendent $\widetilde{W}_s$ with dimension h_s+1 by choosing the S-sector intercept $h_s=0$. From the last OPE in Eq. (2.24) follows a commutator similar to that of (3.8). For the picture-changing and spurious state decoupling arguments to go through, though, the contibutions of the dimension $h_s+\frac{4}{3}$ descendents $\widetilde{V}_s^\pm$ and $\widetilde{\widetilde{V}}_s^\pm$ must decouple from the commutator when $h_s=0$. One finds that $\widetilde{V}_s^\pm$ does not decouple. Bosonic sector descendents at higher levels also do not seem to work since they also are created from the W_s primary by quadratic or higher combinations of current modes. This makes a picture-changing argument of the type used to relate (3.5) to (3.13) problematical.

To summarize, our prescription for dual N-point tree amplitudes satisfying spurious state decoupling can be written in either of the equivalent "pictures"

$$\begin{aligned}\mathcal{A}_N &= 2\langle W_d^+|V_d^{(+)}(1)\frac{1}{L_0-\frac{1}{3}}V_d^{(+)}(1)\cdots\frac{1}{L_0-\frac{1}{3}}V_d^{(+)}(1)|W_d^-\rangle\\ &= \langle V_d^{(+)}|V_d^{(+)}(1)\frac{1}{L_0-1}V_d^{(+)}(1)\cdots\frac{1}{L_0-1}V_d^{(+)}(1)|V_d^{(+)}\rangle,\end{aligned} \quad (3.16)$$

where $W_d^\pm$ are D-module physical states which are required to have an L_0 intercept,

$$L_0|W_d^\pm\rangle = \tfrac{1}{3}|W_d^\pm\rangle, \quad (3.17)$$

and their relative normalizations are fixed by

$$G_0^\pm|W_d^\pm\rangle = \Lambda^\pm|W_d^\mp\rangle, \quad (3.18)$$

where $\Lambda^+\Lambda^- = \frac{1}{3}-\frac{c}{24}$ is fixed by associativity of the FSC algebra, and we choose $\Lambda^+ = \sqrt{|\frac{1}{3}-\frac{c}{24}|}$. With this convention the $V_d^{(+)}(x)$ fields are the $q=0$ descendent states

$$|V_d^{(+)}\rangle = G_{-2/3}^+|W_d^-\rangle + G_{-2/3}^-|W_d^+\rangle. \quad (3.19)$$

(Each state or vertex in the amplitude can correspond to a different physical state, of course.)

This prescription can be extended to include two S-module physical states by simply replacing the "in" and "out" $W_d^\pm$ states in Eq. (3.13) with S-module states W_s:

$$\mathcal{A}_N = \langle W_s|V_d^{(+)}(1)\frac{1}{L_0-h_s}\cdots\frac{1}{L_0-h_s}V_d^{(+)}(1)|W_s\rangle. \quad (3.20)$$

The argument for spurious state decoupling then goes through unchanged. However, since there is no appropriate dimension-1 commuting vertex in the S module to play the role of the $V_d^{(+)}$ vertices, we cannot prove cyclic symmetry of the amplitudes with two S-module states, nor can we extend the prescription to include scattering of three or more S-module states. This situation is closely analogous to what happens in the old covariant formalism in the ordinary superstring. There, dual amplitudes with spurious state decoupling can be formulated for scattering of Neveu-Schwarz sector states, and can only be extended to include two Ramond-sector states as the "in" and "out" states in the correlator, thus losing manifest cyclic symmetry. So presumably, just as in the Ramond sector of the superstring, our inability to incorporate more than two S-module physical states in our scattering prescription means that there is a nontrivial contribution to S-module scattering amplitudes coming from the "fractional ghost" fields on the world sheet.

Note, however, that upon factorizing the D-module scattering amplitude in Eq. (3.16) on any propagator, we can never obtain an S-module intermediate state. The reason is simply that the D-module W_d^- "in" state has $\mathbb{Z}_3$ charge $q = -1$ and their descendent $V_d^{(+)}$ vertices have charge $q = 0$. By conservation of $\mathbb{Z}_3$ charge, only $q = -1$ intermediate states can contribute, whereas the S-module physical states W_s have $q = 0$. This means, for example, that

$$\langle W_s | V_d^{(+)}(1) | W_d^- \rangle = 0. \tag{3.21}$$

This selection rule makes it consistent at tree level to drop the S-module physical states altogether, a desirable feature if it turns out that the S-module physical states include tachyons. This projection is closely analogous to the GSO projection in the Neveu-Schwarz sector of the ordinary superstring [8].

IV. A THREE-DIMENSIONAL FRACTIONAL SUPERSTRING

Any string propagating in D flat space-time dimensions will be described by a world sheet CFT which includes D massless scalar fields $X^\mu(\sigma,\tau)$. These fields are interpreted as giving the space-time coordinates X^μ of the point (σ,τ) on the string world sheet. The idea behind the construction of the spin-4/3 fractional superstring is to require that the world-sheet CFT also include a set of fields $\epsilon_\mu^\pm(\sigma,\tau)$ of right-moving (holomorphic) conformal dimension 1/3, transforming as vectors under space-time Lorentz transformations. In addition, we demand that there exist currents $G^\pm(\sigma,\tau)$ of conformal dimension 4/3 on the world sheet of the form $G^\pm(\sigma,\tau) = \epsilon_\mu^\pm \partial X^\mu + \cdots$, obeying the FSC algebra (2.1). As described in the last section, this algebra organizes the world-sheet CFT, allowing us to identify vertex operators corresponding to physical states.

In this section we construct an example of a CFT satisfying these requirements, and having a three-dimensional space-time interpretation. We compute a few low-lying states in the spectrum of this fractional superstring and calculate some of their scattering amplitudes. The CFT in question is particularly simple, being constructed from five free massless scalar fields on the world sheet. In describing this theory, we only write the right-moving parts of the world-sheet fields, i.e., those holomorphic in $z = \tau + i\sigma$. By itself this is suitable for describing an open string; if matched with an appropriate left-moving theory it will describe a closed or heterotic string.

Since we construct this CFT from five free massless scalars, it will have central charge $c = 5$. Three of the scalars are just coordinate boson fields $X^\mu(z)$, $\mu = 0,1,2$, with the standard operator products

$$X^\mu(z) X^\nu(w) = -g^{\mu\nu} \ln(z-w) , \tag{4.1}$$

where $g^{\mu\nu}$ is the three-dimensional Minkowski metric with signature $(-++)$. The stress-energy tensor for these fields is given by

$$T_X = -\tfrac{1}{2} g_{\mu\nu} \partial X^\mu \partial X^\nu. \tag{4.2}$$

This X^μ CFT has a global so(2,1) Lorentz symmetry, generated by the charges

$$M_X^{\mu\nu} = \oint \frac{dz}{2\pi i} \left(X^\mu \partial X^\nu - X^\nu \partial X^\mu \right) . \tag{4.3}$$

We assign Hermiticities and choose conventions so that $(X^\mu)^\dagger = X_\mu$ and $(\partial X^\mu)^\dagger = -\partial X_\mu$.

The remaining two fields, $\varphi^i(z)$, $i = 1,2$, describe a map from the string world sheet to a torus. In a basis in which the target space φ^i boundary conditions are diagonalized,

$$\varphi^i(z) = \varphi^i(z) + 2\pi , \tag{4.4}$$

their operator product expansion (OPE) is

$$\varphi^i(z) \varphi^j(w) = -g^{ij} \ln(z-w) , \tag{4.5}$$

where

$$g^{ij} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{4.6}$$

Alternatively, we could have chosen different linear combinations of the φ^i, say $\tilde{\varphi}^i$, which have the standard OPE's $\tilde{\varphi}^i(z)\tilde{\varphi}^j(w) = -\delta^{ij}\ln(z-w)$, but which would then have boundary conditions more complicated than those in (4.4). Introduce two pairs of vectors $\mathbf{e}^i$ and $\mathbf{e}_i$ satisfying $\mathbf{e}^i \cdot \mathbf{e}^j = g^{ij}$, $\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij}$, and $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$, where g_{ij} is the matrix inverse of g^{ij}, and define the vector-valued field $\boldsymbol{\varphi} = \varphi^i \mathbf{e}_i$. The stress-energy tensor for scalars obeying the OPE's (4.5) is

$$T_\varphi = -\tfrac{1}{2} \partial\boldsymbol{\varphi} \cdot \partial\boldsymbol{\varphi} = -(\partial\varphi^1)^2 - (\partial\varphi^2)^2 - (\partial\varphi^1)(\partial\varphi^2). \tag{4.7}$$

The $\boldsymbol{\varphi}$ CFT also has a global so(2,1) Lorentz symmetry, and there are two dimension-1/3 fields, ϵ_μ^+ and ϵ_μ^-, which transform as vectors under this symmetry. To see this, consider the vertex operators

$$V_{\mathbf{m}} = c(\mathbf{m}) \exp\{i\mathbf{m} \cdot \boldsymbol{\varphi}\}, \tag{4.8}$$

where $\mathbf{m} = m_i \mathbf{e}^i$ and $c(\mathbf{m})$ is an appropriate cocycle operator, described in more detail in Appendix A. Because of the boundary conditions (4.4), these are well-defined fields for integer m_i. The $V_{\mathbf{m}}$ vertex operators are Virasoro primary fields of conformal dimension

$$h(V_{\mathbf{m}}) = \tfrac{1}{2} \mathbf{m} \cdot \mathbf{m} = \tfrac{1}{3} \left(m_1^2 + m_2^2 - m_1 m_2 \right) . \tag{4.9}$$

It follows that all vertex operators have dimensions either an integer or an integer plus 1/3. Note also that $m_1 + m_2$ is (mod 3) just the $\mathbb{Z}_3$ charge of the $V_{\mathbf{m}}$ operator. The "momenta" $\mathbf{m}$ of the vertex operators $V_{\mathbf{m}}$ take values in a triangular lattice, as shown in Fig. 4.

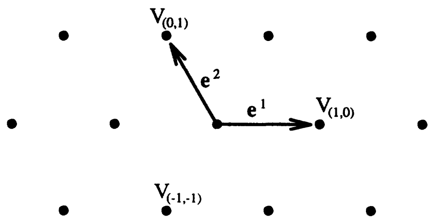

FIG. 4. Triangular su(3) lattice of vertex "momenta" $\mathbf{m} = m_1\mathbf{e}^1 + m_2\mathbf{e}^2$.

This lattice is actually the su(3) weight lattice (g_{ij} is equivalent to the su(3) Cartan matrix), and thus the φ CFT is the standard free boson realization of the su(3)$_1$ Wess-Zumino-Witten model [15]. An so(3) current algebra arises because so(3)$_2$ is conformally embedded in su(3)$_1$, corresponding to the embedding of so(3) as a nonregular subalgebra of su(3). The difference between so(3) and so(2,1) is just a matter of the appropriate choice of cocycles. With these cocycles, the vertex operators satisfy the basic operator product expansion

$$V_{\mathbf{m}}(z)V_{\mathbf{n}}(w) = (-1)^{m_2 n_1}(z-w)^{\mathbf{m}\cdot\mathbf{n}}V_{\mathbf{m}+\mathbf{n}}(w) + \cdots . \tag{4.10}$$

More concretely, consider the triplet of dimension-one fields U_μ:

$$\begin{aligned} U_0 &= V_{(1,-1)} + V_{(-1,1)}, \\ U_1 &= V_{(1,2)} + V_{(-1,-2)}, \\ U_2 &= V_{(-2,-1)} + V_{(2,1)}, \end{aligned} \tag{4.11}$$

where we denote the vertex operator $V_{\mathbf{m}}$ by the components of $\mathbf{m} = m_i\mathbf{e}^i$ as $V_{(m_1,m_2)}$. The U_μ generate the so(2,1)$_2$ Kac-Moody current algebra

$$U^\mu(z)U^\nu(w) = \frac{2g^{\mu\nu}}{(z-w)^2} + \frac{\varepsilon^{\mu\nu\rho}U_\rho(w)}{(z-w)} + \cdots , \tag{4.12}$$

where $\varepsilon^{\mu\nu\rho}$ is the completely antisymmetric tensor density in three dimensions normalized by $\varepsilon^{012} = 1$. The zero modes of these currents,

$$M_\varphi^{\mu\nu} = \oint \frac{dz}{2\pi i}\varepsilon^{\mu\nu\rho}U_\rho(z), \tag{4.13}$$

generate the global so(2,1) Lorentz rotations.

All the fields in the φ CFT can be organized in so(2,1) representations. For example, some of the simplest Virasoro primary fields in the φ CFT are the so(2,1) vector fields ϵ_μ^+ and ϵ_μ^- of conformal dimension 1/3 and the so(2,1) scalars s^+ and s^- of dimension 4/3, given by

$$\begin{aligned} \epsilon_\mu^+ &= \left(V_{(-1,-1)}\, ,\, V_{(1,0)}\, ,\, V_{(0,1)}\right), \\ \epsilon_\mu^- &= \left(V_{(1,1)}\, ,\, V_{(-1,0)}\, ,\, V_{(0,-1)}\right), \\ s^+ &= \tfrac{1}{3}\left[V_{(2,2)} + V_{(-2,0)} + V_{(0,-2)}\right], \\ s^- &= \tfrac{1}{3}\left[V_{(-2,-2)} + V_{(2,0)} + V_{(0,2)}\right]. \end{aligned} \tag{4.14}$$

The other Virasoro primary fields up to dimension 4/3 are the U_μ and a symmetric-traceless $W_{\mu\nu}$ both of dimension 1, and a pair of symmetic-traceless dimension 4/3 fields $t^+_{\mu\nu}$ and $t^-_{\mu\nu}$. (The precise vertex operator definitions of these fields are given in Appendix A.) There are, of course, an infinite tower of Virasoro primary fields of higher dimension on the world sheet. The dimension-1/3 vector fields, ϵ^+_μ and ϵ^-_μ, are the analogues of the dimension-1/2 fermion fields ψ^μ of the ordinary ten-dimensional superstring. Likewise, the dimension-1 adjoint field $\varepsilon^{\mu\nu\rho}U_\rho$ is the analogue of the dimension-1 $\psi^\mu\psi^\nu$ superstring field. On the other hand, the dimension-4/3 scalars, $s^\pm$, and the dimension-1 and -4/3 spin-2 fields, $W_{\mu\nu}$ and $t^\pm_{\mu\nu}$, have no superstring analogues. It is a straightforward exercise to work out the OPE's satisfied by the above fields using the free boson operator product (4.5). The results are collected in Appendix A, where attention has also been paid to the cocycle algebra, needed to get the signs right.

We now construct the fractional supercurrents $G^\pm$. This current must be a dimension-4/3 Virasoro primary field, and a scalar with respect to the global so(2,1) Lorentz symmetry generated by $M^{\mu\nu} = M_X^{\mu\nu} + M_\varphi^{\mu\nu}$. In addition, it should be invariant under translations along the X^μ directions, generated by the momentum $P^\mu = i\oint \frac{dz}{2\pi i}\partial X^\mu$, which together with $M^{\mu\nu}$ generates the full three-dimensional Poincaré group. This implies that $G^\pm$ can only depend on derivatives of X^μ and not on the $e^{ik\cdot X}$ vertex operators. There are clearly only four fields which obey these requirements: $\epsilon^\pm_\mu\partial X^\mu$ and $s^\pm$. The coefficients with which these four fields contribute to $G^\pm$ are fixed by requiring that the OPE's of $G^\pm$ with themselves close only on the stress-energy tensor $T = T_X + T_\varphi$ and $G^\pm$ among its singular terms. Using the OPE's tabulated in Appendix A, the fractional supercurrent is found to be

$$\begin{aligned} G^+ &= \frac{1}{\sqrt{2}}\left(\epsilon^+\cdot\partial X - \frac{3}{2}s^+\right), \\ G^- &= \frac{1}{\sqrt{2}}\left(-\epsilon^-\cdot\partial X - \frac{3}{2}s^-\right). \end{aligned} \tag{4.15}$$

In fact, the coefficents of the terms in $G^\pm$ are overconstrained by the condition that a fractional superconformal algebra closing only on $G^\pm$ and T should exist. The existence of the solution (4.15) is an indication that we have in fact chosen a special world-sheet CFT: the generic CFT with a global Lorentz symmetry and dimension-1/3 vector fields would not have a fractional superconformal symmetry. $G^\pm$ and T satisfy the FSC operator product algebra (2.1,2.2) with $c = 5$.

We should comment on the uniqueness of the expression (4.15) for the fractional supercurrent. First, note that the replacement $V_{\mathbf{m}} \to \widetilde{V}_{\mathbf{m}} = \beta^{m_1}\gamma^{m_2}V_{\mathbf{m}}$ is a symmetry of the basic operator product (4.10), where β and γ can be arbitrary complex numbers. (This can be thought of as the result of a complex shift in the origin of the φ^j boson fields.) Thus, making this replacement in the expressions (4.15) for $G^\pm$ and then reexpressing the $\widetilde{V}_{\mathbf{m}}$'s in terms of β, γ, and the old $V_{\mathbf{m}}$'s will give new expressions for $G^{\pm}$ which will automatically obey the same operator product algebra. Similarly, the X^{μ} OPE's are preserved under the replacement $X^{\mu} \to \Lambda^{\mu}_{\nu} X^{\nu}$ where Λ^{μ}_{ν} is any so(2,1) rotation, and so any such replacement will preserve the FSC algebra OPE's. However, such transformations will not, in general, preserve so(2,1) invariance. Indeed, it is easy to check that there are only six such transformations that *do* preserve the space-time Lorentz invariance. The resulting six solutions for $G^{\pm}$ are

$$G^{\pm} \to \omega^{\pm q} G^{\pm},$$

or

$$G^{\pm} \to \omega^{\pm q} \widetilde{G}^{\pm}, \tag{4.16}$$

where $\omega = e^{2\pi i/3}$, $q \in \mathbb{Z}_3$, and $\widetilde{G}^{\pm}$ are given by

$$\begin{aligned} \widetilde{G}^{+} &= \frac{1}{\sqrt{2}}\left(-\epsilon^{+} \cdot \partial X - \frac{3}{2} s^{+}\right), \\ \widetilde{G}^{-} &= \frac{1}{\sqrt{2}}\left(\epsilon^{-} \cdot \partial X - \frac{3}{2} s^{-}\right), \end{aligned} \tag{4.17}$$

which differs from the solution in (4.15) by a sign change in the $\epsilon^{\pm} \cdot \partial X$ terms. The existence of these six solutions is a consequence of the $\mathbb{Z}_2 \times S_3$ automorphism group of the $c = 5$ CFT generated by $X^{\mu} \to -X^{\mu}$, $V_{\mathbf{m}} \to \exp\{2\pi i(m_1 + m_2)/3\} V_{\mathbf{m}}$, and $V_{\mathbf{m}} \to V_{-\mathbf{m}}$, which leaves the so(2,1) generators invariant. These six solutions for $G^{\pm}$ give rise to equivalent representation theories and thus it is immaterial which of them we choose to be the generators of our physical state conditions.

A. Physical state conditions for general vertex operators

In this subsection we will set up an efficient formalism for computing the action of the modes of the currents $G^{\pm}$ and T on a general state. This enables one to determine, in principle, the physical states at arbitrarily high levels. The method we use involves generalized commutation relations similar to those derived in Sec. II. The reader unfamilar with generalized commutators may safely skip to the next subsection where the lowest-lying physical states and their scattering amplitudes are computed using only the (free boson) OPE's collected in Appendix A.

It is a complicated problem to identify the Virasoro primary so(2,1) covariant combinations of fields of high dimension in the φ CFT. In addition, to compute the action of the physical state conditions on these fields, one must calculate their OPE's with the $G^{\pm}$ currents, which can be a lengthy procedure. A way around this is to express all the states in the φ CFT in terms of the modes of the $\epsilon^{\pm}_{\mu}$ fields. In this basis so(2,1) covariance is manifest. Also, since the modes of the currents $G^{\pm}$ and T can be written in terms of $\epsilon^{\pm}_{\mu}$ modes, all that is needed to compute the physical state conditions on a given state are the generalized commutation relations of the $\epsilon^{\pm}_{\mu}$ modes.

These generalized commutation relations (GCR's) can be derived from the $\epsilon^{\pm}_{\mu}$ OPE's given in Appendix A in the same way that the $G^{\pm}$ GCR's were derived in Sec. II. In particular, picking up just the first term of the $\epsilon^{\pm}_{\mu}\epsilon^{\pm}_{\nu}$ OPE gives

$$\sum_{\ell=0}^{\infty} C_{\ell}^{(-2/3)} \left\{ \epsilon^{\mu\pm}_{m-\ell\pm\frac{q}{3}} \epsilon^{\nu\pm}_{n+\ell+\frac{2\pm q}{3}} - \epsilon^{\nu\pm}_{n-\ell\pm\frac{q}{3}} \epsilon^{\mu\pm}_{m+\ell+\frac{2\pm q}{3}} \right\} = \varepsilon^{\mu\nu}{}_{\rho}\, \epsilon^{\rho\mp}_{m+n+\frac{2\pm 2q}{3}} \tag{4.18}$$

when acting on any state with $\mathbb{Z}_3$ charge q. (The rules for the allowed modings of the $\epsilon^{\pm}$ fields are the same as that for the $G^{\pm}$ currents summarized in Fig. 2.) The binomial coefficient $C_{\ell}^{(\alpha)}$ is given in Eq. (2.9). Picking up only the leading term of the $\epsilon^{\pm}_{\mu}\epsilon^{\mp}_{\nu}$ OPE gives

$$\sum_{\ell=0}^{\infty} C_{\ell}^{(-1/3)} \left\{ \epsilon^{\mu\pm}_{m-\ell+\frac{1\pm q}{3}} \epsilon^{\nu\mp}_{n+\ell+\frac{2\mp q}{3}} + \epsilon^{\nu\mp}_{n-\ell+\frac{1\mp q}{3}} \epsilon^{\mu\pm}_{m+\ell+\frac{2\pm q}{3}} \right\} = g^{\mu\nu} \delta_{m+n+1}. \tag{4.19}$$

Any state in the φ CFT can be written as a polynomial in the $\epsilon^{\pm}$ creation modes acting on the vacuum. The GCR's (4.18) and (4.19) are sufficient to reduce any set of such states to a linearly independent basis.

The current modes can be expressed in terms of $\epsilon^{\pm}$ modes as follows. Since $\epsilon^{\pm} \cdot \epsilon^{\pm} = 3z^{2/3} s^{\mp} + \cdots$, one derives

$$s^{\pm}_{2m-\frac{1\mp q}{3}} = \frac{1}{3} \sum_{\ell=0}^{\infty} C_{\ell}^{(-5/3)} \left\{ \epsilon^{\mp}_{m-\ell-1\mp\frac{q}{3}} \cdot \epsilon^{\mp}_{m+\ell+\frac{2\pm 2q}{3}} + \epsilon^{\mp}_{m-\ell-1\pm\frac{2q}{3}} \cdot \epsilon^{\mp}_{m+l+\frac{2\mp q|}{3}} \right\} \tag{4.20}$$

$$s^{\pm}_{2m+1-\frac{1\mp q}{3}} = \frac{1}{3} \sum_{\ell=0}^{\infty} C_{\ell}^{(-5/3)} \left\{ \epsilon^{\mp}_{m-\ell\mp\frac{q}{3}} \cdot \epsilon^{\mp}_{m+\ell+\frac{2\pm 2q}{3}} + \epsilon^{\mp}_{m-\ell-1\pm\frac{2q}{3}} \cdot \epsilon^{\mp}_{m+\ell+1+\frac{2\mp q}{3}} \right\},$$

and since $\epsilon^{\pm} \cdot \epsilon^{\mp} = 3z^{-2/3} + z^{4/3} T_{\varphi} + \cdots$,

$$L^{\varphi}_{2m} = \sum_{\ell=0}^{\infty} C_{\ell}^{(-7/3)} \left\{ \epsilon^{+}_{m-\ell-\frac{2-q}{3}} \cdot \epsilon^{-}_{m+\ell+\frac{2-q}{3}} + \epsilon^{-}_{m-\ell-\frac{5+q}{3}} \cdot \epsilon^{+}_{m+\ell+\frac{5+q}{3}} \right\} - \frac{q(q+3)}{6} \delta_{m}, \tag{4.21}$$

$$L^{\varphi}_{2m+1} = \sum_{\ell=0}^{\infty} C_{\ell}^{(-7/3)} \left\{ \epsilon^{+}_{m-\ell-\frac{2-q}{3}} \cdot \epsilon^{-}_{m+\ell+\frac{5-q}{3}} + \epsilon^{-}_{m-\ell-\frac{2+q}{3}} \cdot \epsilon^{+}_{m+\ell+\frac{5+q}{3}} \right\},$$

where L^{φ}_{n} are the modes of T_{φ}. Introduce also the usual mode expansion for the X^{μ} fields:

$$X^{\mu}(z) = x^{\mu} - i\alpha^{\mu}_{0}\ln(z) + i\sum_{n\neq 0} \frac{1}{n}\alpha^{\mu}_{n} z^{-n}, \qquad (4.22)$$

satisfying the standard commutation relations $[x^{\mu}, \alpha^{\nu}_{0}] = ig^{\mu\nu}$ and $[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = m\delta_{m+n}g^{\mu\nu}$. Then, from the expressions for the currents $G^{\pm}$ and T worked out earlier in this section, one finds

$$L_m = \frac{1}{2}\sum_{\ell=-\infty}^{\infty} \alpha_{m-\ell}\cdot\alpha_{\ell} + L^{\varphi}_{m},$$
$$G^{\pm}_{r} = \mp\frac{i}{\sqrt{2}}\sum_{\ell=-\infty}^{\infty} \epsilon^{\pm}_{r-\ell}\cdot\alpha_{\ell} - \frac{3}{2\sqrt{2}}s^{\pm}_{r}, \qquad (4.23)$$

which, using (4.20) and (4.21), gives the current modes solely in terms of $\epsilon^{\pm}$ and α modes.

B. Simple vertex operators

The fields of our $c = 5$ CFT are organized into highest-weight modules of the fractional superconformal algebra. We refer to the two fields of lowest conformal dimension in a module as a "fractional superconformal multiplet." The modules are characterized by the dimensions and $\mathbb{Z}_3$ charges of the multiplet fields. These and other properties of the fractional superconformal modules were derived in a general way, independent of any particular CFT representation of the fractional superconformal algebra, in Sec. II. For the purposes of this section, we just illustrate these facts by considering two of the simplest vertex operators in the $c = 5$ theory,

$$W_s = e^{ik\cdot X}, \quad W^{\pm}_{d} = \zeta^{\pm\mu}\epsilon^{\pm}_{\mu}e^{ik\cdot X}, \qquad (4.24)$$

which describe, respectively, scalar and vector particles in space-time as shown by their so(2,1) transformation properties. The $\zeta^{\pm}_{\mu}$ coefficients in W_d are polarization vectors, and the k^{μ} are interpreted as space-time momenta. Both vertices are Virasoro primary fields of conformal dimensions $h(W_s) = \frac{1}{2}k^2$ and $h(W^{\pm}_{d}) = \frac{1}{2}k^2 + \frac{1}{3}$.

Before deriving the properties of these vertex operators in detail, let us summarize the results relevant for the computation of scattering amplitudes as discussed in Sec. III. The properly normalized $W^{\pm}_{d}$ vertices satisfying the physical state conditions are

$$W^{\pm}_{d} = \left(\pm\xi\cdot\epsilon^{\pm} - ik\wedge\xi\cdot\epsilon^{\pm}\right)e^{ik\cdot X}, \qquad (4.25)$$

where $(A\wedge B)^{\mu} = \varepsilon^{\mu\nu\rho}A_{\nu}B_{\rho}$, the polarization ξ^{μ} is transverse $\xi\cdot k = 0$, and the state is massless $k^2 = 0$. The first FSC algebra descendent of this state is

$$V^{(+)}_{d} = -\sqrt{2}\Big[\xi\cdot\partial X - ik\wedge\xi\cdot U + k^{\mu}(k\wedge\xi)^{\nu}W_{\mu\nu}\Big]e^{ik\cdot X}. \qquad (4.26)$$

$W^{\pm}_{d}$ and $V^{(+)}_{d}$ are the vertices appropriate for computing scattering amplitudes using the prescription (3.13) derived in Sec. III.

The operator product algebra of the fractional superconformal currents $G^{\pm}$ with W_s is easily worked out:

$$G^{\pm}(z)W_s(w) = \frac{V^{\pm}_{s}(w)}{(z-w)} + \text{regular terms}, \qquad (4.27)$$

where

$$V^{\pm}_{s} = \mp\frac{i}{\sqrt{2}}k\cdot\epsilon^{\pm}e^{ik\cdot X} \qquad (4.28)$$

are Virasoro primary fields of dimension $\frac{1}{2}k^2 + \frac{1}{3}$. We should think of $V^{\pm}_{s}$ as forming a "fractional supermultiplet" with W_s. Note that no cuts occur in the OPE (4.27) reflecting the fact that W_s is single valued with respect to the currents T and $G^{\pm}$. We describe this situation by saying that W_s is a "bosonic" field on the world sheet. Computing, say, the $G^{\pm}V^{\mp}_{s}$ OPE,

$$G^{\pm}(z)V^{\mp}_{s}(w) = \frac{\frac{1}{2}k^2W_s(w)}{(z-w)^{5/3}} + \frac{\frac{1}{2}\partial W_s(w) \pm \widetilde{W}_s(w)}{(z-w)^{2/3}} + \cdots, \qquad (4.29)$$

we see that it closes back on W_s, along with the higher-dimension Virasoro primary operator $\widetilde{W}_s$ whose form will not be important to us. The fractional powers of $(z-w)$ appearing in (4.29) reflect the "fractional statistics" of $V^{\pm}_{s}$ on the world sheet. Summarizing, we have found that the pair of fields $(W_s, V^{\pm}_{s})$ belong to a fractional superconformal multiplet with conformal dimensions $(\frac{1}{2}k^2, \frac{1}{2}k^2 + \frac{1}{3})$ and with world-sheet statistics *(bosonic, fractional)*.

We can perform a similar analysis for the $W^{\pm}_{d}$ field in (4.27). The first few terms of the $G^{\pm}W^{\pm}_{d}$ OPE's are

$$G^{\mp}(z)W^{\pm}_{d}(w) = \frac{\pm ik\cdot\zeta^{\pm}}{\sqrt{2}(z-w)^{5/3}}e^{ik\cdot X}(w) + \cdots,$$
$$G^{\pm}(z)W^{\pm}_{d}(w) = \frac{(-\zeta^{\pm}\mp 2ik\wedge\zeta^{\pm})\cdot\epsilon^{\mp}}{2\sqrt{2}(z-w)^{4/3}}e^{ik\cdot X}(w) + \cdots. \qquad (4.30)$$

Since the operator $e^{ik\cdot X}$ in the first term of the first OPE has lower conformal dimension than $W^{\pm}_{d}$, it follows that $W^{\pm}_{d}$ is, in general, not a primary fractional superconformal field. In fact, if the polarization vectors take the form $\zeta^{\pm}_{\mu} \sim k_{\mu}$, then we recognize $W^{\pm}_{d}$ as the $V^{\pm}_{s}$ member of the W_s fractional supermultiplet. For $W^{\pm}_{d}$ to be the highest member of its own fractional supermultiplet, we must require the coefficient of the $(z-w)^{-5/3}$ term in (4.30) to vanish:

$$k \cdot \zeta^{\pm} = 0. \tag{4.31}$$

In addition, to normalize the $W_d^{\pm}$ vertices according to the prescription (3.18) used to define scattering amplitudes in Sec. III, we demand that the coefficients of the $(z-w)^{-4/3}$ terms satisfy

$$-\zeta_\mu^{\pm} \mp 2i(k \wedge \zeta^{\pm})_\mu = 2\sqrt{2}\Lambda\zeta_\mu^{\mp}, \tag{4.32}$$

where $2\sqrt{2}\Lambda = \sqrt{4k^2+1}$. (Actually, Λ is fixed to this value by consistency of these equations, and does not, therefore, represent an independent requirement on $\zeta_\mu^{\pm}$.) The solution to (4.31) and (4.32) can be expressed in terms of a single transverse polarization vector ξ^μ: $\xi \cdot k = 0$ and

$$\zeta_\mu^{\pm} = \pm\xi_\mu - \frac{2i(k\wedge\xi)_\mu}{1+\sqrt{4k^2+1}}. \tag{4.33}$$

The $G^{\pm}W_d^{\pm}$ OPE's can then be computed:

$$\begin{aligned} G^{\pm}(z)W_d^{\pm}(w) &= \left(\frac{\sqrt{4k^2+1}}{2\sqrt{2}}\right)\frac{W_d^{\mp}(w)}{(z-w)^{4/3}} + \cdots, \quad G^{\mp}(z)W_d^{\pm}(w) = \frac{1}{2}\frac{\left\{V_d^{(+)}(w) \mp V_d^{(-)}(w)\right\}}{(z-w)^{2/3}} + \cdots, \\ G^{\pm}(z)V_d^{(+)}(w) &= \frac{(2k^2+3+\sqrt{4k^2+1})}{4(z-w)^2}\left\{W_d^{\mp}(w) + \frac{2(z-w)}{3k^2+2}\partial W_d^{\mp}(w)\right\} - \left(\frac{\sqrt{4k^2+1}-1}{2\sqrt{2}}\right)\frac{\widetilde{W}_d^{\mp}(w)}{(z-w)} + \cdots, \\ G^{\pm}(z)V_d^{(-)}(w) &= \mp\frac{(2k^2+3-\sqrt{4k^2+1})}{4(z-w)^2}\left\{W_d^{\mp}(w) + \frac{2(z-w)}{3k^2+2}\partial W_d^{\mp}(w)\right\} \pm \left(\frac{\sqrt{4k^2+1}+1}{2\sqrt{2}}\right)\frac{\widetilde{W}_d^{\mp}(w)}{(z-w)} + \cdots, \end{aligned} \tag{4.34}$$

where we have also written the OPE's of $G^{\pm}$ with the $V_d^{(\pm)}$ descendents of $W_d^{\pm}$. These OPE's are, of course, special cases of the D-module result (2.29) derived in a more general context in Sec. II, and shown to be crucial for spurious state decoupling in tree-level scattering amplitudes in Sec. III. The form of the $V_d^{(\pm)}$ and $\widetilde{W}_d^{\pm}$ descendents can be easily worked out; in the case $k^2 = 0$, the explicit forms of the $V_d^{(\pm)}$ vertices are

$$\begin{aligned} V_d^{(+)} &= -\sqrt{2}\Big[\xi\cdot\partial X - ik\wedge\xi\cdot U + k^\mu(k\wedge\xi)^\nu W_{\mu\nu}\Big]e^{ik\cdot X}, \\ V_d^{(-)} &= +\sqrt{2}\Big[ik\wedge\xi\cdot\partial X - \tfrac{1}{2}\xi\cdot U - ik^\mu\xi^\nu W_{\mu\nu}\Big]e^{ik\cdot X}. \end{aligned} \tag{4.35}$$

Note that in Minkowski space-time $(k\wedge\xi)^\mu$ is proportional to k^μ for lightlike k^μ and transverse ξ^μ. From (4.34) the highest-weight states $W_d^{\pm}$ have cuts in their operator products with the currents $G^{\pm}$, reflecting these states' fractional statistics on the world sheet. On the other hand, the $V_d^{(\pm)}$ states are world-sheet bosons since they have no cuts with $G^{\pm}$. Just as with the W_s state, we say that $(W_d^{\pm}, V_d^{(\pm)})$ form a fractional supermultiplet, but with conformal dimensions $(\frac{1}{2}k^2+\frac{1}{3}, \frac{1}{2}k^2+1)$ and world-sheet statistics (*fractional, bosonic*).

C. Three-point couplings and scattering amplitudes

We will now calculate some tree-level scattering amplitudes of the vector and scalar particles described above. The prescription for computing these amplitudes was worked out in Sec. III, and is summarized in Eqs. (3.16)–(3.21). The intercept condition (3.17) for the D-module physical states implies that our vector vertex must have dimension 1/3. $W_d^{\pm}$ meet this requirement if $k^2 = 0$, thus describing massless vector particles. Furthermore, the operators ∂X_μ, U_μ, and $W_{\mu\nu}$ appearing in $V_d^{(+)}$ are all bosonic fields on the world sheet—they have no cuts in their OPE's with any other field—making them suitable for vertices in dual amplitudes by the arguments presented in Sec. III. The intercept for the S-module vertex W_s describing scalar particles is not fixed by our considerations so far. If, for example, its intercept were 1/3, the same as that of the D module, then W_s would describe a tachyon. We will mention below some considerations which may fix the S-module intercept, but for the present discussion we will leave it arbitrary.

The simplest amplitude to calculate is the three-point coupling of two scalar states to a vector state given by the formula (3.20):

$$\mathcal{A}_{ssv} = \langle W_s(k_3)|V_d^{(+)}(k_2,\xi_2;1)|W_s(k_1)\rangle, \tag{4.36}$$

where we have indicated the momenta and polarization vectors associated to each vertex. Inserting the explicit expressions for the vertices given in Eqs. (4.24) and (4.35), we find

$$\begin{aligned} \mathcal{A}_{ssv} = -\sqrt{2}\langle k_3,0|\xi_2\cdot[\partial X(1) + ik_2\wedge U(1) \\ - k_2\wedge W\cdot k_2]e^{ik_2\cdot X}(1)|k_1,0\rangle. \end{aligned} \tag{4.37}$$

The U_μ and $W_{\mu\nu}$ fields of the φ CFT give no contribution by Lorentz invariance and all that survives is a standard free-boson correlator in the X^μ CFT, giving

$$\mathcal{A}_{ssv} = i\sqrt{2}\xi_2\cdot k_1\delta^3(k_1+k_2+k_3). \tag{4.38}$$

It should be clear that the calculation of any N-point function will reduce to free-field correlators in the φ and X^μ CFTs.

Another three-point amplitude that can be calculated in our formalism is the coupling between one scalar and two vector particles. It is trivial to check that it vanishes identically, illustrating the selection rule (3.21). This implies that the scalar particle can be consistently decoupled (at tree level) from the scattering of vector particles, in close analogy to the way the tachyonic state in the Neveu-Schwarz sector of the ordinary superstring can be decoupled from scattering of the massless vector states. In general, the selection rule (3.21) allows the tree-level decoupling of world-sheet S-module physical states from the scattering of D-module ones, in close analogy to the GSO projection [8] in the Neveu-Schwarz sector of the ordinary superstring.

A less trivial amplitude is the coupling $\mathcal{A}_{vvv}$ of three massless vector states. One expects such a coupling to be gauge invariant since the $V_d^{(+)}$ vertices describe gauge bosons in the transverse gauge $\xi \cdot k = 0$. Indeed, upon making a gauge transformation $\delta\xi^\mu \sim k^\mu$, one finds $\delta V_d^{(+)} \sim \partial(\exp\{ik\cdot X\})$, a spurious state which decouples by the arguments of Sec. III. In fact, with some straightforward algebra using the kinematics of three massless particles one computes explicitly

$$\begin{aligned}\mathcal{A}_{vvv} &= 2\langle W_d^+(k_3,\xi_3)|V_d^{(+)}(k_2,\xi_2;1)|W_d^-(k_1,\xi_1)\rangle \\ &= i2\sqrt{2}\Big[(k_1\cdot\xi_3)(\xi_2\cdot\xi_1)+(k_2\cdot\xi_1)(\xi_3\cdot\xi_2)+(k_3\cdot\xi_2)(\xi_1\cdot\xi_3) \\ &\qquad -(\xi_1\cdot k_2)(\xi_2\cdot k_3)(\xi_3\cdot k_1)\Big]\delta^3(k_1+k_2+k_3). \qquad (4.39)\end{aligned}$$

The first three terms are precisely the expected Yang-Mills coupling; gauge group charges can be introduced by Chan-Paton factors [16] in the usual way. The last term in (4.39) represents a nonlinear correction to the Yang-Mills action which is higher-order in the string tension, and therefore is suppressed at energies far below the Planck scale. The nonlinear term also appears in the three-vector coupling in the bosonic string, though with the opposite sign; in the superstring no such term appears in the three-point coupling (though string correction terms do appear in higher-point functions).

Higher-point amplitudes can also be calculated using the prescription of the last section. The main features of these amplitudes can be easily understood without detailed computation. Consider, for example, the four-point vector particle amplitude

$$\mathcal{A}_{4v} = \int_1^\infty dx\langle V_d^{(+)}(k_4,\xi_4)|V_d^{(+)}(k_3,\xi_3;x) \times V_d^{(+)}(k_2,\xi_2;1)|V_d^{(+)}(k_1,\xi_1)\rangle, \qquad (4.40)$$

in the "picture" of Eq. (3.4). Inserting the expression (4.35) for $V_d^{(+)}$ leads to a sum of terms, each of which is a product of a correlator in the φ CFT and a correlator in the X_μ CFT. Now, only in the X_μ CFT correlators is the dependence on the momenta k_i nonpolynomial, entering through the exponentials as

$$\langle k_4|e^{ik_3\cdot X}(x)e^{ik_2\cdot X}(1)|k_1\rangle \qquad (4.41)$$

(perhaps with extra ∂X^μ insertions as well). These correlators are precisely the ones that enter into bosonic and superstring scattering amplitudes, and give rise to Γ-function dependence on the Mandelstam invariants similar to that which appears in the Veneziano amplitude. These factors result in the extremely soft high-energy Regge behavior characteristic of string amplitudes. Fractional superstring amplitudes will differ from ordinary superstring amplitudes only by the φ CFT correlators which are polynomial in the momenta, and so cannot change the soft high-energy behavior of the amplitudes. This implies, in particular, that the canceled propagator argument used in the last section is justified.

It would be an interesting exercise to calculate the explicit expression for some four-point functions in this three-dimensional fractional superstring model. As an example of what one could learn from such a computation, consider the four-point correlator $\langle W_s|V_d^{(+)}V_d^{(+)}|W_s\rangle$ of two vector and two scalar particles. Although our prescription for including two S-module physical states in scattering amplitudes is not manifestly dual, the final expression should be. That means in practice that one could factorize the expression in the s and t channels and check that the appropriate spectra of intermediate states is recovered. This should place restrictions on the allowed intercepts for S-module physical states.

A no-ghost theorem for this three-dimensional model of fractional superstrings is discussed in Ref. [9], where it is argued that the space of physical states has nonnegative norm. Combined with the spurious state decoupling theorem for tree-scattering amplitudes shown in Sec. III this implies that tree-level amplitudes in the three-dimensional model of spin-4/3 fractional superstrings are unitary.

Higher-point closed fractional superstring or heterotic-type scattering amplitudes can be easily obtained by combining appropriate open string amplitudes [14]. For example, in a closed string we could match a left-moving and a right-moving version of, say, W_d^+ to form the massless physical state

$$W_d^+(z,\bar z) = \xi^{\mu\nu}\Big[\epsilon_\mu^+\bar\epsilon_\nu^+ + g_{\mu\nu}(k\cdot\epsilon^+)(k\cdot\bar\epsilon^+) + \epsilon_\mu^+(ik\wedge\bar\epsilon^+)_\nu + (ik\wedge\epsilon^+)_\mu\bar\epsilon_\nu^+\Big]e^{ik\cdot X} \qquad (4.42)$$

with $k_\mu\xi^{\mu\nu} = k_\nu\xi^{\mu\nu} = 0$. The symmetric-traceless, antisymmetric, and trace parts of $\xi^{\mu\nu}$ will then describe the graviton, the antisymmetric tensor field, and the dilaton (in covariant gauge), respectively, just as in bosonic strings and ordinary superstrings.

V. DISCUSSION AND OUTLOOK

In this paper we have shown how to construct tree-level scattering amplitudes for the spin-4/3 fractional superstring which are dual and obey spurious state decoupling. We have illustrated these properties in an explicit three-dimensional model of the spin-4/3 fractional superstring, and found that it has a sensible space-time spectrum including gauge bosons and a graviton (for closed strings). Tree-level unitarity follows from the spurious state decoupling property once a "no-ghost" theorem for the physical state spectrum in a given representation is proven. A no-ghost theorem has been presented in Ref. [9] for the three-dimensional model discussed in this paper. Space-time fermion states for the $c = 5$ representation of the spin-4/3 algebra are constructed in Ref. [10], where general (representation-independent) spurious state decoupling arguments are also presented for scattering amplitudes involving twisted-sector states.

The tree-level considerations of this paper and Ref. [10] leave us with a certain amount of arbitrariness in constructing spin-4/3 fractional superstrings. In particular, we are free to include or not the world-sheet S-module untwisted-sector states; we can couple left- and right-moving theories at will on the world sheet in type II and heterotic constructions; and the choice of CFT representation of the spin-4/3 FSC algebra is presumably constrained by tree unitarity only to have central charge less than or equal to its critical value $c = 10$. The inclusion of string loop amplitudes should remove much of this arbitrariness. As is the case with the bosonic and superstrings, one expects that loop amplitudes will only be consistent at the critical central charge, and modular invariance will determine which left- and right-moving sectors, and at which values of their intercepts, can be consistently coupled together.

The main difficulty in constructing a critical ($c = 10$) representation of the FSC algebra is its nonlinearity discussed in Sec. II: the tensor product of two representations of this algebra is not itself a representation. In particular, the tensor product of two copies of the $c = 5$ representation described above will not make a $c = 10$ representation of the FSC algebra. For certain representations, one can, however, construct higher-c representations from a given representation by turning on a background charge for one of the $X^\mu(z)$ coordinate boson fields, corresponding to turning on a linear dilaton background in space-time. Also, a set of representations constructed from free bosons have been found, all with central charges $c \leq 8$. These representations are briefly described in Appendix B. It may be that some generalization of these constructions will yield $c > 8$ (and in particular $c = 10$) representations.

Once given a $c = 10$ representation of the FSC algebra, one can imagine "sewing" tree amplitudes in the old covariant formalism described above to form one-loop amplitudes by a suitable generalization of the sewing procedure for the bosonic string [17]. Such an amplitude would not only have to be unitary, but also modular invariant. The construction of a consistent one loop amplitude is a crucial test of the existence of the spin-4/3 fractional superstring theory.

At higher loops it seems likely that a clearer understanding of the "fractional moduli" describing the sewing of tree amplitudes will be necessary. This is essentially the question of determining the local world-sheet symmetry underlying the FSC constraint algebra. Although the form of the FSC algebra provides a rigid guide to such a symmetry, its identification remains an open question. One approach to this problem is to construct a world-sheet ghost system with a nilpotent Becchi-Rouet-Stora-Tyutin (BRST) charge whose cohomology reproduces the physical state conditions analyzed in this paper. The BRST charge would be expected to have the form $Q = cT_m + \gamma^+ G_m^- + \gamma^- G_m^+ + \cdots$ where T_m and $G_m^\pm$ are the "matter" FSC currents, c is the dimension-(-1) reparametrization ghost, and $\gamma^\pm$ are dimension-$(-\frac{1}{3})$ fractional superghost fields. No such ghost system and BRST charge have been constructed. Another possibility is that the BRST ghosts of the fractional superstring and the matter fields are inherently coupled. In this case one should seek a $c = 0$ representation of the FSC algebra that contains both the matter and the ghost fields and permits the construction of a nilpotent BRST charge.

In this paper we considered the fractional superstring based on the spin-4/3 FSC algebra. This world-sheet algebra is associated with the $su(2)_4$ Wess-Zumino-Witten (WZW) model as explained in Appendix B. In general, one can construct fractional algebras associated in the same way to WZW models based on any Lie algebra [6,1,12]. For example, the algebra based on $su(2)_1$ is simply the Virasoro algebra, and its associated string is the bosonic string; associated with $su(2)_2$ is the super-Virasoro algebra which underlies the ordinary superstring. These are special examples in that the resulting algebras are local on the world sheet. Other local algebras underlie the W strings associated with any level-one WZW model. Given the results of this paper, it is natural to speculate that there exist strings corresponding to the nonlocal algebras associated with WZW models at arbitrary levels.

The generic such fractional string, however, will be technically more difficult to work with than the spin-4/3 fractional string. The main reason is that the general fractional chiral algebra does not admit a splitting into Abelianly braided currents as the spin-4/3 FSC algebra did. This splitting was the main technical crutch that allowed us to understand the properties of the FSC modules in a representation-independent way. To deal similarly with an inherently non-Abelianly braided current algebra will require a more thorough understanding of the braiding properties of their currents and the development of the conformal field theory techniques needed to derive their generalized commutation relations.

Two particularly simple series of fractional superconformal algebras are those based on the $so(N)_2$ models for arbitrary N, and those based on the $su(2)_K$ models for arbitrary K. Since conformally, $su(2)_4 = so(3)_2$, the former series includes the spin-4/3 fractional superstring as a special case. The merit of this series is that all the resulting fractional world-sheet algebras are

Abelianly braided. Also they clearly have representations with global so(N) symmetry groups; however it is not clear how to construct representations with N-dimensional Poincaré invariance. It is interesting to note that, since so(4)=so(3)⊗so(3), the world-sheet symmetry algebra corresponding to so$(4)_2$ is simply two copies of the spin-4/3 algebra; however, the coordinate bosons coupled to the so$(3)_2$ model in our $c = 5$ representation do not transform in the vector representation of so(4), and so cannot give a flat space-time interpretation.

The representation theory of the other simple series of algebras based on the su$(2)_K$ models, though non-Abelianly braided in general, have been more intensively studied [12]. Since su$(2)_K$ =so$(3)_{K/2}$ all these models (trivially) have representations with so(2,1) Lorentz symmetry. Whether there are any representations in which this Lorentz symmetry can be extended to the Poincaré symmetry of three- (or higher-) dimensional space-time is an open problem. One indication that such representations really may exist comes from the modular-invariant fractional superstring partition functions proposed in Ref. [2]. Although the precise connection between these partition functions and fractional superstrings defined by a fractional superconformal algebra as discussed above is not clear, there are many suggestive points of contact; indeed, the construction of the explicit three-dimensional $c = 5$ representation of the spin-4/3 algebra was originally motivated by consideration of the "internal projection" appearing in these partition functions [3,18]. Some hints of the space-time structure of the critical su$(2)_K$ fractional superstrings have been gleaned from fractional superstring partition functions [2,3]. For example, the low-energy physics of these strings is believed to describe supergravity in six and four dimensions for $K = 4$ and 8, respectively. If this is true, then the critical spin-4/3 fractional superstring should have a six flat space-time dimensional representation. An interesting question in connection with these new strings is whether their fractional world-sheet structures "translate" into some novel symmetries or physics in space-time. In this connection, there are some suggestive hints from the fractional superstring partition functions [18,19].

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APPENDIX A: UNTWISTED SECTOR OF SO$(2,1)_2$

We add standard cocycles to the free boson theory compactified on the su(3) root lattice considered in Sec. II. Following the notation of that section, define the vertex operators $V_{\mathbf{m}}$ as

$$V_{\mathbf{m}} = c(\mathbf{m}) : e^{i\mathbf{m}\cdot\boldsymbol{\varphi}} :, \tag{A1}$$

where the colons denote normal ordering with respect to the conformal vacuum. The cocycles $c(\mathbf{m})$ can be chosen to obey the properties [15]

$$\begin{aligned} c(\mathbf{m})c(\mathbf{n}) &= c(\mathbf{m}+\mathbf{n}), \\ c(\mathbf{m})e^{i\mathbf{n}\cdot\boldsymbol{\varphi}} &= (-1)^{m_1 n_2} e^{i\mathbf{n}\cdot\boldsymbol{\varphi}} c(\mathbf{m}), \\ [c(\mathbf{m})]^\dagger &= c(-\mathbf{m}), \\ c(\mathbf{0}) &= 1. \end{aligned} \tag{A2}$$

These properties imply, in particular, that

$$(V_{\mathbf{m}})^\dagger = (-1)^{m_1 m_2} V_{-\mathbf{m}}. \tag{A3}$$

Using these definitions and the free field operator products (4.6), the basic vertex operator product expansion

$$V_{\mathbf{m}}(z)V_{\mathbf{n}}(w) = (-1)^{m_2 n_1}(z-w)^{\mathbf{m}\cdot\mathbf{n}} V_{\mathbf{m}+\mathbf{n}}(w) + \cdots \tag{A4}$$

can be derived.

Alternatively, the Hilbert space can be explicitly constructed in terms of the modes of the $\boldsymbol{\varphi}(z)$ fields, defined by the expansion

$$\varphi^j(z) = \phi^j - ip^j \ln(z) + i\sum_{n\neq 0} \frac{1}{n}\alpha^j_n z^{-n}, \tag{A5}$$

and satisfying the commutation relations $[\phi^i, p^j] = ig^{ij}$, $[\alpha^i_n, \alpha^j_m] = ng^{ij}\delta_{m+n}$, and the Hermiticity assignments $(\phi^j)^\dagger = \phi^j$, $(p^j)^\dagger = p^j$, and $(\alpha^j_n)^\dagger = \alpha^j_{-n}$. The cocycles can be explicitly realized [14] by $c(\mathbf{m}) = (-1)^{m_1 p_2}$, where $p_2 = g_{2j}p^j$ is a component of the momentum zero-mode of $\boldsymbol{\varphi}(z)$. In terms of the modes, the normal-ordered vertex operators can be written

$$V_{\mathbf{m}}(z) = (-1)^{m_1 p_2} e^{i\mathbf{m}\cdot\boldsymbol{\phi}} z^{\mathbf{m}\cdot\mathbf{p}} \exp\left\{-\sum_{n<0}\frac{1}{n}\mathbf{m}\cdot\boldsymbol{\alpha}_n z^{-n}\right\} \exp\left\{-\sum_{n>0}\frac{1}{n}\mathbf{m}\cdot\boldsymbol{\alpha}_n z^{-n}\right\}. \tag{A6}$$

The basic operator product expansion (A4) follows easily.

All fields in the $\boldsymbol{\varphi}$ CFT can be organized in so(2,1) representations. All the Virasoro primary fields in the $\boldsymbol{\varphi}$ CFT up to dimension 4/3 are

$$
\begin{aligned}
h = 1/3:\quad & \epsilon^+_\mu = \left(V_{(-1,-1)}, V_{(1,0)}, V_{(0,1)}\right), \quad \epsilon^-_\mu = \left(V_{(1,1)}, V_{(-1,0)}, V_{(0,-1)}\right), \\
h = 1:\quad & U_\mu = \left(V_{(1,-1)} + V_{(-1,1)}\,,\ V_{(1,2)} + V_{(-1,-2)}\,,\ V_{(-2,-1)} + V_{(2,1)}\right), \\
& W_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 2i\partial\varphi_{(1,1)} & V_{(2,1)} - V_{(-2,-1)} & V_{(-1,-2)} - V_{(1,2)} \\ & 2i\partial\varphi_{(1,0)} & V_{(1,-1)} - V_{(-1,1)} \\ & & 2i\partial\varphi_{(0,1)} \end{pmatrix}, \qquad \text{(A7)} \\
h = 4/3:\quad & s^+ = \tfrac{1}{3}\left[V_{(2,2)} + V_{(-2,0)} + V_{(0,-2)}\right], \quad s^- = \tfrac{1}{3}\left[V_{(-2,-2)} + V_{(2,0)} + V_{(0,2)}\right], \\
& t^+_{\mu\nu} = \begin{pmatrix} -V_{(2,2)} + s^+ & -\tfrac{i}{2}\partial\varphi_{(2,1)}V_{(0,1)} & \tfrac{i}{2}\partial\varphi_{(1,2)}V_{(1,0)} \\ & V_{(-2,0)} - s^+ & \tfrac{i}{2}\partial\varphi_{(-1,1)}V_{(-1,-1)} \\ & & V_{(0,-2)} - s^+ \end{pmatrix}, \\
& t^-_{\mu\nu} = \begin{pmatrix} -V_{(-2,-2)} + s^- & \tfrac{i}{2}\partial\varphi_{(2,1)}V_{(0,-1)} & -\tfrac{i}{2}\partial\varphi_{(1,2)}V_{(-1,0)} \\ & V_{(2,0)} - s^- & -\tfrac{i}{2}\partial\varphi_{(-1,1)}V_{(1,1)} \\ & & V_{(0,2)} - s^- \end{pmatrix}.
\end{aligned}
$$

Here we have defined the combination $\varphi_{\mathbf{m}} = \mathbf{m}\cdot\boldsymbol{\varphi}$, so that, for example, $\partial\varphi_{(2,1)} = 2\partial\varphi^1 + \partial\varphi^2$. We have also only written half of the entries for the spin-2 fields $W_{\mu\nu}$ and $t^\pm_{\mu\nu}$, since they are symmetric-traceless tensors.

From (A3) it follows that Hermitian conjugation is accompanied by lowering (raising) upper (lower) space-time indices. For example, $(\epsilon^{+\mu})^\dagger = \epsilon^-_\mu$ and $(W_{\mu\nu})^\dagger = W^{\mu\nu}$.

The vertex operator OPE's can be worked out using the free field operator products (4.6) and the vertex OPE's (A4). The results for some leading terms are listed below. For ease of writing, all the operator products are of the form $A(z)B(0)$, the right-hand sides of the OPE's are all evaluated at 0, and the dependence of the fields on their arguments is suppressed:

$$
\begin{aligned}
\epsilon^\pm_\mu\epsilon^\pm_\nu &= z^{-1/3}\varepsilon_{\mu\nu}{}^\rho\epsilon^\mp_\rho + z^{2/3}\left(\tfrac{1}{2}\varepsilon_{\mu\nu}{}^\rho\partial\epsilon^\mp_\rho + g_{\mu\nu}s^\mp + t^\mp_{\mu\nu}\right), \\
\epsilon^\pm_\mu\epsilon^\mp_\nu &= z^{-2/3}g_{\mu\nu} + z^{1/3}\left(\tfrac{1}{2}\varepsilon_{\mu\nu\rho}U^\rho \pm W_{\mu\nu}\right) + z^{4/3}\left(\tfrac{1}{3}g_{\mu\nu}T_\varphi + \tfrac{1}{4}\varepsilon_{\mu\nu\rho}\partial U^\rho \pm \tfrac{1}{2}\partial W_{\mu\nu} + H_{\mu\nu} + \varepsilon_{\mu\nu\rho}F^\rho\right),
\end{aligned} \qquad \text{(A8)}
$$

$$
\begin{aligned}
\epsilon^\pm_\mu s^\pm &= z^{-4/3}\tfrac{1}{3}\epsilon^\mp_\mu - z^{-1/3}\tfrac{1}{3}\partial\epsilon^\mp_\mu, \quad s^\pm\epsilon^\pm_\mu = z^{-4/3}\tfrac{1}{3}\epsilon^\mp_\mu + z^{-1/3}\tfrac{2}{3}\partial\epsilon^\mp_\mu, \\
\epsilon^\pm_\mu s^\mp &= z^{-2/3}\tfrac{1}{3}U_\mu - z^{1/3}\tfrac{1}{3}F_\mu, \quad s^\pm\epsilon^\mp_\mu = z^{-2/3}\tfrac{1}{3}U_\mu + z^{1/3}\tfrac{1}{3}\left(\partial U_\mu + F_\mu\right),
\end{aligned} \qquad \text{(A9)}
$$

$$
s^\pm s^\pm = z^{-4/3}\tfrac{2}{3}s^\mp + z^{-1/3}\tfrac{1}{3}\partial s^\mp, \quad s^\pm s^\mp = z^{-8/3}\tfrac{1}{3} + z^{-2/3}\tfrac{4}{9}T_\varphi, \qquad \text{(A10)}
$$

$$
\begin{aligned}
\epsilon^\pm_\mu U_\nu &= z^{-1}\varepsilon_{\mu\nu}{}^\rho\epsilon^\pm_\rho - \tfrac{1}{2}\varepsilon_{\mu\nu}{}^\rho\partial\epsilon^\pm_\rho + 2g_{\mu\nu}s^\pm - t^\pm_{\mu\nu}, \\
U_\mu\epsilon^\pm_\nu &= z^{-1}\varepsilon_{\mu\nu}{}^\rho\epsilon^\pm_\rho + \tfrac{3}{2}\varepsilon_{\mu\nu}{}^\rho\partial\epsilon^\pm_\rho + 2g_{\mu\nu}s^\pm - t^\pm_{\mu\nu}, \\
s^\pm U_\mu &= z^{-2}\tfrac{2}{3}\epsilon^\pm_\mu + z^{-1}\tfrac{2}{3}\partial\epsilon^\pm_\mu, \quad U_\mu s^\pm = z^{-2}\tfrac{2}{3}\epsilon^\pm_\mu + \mathcal{O}(z^0),
\end{aligned} \qquad \text{(A11)}
$$

$$
\begin{aligned}
\epsilon^\pm_\rho W_{\mu\nu} &= \mp z^{-1}\tfrac{1}{2}\delta_{\mu\nu\rho}{}^\sigma\left(\epsilon^\pm_\sigma - z\tfrac{1}{2}\partial\epsilon^\pm_\sigma\right) \mp \tfrac{1}{2}\left(\varepsilon_{\rho\mu}{}^\sigma t^\pm_{\sigma\nu} + \varepsilon_{\rho\nu}{}^\sigma t^\pm_{\sigma\mu}\right), \\
W_{\mu\nu}\epsilon^\pm_\rho &= \pm z^{-1}\tfrac{1}{2}\delta_{\mu\nu\rho}{}^\sigma\left(\epsilon^\pm_\sigma + z\tfrac{3}{2}\partial\epsilon^\pm_\sigma\right) \pm \tfrac{1}{2}\left(\varepsilon_{\rho\mu}{}^\sigma t^\pm_{\sigma\nu} + \varepsilon_{\rho\nu}{}^\sigma t^\pm_{\sigma\mu}\right), \\
s^\pm W_{\mu\nu} &= \pm z^{-1}\tfrac{2}{3}t^\pm_{\mu\nu}, \quad W_{\mu\nu}s^\pm = \mp z^{-1}\tfrac{2}{3}t^\pm_{\mu\nu},
\end{aligned} \qquad \text{(A12)}
$$

where $g_{\mu\nu}$ is the Minkowski metric in three dimensions with signature $(-++)$, $\varepsilon_{\mu\nu\rho}$ is the antisymmetric tensor in three dimensions normalized by $\varepsilon_{012} = -1$, obeying $\varepsilon_{\mu\nu\rho}\varepsilon^\mu{}_{\alpha\beta} = -g_{\nu\alpha}g_{\rho\beta} + g_{\nu\beta}g_{\rho\alpha}$, and we have defined $\delta_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - \frac{2}{3}g_{\mu\nu}g_{\rho\sigma}$. The fields F_μ and symmetric-traceless $H_{\mu\nu}$ appearing in (A8) and (A9) are dimension-2 Virasoro primary fields. We can take (A8) as their definition.

APPENDIX B: OTHER KNOWN SPIN-4/3 FSC REPRESENTATIONS

The representation theory of the spin-4/3 FSC algebra is related to the su(2)$_4$ Wess-Zumino-Witten (WZW) model in the same way as the Virasoro algebra representation theory is related to the su(2)$_1$ model. In particular, the FSC algebra has a series of unitary minimal representations realized by the $su(2)_4 \otimes su(2)_L/su(2)_{4+L}$ coset models with central charges

$$c = 2 - \frac{24}{(L+2)(L+6)} \qquad \text{for } L = 1, 2, \ldots, \tag{B1}$$

accumulating at the $c = 2$ $su(2)_4$ WZW model. The FSC algebra can be realized in this model as follows. Let $J^a(z) = \sum_n J^a_n z^{-n-1}$ denote the $su(2)_4$ Kač-Moody currents, $\Phi^a(z)$ the dimension-1/3 chiral primary field in the adjoint representation, and q_{ab} the su(2) Killing form. The FSC current is [6,1]

$$G^+(z) + G^-(z) = \sum_{a,b} q_{ab} J^a_{-1} \Phi^b(z). \tag{B2}$$

The $su(2)_4$ theory can be bosonized in terms of one free boson and a $\mathbb{Z}_4$ parafermion theory [4], which is itself equivalent to a single compactified boson. Thus the $c = 2$ representation can be written in terms of two free bosons X and ρ satisfying $X(z)X(w) = -\ln(z-w)$ and $\rho(z)\rho(w) = -\frac{1}{6}\ln(z-w)$, with ρ compactified on the unit circle $\rho = \rho + 2\pi$. The FSC algebra currents are given by [13]

$$G^\pm = \frac{i}{\sqrt{2}} e^{\pm 2i\rho} \partial X + \frac{1}{2} e^{\mp 4i\rho}. \tag{B3}$$

The coset models can be realized in terms of this bosonized theory by turning on a background charge for the X boson [6], $T_X = -\frac{1}{2}(\partial X \partial X + iQ\partial^2 X)$, so that the total central charge of the representation becomes $c = 2 - 3Q^2$. The expression for the FSC current with background charge is given in Ref. [1].

The $su(2)_4$ WZW model is equivalent to the $so(3)_2$ model, whose two-boson construction was explained in Sec. IV. This equivalence suggests other free field representations of the spin-4/3 FSC algebra. In particular, an inequivalent $c = 2$ representation can be realized in terms of the two bosons of the $so(3)_2$ model. The FSC algebra currents in this representation are simply [11]

$$G^\pm = \tfrac{3}{2} s^\pm, \tag{B4}$$

where the $s^\pm$ fields are defined in Appendix A. Other free boson representations can be formed by taking various tensor products of free uncompactified bosons X^μ and copies of the $so(3)_2$ WZW model.

A $c = 4$ representation is $so(3)_2 \otimes so(3)_2$ with the FSC lgebra currents given by

$$G^\pm = \frac{1}{\sqrt{6}} \left(e^{\pm 2\pi i/9} U^\mu \otimes \epsilon^\pm_\mu + 3 e^{\mp \pi i/9} \cdot \mathbb{1} \otimes s^\pm \right), \tag{B5}$$

where $\mathbb{1}$ denotes the identity operator. This representation is related to the Goddard-Schwimmer construction [11] of the subset of the spin-4/3 FSC algebra minimal models with $L = 2K$ in (B1) in terms of an $so(3)_K \otimes so(3)_2$ theory with a $U^\mu \otimes U_\mu$ term added to the stress-energy tensor.

The $c = 5$ representation discussed in detail in this paper is a tensor product of one $so(3)_2$ with three free bosons X^μ. We repeat here the resulting form of the FSC algebra currents:

$$G^\pm = \frac{1}{\sqrt{2}} \left(\pm \partial X^\mu \epsilon^\pm_\mu - \frac{3}{2} s^\pm \right). \tag{B6}$$

It is a nontrivial fact that when a background charge is turned on for the X^μ fields in this representation, the form of the fractional current can be modified in such a way as to still satisfy the FSC algebra [20].

A $c = 6$ representation is the threefold tensor product $so(3)_2 \otimes so(3)_2 \otimes so(3)_2$ with FSC algebra currents given by

$$G^\pm = \frac{1}{\sqrt{6}} \Big(e^{\pm \pi i/4} U^\mu \otimes \mathbf{1} \otimes \epsilon^\pm_\mu + e^{\mp \pi i/4} \mathbb{1} \otimes U^\mu \otimes \epsilon^\pm_\mu + \frac{3}{\sqrt{2}} \mathbb{1} \otimes \mathbf{1} \otimes s^\pm \Big). \tag{B7}$$

A $c = 7$ representation consists of two copies of $so(3)_2$ and one set of three free bosons X^μ with

$$G^\pm = \frac{1}{\sqrt{6}} \left(\pm \sqrt{3} \partial X^\mu \cdot \mathbb{1} \otimes \epsilon^\pm_\mu - U^\mu \otimes \epsilon^\pm_\mu - \frac{3}{2} \mathbb{1} \otimes s^\pm \right). \tag{B8}$$

It is interesting to note that this representation has a three-dimensional Poincaré symmetry, and so, like the $c = 5$ representation discussed in this paper, is a suitable model for constructing spin-4/3 fractional string tree-level scattering amplitudes.

A $c = 8$ representation consists of one $so(3)_2$ and two sets of three free bosons, X^μ and Y^μ. Its FSC algebra currents have the especially simple form

$$G^\pm = \pm \frac{1}{\sqrt{2}} \partial \left(X^\mu \pm i Y^\mu \right) \epsilon^\pm_\mu. \tag{B9}$$

Note that at $c = 8$ the FSC structure constants vanish: $\lambda^\pm = 0$. Although this representation has a six-dimensional global translation group, its largest global "rotation" group is only su(3), and thus cannot describe string scattering in six dimensions. Furthermore, since the uncompactified boson fields appear only in the complex combinations $X^\mu \pm iY^\mu$, there are effectively three timelike directions in this representation.

No $c > 8$ free field representations (i.e., with no background charges) are known for the spin-4/3 FSC algebra.

Finally, note that the Hermiticity properties of the currents are constrained by the form of the FSC algebra. Assuming that G^+ and G^- are related by some Hermiticity relations (which by no means has to be the case), it is not hard to show that, up to rescalings of the currents, the algebra (2.1) admits four inequivalent Hermiticity assignments:

$$(G^\pm)^\dagger = G^\mp, \qquad \lambda^+ = \lambda^-, \tag{B10a}$$

$$(G^\pm)^\dagger = G^\pm, \qquad \lambda^+ = \lambda^-, \tag{B10b}$$

$$(G^\pm)^\dagger = -G^\mp, \qquad \lambda^+ = -\lambda^-, \tag{B10c}$$

$$(G^\pm)^\dagger = G^\pm, \qquad \lambda^+ = -\lambda^-, \tag{B10d}$$

where in all cases λ^+ can be taken to be a positive real number. Note that, by (2.2), $\lambda^+\lambda^-$ changes sign at $c = 8$, so the Hermiticity assignments (B10a) and (B10b) apply only when $c \leq 8$, while the assignments (B10c) and (B10d) are allowed only for $c \geq 8$.

For all the Hermiticity assignments one can construct the Hermitian current $G \equiv G^+ + \text{sgn}(8 - c)G^-$ which satisfies for $c > 8$, $GG \sim -1 + \cdots$. This shows that, because G is Hermitian, such $c > 8$ representations of the FSC algebra are necessarily nonunitary. As mentioned in Sec. I, the critical central charge of the spin-4/3 fractional superstring is $c = 10$, and thus any critical representation of the FSC algebra with simple Hermiticity properties for the fractional currents will be nonunitary. This, of course, is perfectly consistent for strings describing propagation in Minkowski space-times; however, it is different from what occurs in bosonic and ordinary superstrings where there is no such automatic requirement of nonunitarity at the critical central charge.

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