New aspects of the Casimir energy theory for a piecewise uniform string

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(Received 2 September 1993)

The Casimir energy for the transverse oscillations of a piecewise uniform closed string is calculated. The string consists of two parts I and II each having in general different tension and mass density but adjusted in such a way that the velocity of sound always equals the velocity of light. This model was introduced by Brevik and Nielsen, and the present paper contains new developments of the theory, in particular, a very simple regularization of the energy density. Using the technique introduced by van Kampen, Nijboer, and Schram, the Casimir energy is written as a contour integral, from which the energy can be readily calculated, for arbitrary length $s = L_{\rm II}/L_{\rm I}$ and tension $x = T_{\rm I}/T_{\rm II}$ ratios. Also, the finite temperature version of the theory is constructed.

PACS number(s): 03.70.+k, 11.10.Gh, 11.10.Wx

I. INTRODUCTION

Consider, in Minkowski space, a closed string of length L composed of two parts, of lengths $L_{\rm I}$ and $L_{\rm II}$, respectively [1]. The tensions $T_{\rm I}$ and $T_{\rm II}$ and mass densities $\rho_{\rm I}$ and $\rho_{\rm II}$ corresponding to the two pieces are in general different, but they will be required to satisfy the condition that the sound velocity be always equal to the light velocity, i.e.,

$$v_s = (T_{\rm I}/\rho_{\rm I})^{1/2} = (T_{\rm II}/\rho_{\rm II})^{1/2} = c. \tag{1}$$

Our purpose is to study the Casimir energy associated with the transverse oscillations of this piecewise uniform string. It turns out that the present model is, at least from a formal point of view, a very interesting one; the calculations can be carried out without encountering the annoying divergences in the regularized result which so often plague Casimir calculations when the geometry is nontrivial (curved boundary surfaces, typically). A basic point in this context is condition (1). It renders the string relativistically invariant, and is analogous to requiring the "refractive index $(\epsilon\mu)^{1/2}$" to be equal to unity. In this sense, it is basically of the same kind as the "color medium" proposed by Lee [2] for the region exterior to a hadron. If, by contrast, condition (1) were abandoned, then divergence problems would certainly show up in the formalism.

From a physical point of view, there is well-founded hope that this simple model can help us to understand the issue of the energy of the vacuum state in two-dimensional quantum field theories in general, a quite compelling goal. The model was introduced by Brevik and Nielsen in an earlier paper [1], to which the reader is referred for specific details. There, the zero-point energy was regularized by means of an exponential cutoff. It was also pointed out, that the use of more formal regularization procedures—such as the ζ-function method—might lead to delicate problems; in particular, that (a naively minded, straightforward) use of the *Riemann* ζ function could lead to an incorrect result. This problem was reconsidered and solved in an elegant way by Li, Shi, and Zhang [3], who showed that the appropriate ζ function to be used in that case was the generalized form commonly known as the *Hurwitz* ζ function. The final results obtained in [3] were in agreement with those of [1]. The whole situation concerning the use of the ζ-function regularization procedure in this and similar cases is discussed in [4] in great detail.

The main results that will be obtained in the present paper are the following.

(i) The Casimir energy at zero temperature, $T = 0$, will be found as a double function of the length ratio $s \equiv L_{\rm II}/L_{\rm I}$, for any value of s, and of the tension ratio $x \equiv T_{\rm I}/T_{\rm II}$, for arbitrary x. To compare, in Ref. [1] the solution was given explicitly for a few lowest integer values of s and a few selected values of x, only (and this after considerable work). To achieve our goal, we shall employ here a quite elegant technique, based on a well known theorem of complex analysis, and which was first introduced—in a context related to the present one—by van Kampen, Nijboer, and Schram some years ago [5]. It consists in rewriting the Casimir energy under the form of a very simple contour integral. This technique, when applied to the present problem, must be used with some care, in order to avoid an unphysical divergence in the form of a surface term. However, as we shall see below, the suppression of the surface term can be done consistently, through a proper choice of the dispersion function for the system.

(ii) When using this technique, it becomes unnecessary to take the degeneracies of the eigenfrequencies of the system into account explicitly. This comes as a very useful bonus. The reason is that the degeneracies precisely correspond to the multiplicities of the zeros which appear in the *argument principle* [cf. Eq. (6) below]. This fact

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makes the final theory much more simple, as compared with the original procedure of finding and counting the roots of the dispersion equation, that had been used in Ref. [1] (see also [3]).

(iii) The Casimir energy for the string is calculated for finite temperature, $T \neq 0$, also, and the analytic approximation for high T is worked out. This high-temperature limit provides an immediate check of the procedure, since it is easy to find analytically.

Readers interested in the general theory of the Casimir effect can consult the very useful reviews by Plunien, Müller, and Greiner [6], by Barash and Ginzburg [7], and by Mostepanenko and Trunov [8].

II. ZERO TEMPERATURE THEORY

We shall use the same notation as in Ref. [1]. The total length of the string is $L = L_{\rm I} + L_{\rm II}$. Denoting by x the ratio between the two tensions,

$$x = \frac{T_{\rm I}}{T_{\rm II}}, \tag{2}$$

the dispersion equation can be written as

$$\frac{4x}{(1-x)^2}\sin^2\left(\frac{\omega L}{2c}\right) + \sin\left(\frac{\omega L_{\rm I}}{c}\right)\sin\left(\frac{\omega L_{\rm II}}{c}\right) = 0. \tag{3}$$

The physical requirements behind this equation are (i) continuity of the transverse displacements across the two junctions, and (ii) continuity of the transverse elastic forces across the junctions. Since Eq. (3) is invariant under the substitution $x \to 1/x$, we can simply take $0 \le x \le 1$ in what follows (the case $x = 0$ must be considered with some care).

The Casimir energy of the system, E, is constructed such that it describes the nonhomogeneity of the string only, and is thus required to vanish for a uniform string. Therefore, E is equal to the zero-point energy $E_{\rm I+II}$ for the two parts, minus the zero-point energy for the uniform string, i.e.,

$$E = E_{\rm I+II} - E_{\rm uniform}. \tag{4}$$

It should be noted that, when subtracting off $E_{\rm uniform}$, it is completely irrelevant whether the uniform string is composed of type I or type II material. The physical reason for this is that the frequency spectrum for the uniform string is independent of the type of material, as long as the velocity of sound is the same, and thus it is in the present case a consequence of Eq. (1).

The zero-point energy of the composite string is

$$E_{\rm I+II} = \frac{\hbar}{2}\sum \omega_n, \tag{5}$$

where the sum goes over all eigenstates, with account of their degeneracy. Stationarity of the oscillating system implies that all the eigenfrequencies ω_n have to be real. We can let all the ω_n be positive, left-moving waves being associated with negative *wave numbers* (not frequencies). And here comes the first important point in this paper. The sum (5) can be written in the form of a contour integral by means of a well known mathematical theorem called the argument principle [7, 9]. It states that any meromorphic function satisfies the equation

$$\frac{1}{2\pi i}\oint \omega \frac{d}{d\omega}\ln g(\omega)\, d\omega = \sum \omega_0 - \sum \omega_\infty, \tag{6}$$

where ω_0 denotes the zeros and ω_∞ the poles of $g(\omega)$ lying inside the integration contour as shown in Fig. 1. The argument principle is derived from Cauchy's theorem. In the end, the radius R of the contour in (6) is allowed to go to infinity (as is usually the case for theorems of this kind on the complex plane). The multiplicity of the zeros, as well as that of the poles, is automatically taken care of by the two sums in Eq. (6). The argument principle was first applied to the Casimir theory (in the standard configuration with two parallel plates) by van Kampen, Nijboer, and Schram [5].

When applying the argument principle to our present problem, we first notice that the appropriate dispersion function $g(\omega)$ must essentially be the function on the left hand side of Eq. (3)—but it can be modified by a factor not depending on ω, e.g., an arbitrary function of x. But this function $g(\omega)$ has no poles; therefore, the last term in Eq. (6) vanishes and thus the crosses on the real axis in Fig. 1 refer to the zeros of $g(\omega)$ only. As in Ref. [1] we introduce the function

$$F(x) = \frac{4x}{(1-x)^2}, \tag{7}$$

and the variable

$$s = \frac{L_{\rm II}}{L_{\rm I}}, \tag{8}$$

to denote the ratio of the lengths of the two pieces of the string. For definiteness we shall take $L_{\rm I}$ to be the smaller of the two pieces, so that $s \ge 1$. For the dispersion function of the composite system, we now make the ansatz

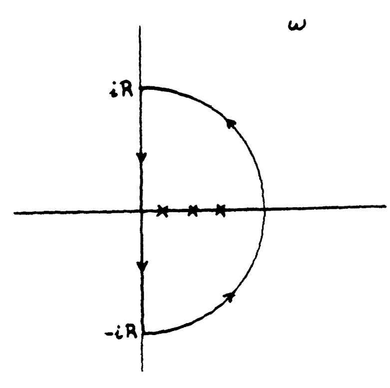

FIG. 1. Integration contour in the complex ω plane.

$$g(\omega)=\frac{F(x)\sin^2[(s+1)\omega L_{\rm I}/(2c)]+\sin(\omega L_{\rm I}/c)\sin(s\omega L_{\rm I}/c)}{F(x)+1} \tag{9}$$

(see the remarks following Eqs. (12) and (15) below). For given values of x and of the total length L, this expression is invariant under the substitution $s\to 1/s$. Thus, the above restriction to values of $s\geq 1$ represents no loss in generality. By making use of the argument principle, for the composite system we have the zero-point energy

$$E_{\rm I+II}=\frac{\hbar}{4\pi i}\oint \omega\frac{d}{d\omega}\ln|g(\omega)|\,d\omega, \tag{10}$$

with the function $g(\omega)$ given by (9). In writing this expression, we have taken advantage of the following important correspondence between degeneracy of the eigenfrequencies ω_n and multiplicity of the zeros of $g(\omega)$. As noted in connection with Eq. (5), it is in general necessary to take into account degeneracies when summing over all states. In Ref. [1], the degeneracies were actually counted explicitly, for each branch of the dispersion equation, in the cases of low integer s for which the solution was worked out. Within the present approach, the handling of the degeneracy problem is, however, much more easy since the argument principle (6), as we have seen, already takes into account the multiplicity of the zeros. There exists a one-to-one correspondence between the degeneracy of the eigenfrequencies and the multiplicity of the zeros. Therefore, the degeneracies are built in automatically, in the integral (10).

III. REGULARIZED CASIMIR ENERGY AND NUMERICAL RESULTS

In spite of these very interesting properties, one should notice that, as it stands, Eq. (10) is not a useful expression. In fact, it is not difficult to see that the contribution of the curved part, the contour of radius R (Fig. 1), to the integral (10) grows without bound as $R\to\infty$. Since, in general, in order to take into account all the modes in the series (5), we must send R to infinity, it follows that a divergence is hidden in the curved contour at infinity. What one has to do is to subtract off the energy of the uniform string. This corresponds to $x=1$ (the value of s need not be specified). Since $F(x)\to\infty$ as $x\to 1$, we obtain using (9)

$$E_{\rm uniform}=\frac{\hbar}{4\pi i}\oint\omega\frac{d}{d\omega}\ln\sin^2\left(\frac{(s+1)\omega L_{\rm I}}{2c}\right)d\omega, \tag{11}$$

and thus the Casimir energy follows from (4),

$$E=\frac{\hbar}{4\pi i}\oint\omega\frac{d}{d\omega}\ln\left|\frac{F(x)+\dfrac{\sin(\omega L_{\rm I}/c)\sin(s\omega L_{\rm I}/c)}{\sin^2[(s+1)\omega L_{\rm I}/(2c)]}}{F(x)+1}\right|d\omega. \tag{12}$$

It is easy to see that when the two pieces have the same length, $L_{\rm I}=L_{\rm II}$ (i.e., $s=1$), then $E=0$, irrespective of the value of x. This is just as it should be according to the detailed considerations in Ref. [1], a fact that actually was the reason behind our particular choice (9) for the dispersion function. Notice also, as a corollary, that Eq. (12) yields $E=0$ when $x=1$.

The contribution from the semicircle of Fig. 1 to the integral in Eq. (12) is now seen to vanish in the limit $R\to\infty$, and the remaining integral along the imaginary axis ($\omega=i\xi$) is integrated by parts, while keeping R finite and taking advantage of the symmetry of the integrand about the origin. We get

$$E=-\frac{\hbar}{2\pi}R\ln\left|\frac{F(x)+\dfrac{\sinh(RL_{\rm I}/c)\sinh(sRL_{\rm I}/c)}{\sinh^2[(s+1)RL_{\rm I}/(2c)]}}{F(x)+1}\right| +\frac{\hbar}{2\pi}\int_0^R\ln\left|\frac{F(x)+\dfrac{\sinh(\xi L_{\rm I}/c)\sinh(s\xi L_{\rm I}/c)}{\sinh^2[(s+1)\xi L_{\rm I}/(2c)]}}{F(x)+1}\right|d\xi. \tag{13}$$

Here, the boundary term is seen to vanish when $R\to\infty$, and thus we obtain, finally,

$$E=\frac{\hbar}{2\pi}\int_0^\infty\ln\left|\frac{F(x)+\dfrac{\sinh(\xi L_{\rm I}/c)\sinh(s\xi L_{\rm I}/c)}{\sinh^2[(s+1)\xi L_{\rm I}/(2c)]}}{F(x)+1}\right|d\xi. \tag{14}$$

We assume that the total length L and the tension ratio x are given quantities. Therefore, $F(x)$ is known, and Eq. (14) gives E as a function of the length ratio s. However, this expression is easily calculable on a computer, and we can equally well give E as a double function of x and s (Fig. 2 below). As a corollary, we have checked that, in the special case $x\to 0$, Eq. (14) gives results which are in agreement with the analytic (well known) expression

$$E=-\frac{\pi\hbar c}{24L}\left(s+\frac{1}{s}-2\right) \tag{15}$$

derived in Ref. [1].

A remark is in order concerning our inclusion of the factor $F(x)[F(x)+1]^{-1}$ in Eq. (9). Had we *not* introduced this factor in $g(\omega)$, namely, had we taken

$$g(\omega)=\sin^2\left[\frac{(s+1)\omega L_{\rm I}}{2c}\right] +F(x)^{-1}\sin\left(\frac{\omega L_{\rm I}}{c}\right)\sin\left(\frac{s\omega L_{\rm I}}{c}\right), \tag{16}$$

we would have obtained

$$E = -\frac{\hbar}{2\pi} \int_0^\infty \xi \frac{d}{d\xi} \ln \left| 1 + \frac{\sinh(\xi L_{\rm I}/c)\, \sinh(s\xi L_{\rm I}/c)}{F(x) \sinh^2 [(s+1)\xi L_{\rm I}/(2c)]} \right| d\xi, \tag{17}$$

which means, for $x = 0$, that

$$E(x=0) = -\frac{\hbar c}{2\pi L_{\rm I}} \int_0^\infty t \frac{d}{dt} \ln \left| \frac{\sinh(t)\, \sinh(st)}{\sinh^2 [(s+1)t/2]} \right| dt. \tag{18}$$

Thus, (17) is valid for the whole range of possible values of x, $0 \le x \le 1$.

As an additional numerical test, by introducing an upper cutoff K,

$$E(K) \equiv -\frac{\hbar c}{2\pi L_{\rm I}} \int_0^K t \frac{d}{dt} \ln \left| 1 + \frac{\sinh(t)\, \sinh(st)}{F(x) \sinh^2 [(s+1)t/2]} \right| dt, \tag{19}$$

it turns out that for values of K between say $K \simeq 30$ and $K \simeq 10^4$, the result for $E(K)$ does not change numerically (to 10 digits) and coincides with the value given by (14). This has been checked for the whole range of values of x, $0 \le x \le 1$, and s, $s \ge 1$. The problem arises if one performs a *partial integration* in Eq. (17): then the resulting first term (i.e., the surface term) turns out to be divergent. In other words, the cutoff K must be kept. This is the drawback associated with expression (17).

As regards the symmetric behavior with respect to the values of s, i.e., the coincidence of the results corresponding to s and $1/s$, for any s, it is easily seen to hold for (17) explicitly [as for (14)]. Therefore, our choice above of letting s to be restricted to values ≥ 1, does not represent any loss in generality.

These conclusions are very nice indeed, and give sense to the introduction of the x-dependent factor in $g(\omega)$, and to the final formula (14) itself as a very simple, regularized expression for the Casimir energy.

The numerical results are collected in Figs. 2 and 3. Figure 2 is a three-dimensional plot of the Casimir energy as obtained from (19) or (14) (both figures are visually undistinguishable). Specifically, it shows how the magnitude $EL/(\hbar c)$ varies versus $x \in [0,1]$ (first axis) and $s \in [0,2]$ (second axis). The Casimir energy is generically seen to be negative. Only for equal lengths ($s = 1$) is the maximum energy $E = 0$ found, irrespective of the value of x. In Fig. 3, several x sections of the previous plot are depicted. It is shown here how the energy varies as a function of s, for several fixed values of x, corresponding to $x = 0$, $x = 0.1$, $x = 0.5$, and $x = 0.9$, respectively. All of them give the magnitude $EL/(\hbar c)$ as a function of s for the range $s \in [0,20]$. These figures must be compared with Fig. 2 of Ref. [1] where, as mentioned above, the solution (obtained after laborious numerical calculation) was given explicitly for a few lowest integer values of s and a few selected values of x. For low integers s up to $s = 7$ and corresponding values for x, we have checked that the results calculated from (14) are in agreement with Fig. 2 of Ref. [1]. The advantages of the present procedure are however unquestionable.

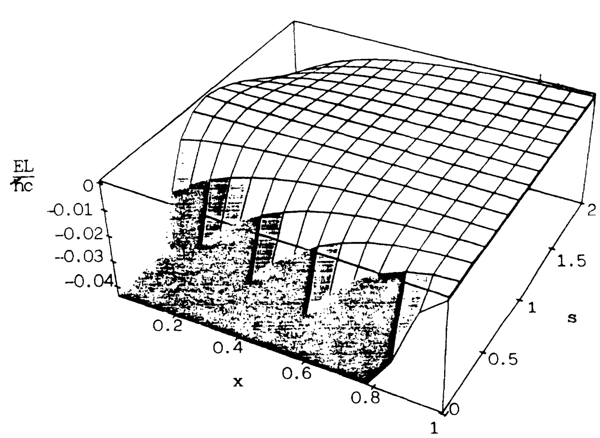

FIG. 2. Three-dimensional plot of the Casimir energy as obtained from (19) or (14), since both figures are visually undistinguishable. The magnitude $EL/(\hbar c)$ is plotted versus the tension ratio $x \in [0,1]$ (first axis) and length ratio $s \in [0,2]$ (second axis). The Casimir energy is generically seen to be negative. Only for equal lengths ($s = 1$) is the maximum energy $E = 0$ attained.

IV. FINITE TEMPERATURE THEORY

A. General formalism

Once the $T = 0$ theory is established, we can readily generalize the situation to the case of finite temperatures, by means of the substitution [10]

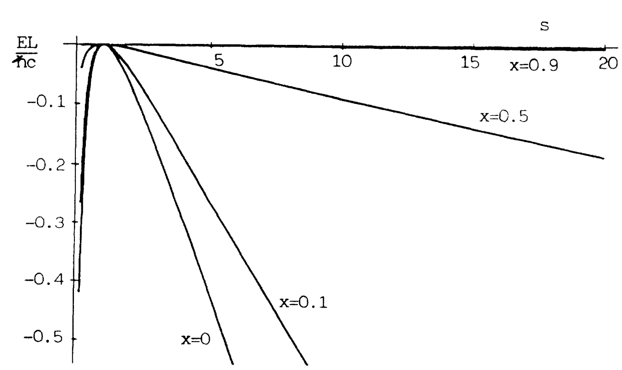

FIG. 3. Several x sections of Fig. 2 are depicted, to show how the energy varies as a function of the length ratio s, for several fixed values of x, namely $x = 0$, $x = 0.1$, $x = 0.5$, and $x = 0.9$, respectively. All curves give the magnitude $EL/(\hbar c)$ as a function of s in the range $s \in [0,20]$.

$$\hbar \int_0^\infty d\xi \longrightarrow 2\pi k_B T \sum_{n=0}^{\infty}{}' , \tag{20}$$

where the prime means that the contribution for $n = 0$ has to be taken with half weight. The discrete Matsubara frequencies are $\xi_n = 2\pi n k_B T/\hbar$, $n = 0, 1, 2, \ldots$. From (14) we then get for the Casimir energy at an arbitrary temperature T,

$$E(T) = k_B T \sum_{n=0}^{\infty}{}' \ln \left| \frac{F(x) + \dfrac{\sinh(\xi_n L_{\rm I}/c)\sinh(s\xi_n L_{\rm I}/c)}{\sinh^2[(s+1)\xi_n L_{\rm I}/(2c)]}}{F(x)+1} \right| . \tag{21}$$

If the string is uniform, $x = 1$ or $F(x) \to \infty$, then Eq. (21) yields $E(T) = 0$. This is just as we would expect, since the Casimir energy is intended to describe the effect of the inhomogeneity of the string only. Moreover, also the case $L_{\rm I} = L_{\rm II}$ is seen to yield $E(T) = 0$, irrespective of the value of x. Both these properties, noted earlier for the $T = 0$ theory, do therefore carry over to the case of arbitrary T.

There are two characteristic frequencies in our system:

(1) The thermal frequency ω_T, which can be defined by $\hbar\omega_T = k_B T$. We may observe that ω_T is related to the $n = 1$ Matsubara frequency ξ_1 through $\omega_T = \xi_1/(2\pi)$.

(2) The *geometric* frequency $\omega_{\rm geom}$, associated with the geometry of the string. We may choose to define $\omega_{\rm geom}$ in terms of $L_{\rm I}$ as fundamental length: $\omega_{\rm geom} = 2\pi c/L_{\rm I}$.

There would also be a third characteristic frequency in the problem, if the *microstructure* of the string were to be taken into account. It would correspond to the absorption frequency(ies) in the dispersion equation for an ordinary dielectric material. But we shall leave out of consideration microstructure effects here. The limiting cases of "high" and "low" temperatures are conveniently discussed in terms of the ratio between ω_T and $\omega_{\rm geom}$.

B. High temperatures

Assume that

$$\frac{\omega_T}{\omega_{\rm geom}} \geq 1. \tag{22}$$

This is the natural condition for applying the high-temperature approximation. Generally, high temperatures are associated with contributions coming from low Matsubara frequencies only. In our case, we have

$$\frac{\xi_n L_{\rm I}}{c} = 4\pi^2 n \frac{\omega_T}{\omega_{\rm geom}} \gg 1, \tag{23}$$

even for the lowest nonvanishing frequency ($n = 1$), so that $\sinh(\xi_n L_{\rm I}/c) \simeq (1/2)\exp(\xi_n L_{\rm I}/c)$, etc., in Eq. (21). The contribution to $E(T)$ from $n \geq 1$, in this approximation, is accordingly seen to vanish and we are left with just the $n = 0$ term. The result is

$$E(T) = \frac{k_B T}{2} \ln \left| \frac{F(x) + 4s/(s+1)^2}{F(x)+1} \right| . \tag{24}$$

The main corrections to this expression come of course from $n = 1$, and are of order $k_B T \exp(-2\xi_1 L_{\rm I}/c)$. Equation (24) is seen to be a classical result, since it is independent of $\hbar$. Notice that $E(T) \leq 0$ always, the equality sign being valid when $s = 1$, as we have pointed out above.

Similarly to the considerations in Ref. [1], we can in pictorial terms associate part I of the string with "our" universe, and part II of it with a "mirror" universe. If our universe is small and the mirror universe large we get, since $s \to \infty$, the very simple expression

$$E(T) = -\frac{k_B T}{2} \ln \left| 1 + F(x)^{-1} \right| . \tag{25}$$

To get a feeling of the numerical magnitudes involved here, let us first choose $L_{\rm I} = 1\ \mu$m, in which case $\omega_{\rm geom} = 2\pi c/L_{\rm I} = 1.88 \times 10^{15}\ {\rm s}^{-1}$. The ratio between the frequencies ω_T and $\omega_{\rm geom}$ becomes then

$$\frac{\omega_T}{\omega_{\rm geom}} = 0.70 \times 10^{-4}\ T, \tag{26}$$

showing that $T \geq 10^4$ K (i.e., 0.86 eV), if the high-temperature approximation is to hold. As another example, let us take the extreme case of $L_{\rm I} = 1.62 \times 10^{-33}$ cm, the Planck length. Then $\omega_{\rm geom} = 1.16 \times 10^{44}\ {\rm s}^{-1}$, and

$$\frac{\omega_T}{\omega_{\rm geom}} = 1.13 \times 10^{-33}\ T. \tag{27}$$

This case thus requires extremely high temperatures, $T \geq 10^{33}$ K (i.e., 0.86×10^{20} GeV), for the high-temperature approximation to be valid.

C. Low temperatures

This limiting case is characterized by

$$\frac{\omega_T}{\omega_{\rm geom}} \ll 1, \tag{28}$$

and a large number of Matsubara frequencies comes into play in Eq. (21). This equation, as it stands, is not written in a convenient form for performing analytical approximations when the temperatures are low. Rather often, when the mathematics is manageable, it is quite useful to exploit the Poisson summation formula (cf., for instance, the paper by Brevik and Clausen of Ref. [2]), whereby the series over n can be handled approximately without much labor. However, in the present case the logarithmic summand in Eq. (21) is too complicated to

TABLE I. Values of the Casimir energy E for some different values of T, assuming $L_{\rm I} = 1\ \mu$m, $s = 2$, and $F = 1$.

$T(K)$	0	10	300	3×10^3	3×10^4	10^6
$\omega_T/\omega_{\rm geom}$	0	6.95×10^{-4}	2.09×10^{-2}	0.209	2.09	69.5
n_T		365	13	2		
E (erg)	-3.3770×10^{-15}	-3.3847×10^{-15}	-3.3848×10^{-15}	-1.18×10^{-14}	-1.18×10^{-13}	-3.94×10^{-12}

permit an efficient use of the Poisson formula. At least for practical purposes, the best way to proceed is to deal directly with the series by means of a computer program, even in the case of low frequencies. The necessary number of terms of the series can be added up in very few seconds, to attain any desired precision. A convenient way of writing the series for low T, for making use of these techniques, is the following:

$$E(T) = k_B T \left\{ \frac{1}{2} \ln \left| \frac{F(x) + 4s/(s+1)^2}{F(x)+1} \right| + \sum_{n=1}^{n_T} \ln \left| \frac{F(x) + \left(1 - e^{-2nb}\right)\left(1 - e^{-2snb}\right)\left(1 - e^{-(s+1)nb}\right)^{-2}}{F(x)+1} \right| \right\}, \qquad (29)$$

where we have called

$$b = \frac{2\pi k_B T L}{\hbar (s+1) c}, \qquad (30)$$

and where, for a given temperature T, the sum can be safely cut at a value n_T such that, e.g., $n_T b \simeq 10$.

Table I shows, as an example, how the Casimir energy changes with the temperature in the case when $L_{\rm I} = 1\ \mu m$, $s = 2$, and $F = 1$ [i.e., $x = 0.1716$, according to (7)]. To distinguish between the low- and the high-temperature regions, the corresponding values of $\omega_T/\omega_{\rm geom}$ are given. For low but finite temperatures, the values of the upper limit n_T occurring in (29), chosen as $n_T = 10/b$, are also given. The $T = 0$ result may be taken directly from Table IV of Ref. [1]. From the present data, it is clearly seen that the Casimir energy becomes more and more negative as the temperature increases.

V. CONCLUSIONS

We have studied in this paper different issues related with a most convenient definition of the Casimir energy for the transverse oscillations of a piecewise uniform string. We have proven that, in fact, the calculations can be carried out in a remarkably easy way, not only because annoying divergences can be completely avoided in the regularized result, but also because the expressions leading to this finite result are very simple [see Eq. (14)], and allow us to calculate the most general case (see Fig. 2). Hence, simplicity is one of the main virtues of the model. Generality of the procedure is another.

Notwithstanding the fact that it is so simple, we do sustain the hope that such a model can actually help us to understand the issue of the energy of the vacuum state in two-dimensional quantum field theories, what is quite a compelling goal by itself. A specific result that has been obtained in the paper is the Casimir energy at zero temperature, as a double function of the length ratio s and of the tension ratio x, for arbitrary s and x. To this end we have devised an elegant technique, based on the argument theorem of complex analysis. This has led us to a formula which, when applied to the present problem, leads to a final result free of any nonphysical divergences (in particular, of the surface divergence that was to be expected). We have shown in detail that the suppression of this divergence can be done consistently, by a proper choice of the dispersion function.

Also, it has become unnecessary to take the degeneracies of the eigenfrequencies of the system into account explicitly, because the degeneracies precisely correspond to the multiplicities of the zeros which appear in the argument principle [see (6)]. This fact makes the final theory much more simple, as compared with the original procedure of finding and counting the roots of the dispersion equation that had been used in Ref. [1].

Finally, the Casimir energy for the string has been calculated at finite temperature $T \neq 0$, the analytic approximation for high T has been obtained, and a formula well suited for numerical calculations in the low-T limit has been given. The specific meaning of these limits in terms of the characteristic frequencies of the system has been discussed numerically.

ACKNOWLEDGMENTS

E.E. is very grateful to Professor Kåre Olaussen and to the members of the Departments of Physics and of Applied Mechanics of Trondheim University, NTH, for the enjoyable and profitable stay. This work has been partly supported through research funds from CERN, from DGICYT (Spain), and from CIRIT (Generalitat de Catalunya).

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