Charged black holes in quadratic theories

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We point out that in general the Reissner-Nordström (RN) charged black holes of general relativity are not solutions of the four-dimensional quadratic gravitational theories. They are, e.g., exact solutions of the $R + R^2$ quadratic theory but not of a theory where a $R_{ab}R^{ab}$ term is present in the gravitational Lagrangian. In the case where such a nonlinear curvature term is present with sufficiently small coupling we obtain an approximate solution for a charged black hole of charge Q and mass M. For $Q \ll M$ the validity of this solution extends down to the horizon. This allows us to explore the thermodynamic properties of the quadratic charged black hole and we find that, to our approximation, its thermodynamics is identical to that of a RN black hole. However our black hole's entropy is not equal to one-fourth of the horizon area. Finally we extend our analysis to the rotating charged black hole and qualitatively similar results are obtained.

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I. INTRODUCTION

In the past few years there has been considerable interest in black hole (BH) solutions and their properties. One of the reasons is the development in our modern physical theories, in particular in gauge field theories and in string theory. On the one hand these developments have enriched the palette of known fields that probably play some role in nature and of course couple to gravity because of the universal character of the gravitational interaction. On the other hand, it is expected that Einstein's theory of gravity gets effectively modified, at some higher energy scale. We may have in the gravitational Lagrangian nonlinear curvature terms and the effective coupling with matter need not to be minimal. Also the space-time dimensionality may be higher than four with the extra dimensions compactified by some appropriate mechanism.

Incorporation of such elements in gravitation turns out to be particularly interesting and has already given rise to new and, sometimes, surprising consequences for black hole physics. In particular the discovery of a new family of BH solutions for Yang-Mills fields [1] has set us free from the so-called "no-hair" conjecture and its related theorems. We now know that black holes do not necessarily belong to the standard black hole families of solutions, namely, the Schwarzschild, Reissner-Nordström (RN), or the Kerr-Newman (KN) ones, and furthermore that in general they are not unique. In other words, they may have "hair." The properties of the Yang-Mills fields that are responsible for the existence of new black hole solutions have been extensively analyzed in Ref. [2]. For comments on the way classical no-hair theorems can be avoided by the presence of some fields, e.g., the dilaton in effective theories inspired on string theory, see Ref. [3].

With the gravity theories modified one may of course expect that also the thermodynamical properties of black hole laws for generalized black holes may significantly differ from that of standard black holes. However, several issues do survive. For example, the first law is still valid for Einstein-Yang-Mills black hole solutions [2]. In this paper we are interested in a somehow accidental property of standard black holes, sometimes referred in the literature as the area law. It is the simple proportionality between the black hole entropy S and its geometric intrinsic quantity, the horizon's area A, namely, the famous $S = A/4$ relation (in appropriate units). Interestingly enough, this relationship is still valid for some generalized black holes, but there are known cases where this is not true. The underlying pattern, if any, is not so clear. For references on cases where this relationship is not valid see [4]. In the same reference an interesting entropy formula is derived which, in some sense, systematizes the known results.

Of particular interest for this paper are cases where the gravitational Lagrangian contains nonlinear curvature terms. For higher dimensionality $d > 4$ it seems that the area law is generically not valid. It is interesting that for Lovelock gravity, the entropy is still given by an intrinsic geometric quantity evaluated at the black hole's horizon [5]. It differs from the $A/4$ value by a sum of intrinsic curvature invariants integrated over a cross section of the horizon. For $d = 4$ in a generic $(\text{Riemann})^2$ gravity the area law is valid, but not in a generic $(\text{Riemann})^3$ theory [4]. We would like, however, to point out that the above statements are true in vacuum. As we shall see in this paper, when matter couples to gravity, e.g., when an electromagnetic field is present, the area law is not valid in a generic four-dimensional gravitational theory with $(\text{Riemann})^2$ terms.

We consider the four-dimensional theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}) + \mathcal{L}_M, \qquad (1)$$

where $R_{\mu\nu}, R$ are the Ricci tensor and the scalar curvature with respect to the metric $g_{\mu\nu}$ and $g := \det|g_{ab}|$. α and β are some coupling constants. We will finally take as the matter part, $\mathcal{L}_M$, the usual electromagnetic Lagrangian.

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It is known that the gravitational part of such a theory can be described in terms of a massless tensorial graviton field, a massive scalar field if $\alpha \neq 0$, and/or a massive tensor field whenever $\beta \neq 0$, see [6–8], and references therein. From the analysis of the field equations at the linearized level, [8] and [9], it follows that the "source" of the additional massive fields is, for the scalar massive field, the trace $T^{(M)}$ of the usual matter stress energy-momentum tensor $T^{(M)}_{\mu\nu}$; while for the massive tensorial field the source is some combination of $T^{(M)}_{\mu\nu}$ components. Using this fact we may understand the motivation and some of the conclusions of this paper: the electromagnetic field, having a traceless stress energy-momentum tensor cannot excite the scalar massive field of the theory. If, furthermore, $\beta = 0$ then the solution may well not differ from that in Einstein's theory, namely, it may be the RN solution. Thus the no-hair conjecture seems to hold in this case. In fact a no-hair theorem can be proven [10]. However, it is immediately clear that this cannot be guaranteed if $\beta \neq 0$. It is the purpose of this paper to study this last case and to show that the charged black holes of the theory with Lagrangian given by (1.1) do not coincide with the RN or, if rotation is included, with the KN metrics. We will have to work in some approximation, which, however, in some limit will allow us to study the thermodynamical properties of these charged black hole solutions of the quadratic theory (1.1). In particular we shall consider only solutions expandable in small α and β.

The paper is organized as follows. In Sec. II we present the field equations in the exact and in some approximate case. In Sec. III we proceed mainly with the solution of the approximate field equations in the static and spherically symmetric case and discuss the range of the validity of the obtained black hole solution. This allows us to study in Sec. IV its thermodynamical properties. In Sec. V we extend the previous analysis to include rotation. Finally, in the Appendix we give the proof of the basic relations used for the solution of the field equations.

We use throughout this paper units in which $\hbar = c = G = k_B = 1$, metric signature $(-+++)$, Riemann tensor $R^a_{bcd} := -\partial_d \Gamma^a_{bc} + \cdots$ and Ricci tensor $R_{ab} := R^c_{acb}$. Finally Gaussian electromagnetic units are employed.

II. FIELD EQUATIONS

A. Exact field equations

The gravitational field equations for the theory in Eq. (1.1) read

$$(1+2\alpha R)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{\alpha}{2}R^2 g_{\mu\nu} + \left(2\alpha + \frac{\beta}{2}\right)g_{\mu\nu}R_{;p}{}^{;p} - (2\alpha+\beta)R_{;\mu\nu} + \beta R_{\mu\nu;p}{}^{;p} - \frac{\beta}{2}R_{pq}R^{pq}g_{\mu\nu} + 2\beta R_{pq}R_\mu{}^p{}_\nu{}^q = \frac{-16\pi}{\sqrt{-g}}\frac{\delta S_M}{\delta(g^{\mu\nu})} := 8\pi T_{\mu\nu}\ , \quad (2.1)$$

where $T_{\mu\nu}$ is the stress-energy-momentum tensor of the matter fields in the Lagrangian of Eq. (1.1). In this paper we will consider as a matter field the electromagnetic field $F_{\mu\nu}$, coupled minimally to gravity. The field equations for $F_{\mu\nu}$ are the Maxwell equations

$$\sqrt{-g}\nabla_\mu F^{\mu\nu} = \partial_\mu(\sqrt{-g}F^{\mu\nu}) = -4\pi\sqrt{-g}j^\nu, \quad (2.2)$$

where j^ν is the electromagnetic current.

Because of the complexity of these equations it is extremely difficult to obtain exact solutions even in cases of high symmetry. Consequently one should be prepared to work with some approximation scheme. We shall now discuss the main elements of our approximation method.

B. Approximate field equations

We concentrate on the case where the quadratic theory is slightly different from Einstein's general relativity. In particular let us consider the case where we can neglect squared Ricci terms in field equations (2.1). These "linearized" field equations read

$$G_{ab} \approx 8\pi T_{ab} - \left(2\alpha + \frac{\beta}{2}\right)g_{ab}\,\Box\, R + (2\alpha+\beta)R_{;ab} - \beta\,\Box\, R_{ab} - 2\beta R_{pq}R_a{}^p{}_b{}^q. \quad (2.3)$$

More specifically, this linearization demands that terms like $R^2, R_{ab}R^{ab}$, which are proportional to α, β parameters, can be neglected on the one hand with respect to R, R_{ab} ones and on the other hand with respect to second derivatives of Ricci curvature terms, which are also proportional to α and β parameters. The first requirement can be achieved by choosing sufficiently small values of α and β. Small values for these parameters is actually what nature has chosen, as it is concluded from all the tests general relativity ($\alpha = 0 = \beta$) has so far successfully passed. The second requirement, however, may not be achievable everywhere in space-time, since it depends on the particular behavior of curvature components. We will discuss it further below, when we will consider the case of charged black holes. Finally, in the case where the (Ricci)×(Riemann) term of Eq. (2.3), being a quadratic curvature term, may seem not to fit so well with the characterization of Eq. (2.3) as "linearized," we would like to remark that such a (Ricci)×(Riemann) term should be retained, since it can arise or absorbed to the second derivatives of the Ricci tensor with a simple commutation of the derivatives.

Assuming that the above requirements are satisfied, the space-time metric g_{ab} in the quadratic theory in the presence of some matter field, is slightly modified from the respective metric $\widehat{g}_{ab}$ in Einstein's theory with the same matter field. We can write

$$g_{ab} = \widehat{g}_{ab} + \chi\widehat{g}_{ab} + \psi_{ab}, \quad (2.4)$$

with $|\chi| \ll 1$, $|\psi_{ab}| \ll |\widehat{g}_{ab}|$. As is shown in the Appendix, by choosing appropriately χ, ψ_{ab} the linearized gravitational field equations Eqs. (2.3) are equivalent to the system of equations

$$G_{\mu\nu}(\widehat{g}_{ab}) = 8\pi T_{\mu\nu}, \tag{2.5}$$

$$(\Box - m_0^2)\chi = -\frac{8\pi}{3}T, \tag{2.6}$$

$$(\Box - m_1^2)\psi_{\mu\nu} + \left(1 - \frac{m_1^2}{m_0^2}\right)\nabla_\mu\nabla_\nu(\psi^\lambda{}_\lambda) = 16\pi\left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}\right), \tag{2.7}$$

where

$$m_0^{-2} = 6\alpha + 2\beta, \qquad m_1^{-2} = -\beta. \tag{2.8}$$

Finally $\psi_{\mu\nu}$ should satisfy the condition

$$(\psi_{ab} - \psi^\lambda_\lambda g_{ab})^{;b} = 0. \tag{2.9}$$

In Eqs. (2.5), (2.6), and (2.7) $G_{\mu\nu}$ is the Einstein tensor for the metric $\widehat{g}_{\mu\nu}$. $T_{\mu\nu}$ is the stress-energy-momentum tensor of the matter fields in the original Lagrangian. However, in our approximation, where squared Ricci terms are neglected, the energy tensor needs to be constructed using only the $\widehat{g}_{ab}$ metric. Terms that are omitted in this way are $(\partial T_{ab}/\partial g_{\mu\nu})\delta g_{\mu\nu}$, which although they are linear in χ and ψ_{ab} fields they are nevertheless quadratic in Ricci tensor. For the same reason the metric $g_{\mu\nu}$ in the right-hand side of Eq. (2.7) can be replaced with $\widehat{g}_{\mu\nu}$. Finally, in our appoximation the derivative operators can also be taken with respect to $\widehat{g}_{\mu\nu}$ metric.

To have positive mass parameters m_0, m_1 we shall hereafter assume the no tachyon constraints

$$3\alpha + \beta \geq 0, \quad \beta \leq 0. \tag{2.10}$$

The above field equations demonstrate the decomposition of the quadratic gravitational theory into a theory with a graviton field $\widehat{g}_{ab}$, a massive scalar field χ and a massive tensorial field $\psi_{\mu\nu}$, all of them coupled with the stress-energy-momentum tensor $T_{\mu\nu}$. In particular one should notice here what we have remarked in the introduction: the scalar field χ has as the source the trace of $T_{\mu\nu}$ and thus will not get excited by the traceless $T_{\mu\nu}$ of an electromagnetic field. On the other hand, the massive spin-2 field $\psi_{\mu\nu}$ in general will. Finally let us remark that there are some issues regarding the equivalence of such a decomposition with the initial theory. These will not be discussed in this paper; see [8] and the references therein. Instead we shall only view these equations as a convenient mathematical reduction step for solving in some approximation the initial fourth-order derivative theory dealing only with second-order derivative equations.

III. SOLUTIONS

We are interested in charged black hole solutions of the quadratic theory with the Lagrangian (1.1). We will consider both static and stationary solutions. However, we prefer to discuss in more detail the static case that will allow the illustration of our method and the exposition of our main results without particular technical complications. Then, in another section, we will extend the analysis to include rotation.

We consider now the case where the matter field is static and spherically symmetric generated by an electric charge Q centered at the origin $r = 0$. The respective space-time will be spherically symmetric and static and therefore there is a coordinate system where the metric can be written as

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.1}$$

Here θ and ϕ are the usual spherical angular coordinates.

The field equations for the Maxwell field $F_{\mu\nu}$ are given by Eq. (2.2) where the current source has only a time component equal to $j^0 = Q\delta(r)/(4\pi r^2\sqrt{AB})$. These electromagnetic field equations are easily solved in the background metric (3.1). It turns out that the only nonvanishing components of the $F^{\mu\nu}$ tensor are the tr (rt) ones with

$$F^{tr} = -\frac{Q}{r^2\sqrt{AB}}. \tag{3.2}$$

The stress-energy-momentum tensor for this field configuration is

$$T_t^t = T_r^r = -T_\theta^\theta = -T_\phi^\phi = -\frac{Q^2}{8\pi r^4}. \tag{3.3}$$

We now look for solutions to the gravitational equations.

A. Exact static black hole solutions

Case (i): $\alpha = 0$, $\beta = 0$. In this case we have Einstein's theory, where the solution is known to be unique. It is the Reissner-Nordström (RN) black hole metric given by Eq. (3.1) with

$$B(r) = A(r)^{-1} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{3.4}$$

where M and Q are, respectively, the mass and the total charge of the black hole.

Case (*ii*): $\alpha \neq 0$, $\beta = 0$. Here we can check directly [12] that the general relativistic RN black hole metric is also an exact solution for this theory. The question is whether it is unique. The answer seems to be yes, since according to Whitt [10] a no-hair theorem can be proven for theories with gravitational Lagrangian $L = R + \alpha R^2$ in vacuum and in the presence of electromagnetic matter in the case where the condition $R \neq -\frac{1}{2\alpha}$ holds everywhere.

Case (*iii*): $\beta \neq 0$. Now the direct check with the RN solution [12], shows that the RN metric is not a solution, except for the vacuum case with zero charge $Q = 0$, where one solution is known to be the general relativistic Schwarzschild black hole. The field equations are very difficult and have not allowed us to find an exact solution.

B. Approximate static black hole solutions

We assume now that there exists some parameter space where one may use the approximate system of second-order field equations (2.5)–(2.7) instead of the original fourth-order derivative equations. After finding solutions to these equations one has to check whether they are compatible with the assumptions that we have made when deriving these field equations.

For the stress-energy-momentum tensor of Eq. (3.3) the solution for $\widehat{g}_{\mu\nu}$ in Eq. (2.5) is of course the RN metric given by Eqs. (3.1) and (3.4). Now, in the background of this $\widehat{g}_{\mu\nu}$ metric we have to solve Eqs. (2.6) and (2.7). Notice, once again, that the χ field is not excited because $T = 0$. Taking into account the symmetries of our problem we are left with the following set of equations for the field $\psi_{\mu\nu}$:

$$(\widehat{\nabla}^2 - m_1^2)\psi_{\mu\nu} \approx 16\pi T_{\mu\nu}, \tag{3.5}$$

where $\widehat{\nabla}^2$ is with respect to the RN metric, and the stress-energy-momentum tensor $T_{\mu\nu}$ is given by Eqs. (3.3).

To proceed we shall consider the case of sufficiently small $|\beta|$ parameter such that $M \gg m_1^{-1}$ (remember $m_1^{-2} := -\beta$). This implies that with $r \geq M$ we will also have $r \gg m_1^{-1}$. Using this fact we observe that we can obtain the first leading term for $\psi_{\mu\nu}$ very simply. Indeed, in the left-hand side of the field equations (3.5) the mass term dominates over the derivative terms for $r \geq M \gg m_1^{-1}$. Thus, to a good approximation we can neglect the derivative terms and arrive at the following diagonal solution for $\psi_{\mu\nu}$:

$$\begin{aligned}
\psi_{tt} &\approx -\frac{2Q^2}{m_1^2 r^4}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right), \\
\psi_{rr} &\approx \frac{2Q^2}{m_1^2 r^4}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}, \\
\psi_{\theta\theta} &\approx -\frac{2Q^2}{m_1^2 r^2}, \\
\psi_{\phi\phi} &\approx -\frac{2Q^2}{m_1^2 r^2}\sin^2\theta.
\end{aligned} \tag{3.6}$$

With a straightforward calculation one can check that the condition (2.9) is satisfied.

Finally replacing in Eq. (2.4) the components of the Reissner-Nordström metric $\widehat{g}_{\mu\nu}$ and of the $\psi_{\mu\nu}$ field from the last equation (3.6) we obtain the metric $g_{\mu\nu}$ for the initial fourth-order gravitational theory. This can be written as

$$\begin{aligned}
ds^2 \approx &-\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(1 + \frac{2Q^2}{m_1^2 r^4}\right)dt^2 \\
&+\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\left(1 + \frac{2Q^2}{m_1^2 r^4}\right)dr^2 \\
&+r^2\left(1 - \frac{2Q^2}{m_1^2 r^4}\right)(d\theta^2 + \sin^2\theta d\phi^2).
\end{aligned} \tag{3.7}$$

It is easy to see that in this metric the "Newtonian" gravitational potential is modified. At large radial distances, particles will feel an additional repulsive force $4Q^2/(m_1^2 r^5)$ in addition to the force that they feel in the RN spacetime. Note, however, that null geodesics on the $\theta = 0$ plane coincide with those of the RN metric.

Up to now we have solved the approximate field equations (2.5)–(2.7) in the case where $r \geq M \gg m_1^{-1}$. However, we still have to check whether the conditions for using the approximate field equations are satisfied. With straightforward but tedious calculations, we find that these conditions are satisfied for the solution (3.7) at sufficiently large distances from the black hole's horizon $r \gg M$. However, if one is interested in a solution that is approximately valid in the region from radial infinity down to the horizon, one has still to restrict it to sufficiently small charges $|Q| \ll M$. We will provide here only a dimensional argument, which leads to this condition. Remember that for using the approximate field equations we needed the Ricci squared terms to be negligible with respect to second derivatives of Ricci terms. For our particular problem the metric is approximately the Reissner-Nordström one, and thus a typical Ricci term will be $\propto Q^2/r^4$. Therefore we must have $(Q^2/r^4)^2 \ll (1/r^2)(Q^2/r^4)$. If this condition is to be satisfied also near the horizon $r \approx M$ one gets the restriction $Q^2 \ll M^2$. And this is what we assume in the following section where we discuss the thermodynamic properties of the approximate solution (3.7).

IV. THERMODYNAMIC PROPERTIES OF THE SOLUTION

As we have seen in the preceding section the metric of a charged black hole in the quadratic gravitational theory (1) is not the Reissner-Nordström solution. To a good approximation the metric is given by Eq. (3.7). Its range of validity extends down to the vicinity of the horizon provided that $M \gg |Q|$ and $M \gg m_1^{-1}$. This will allow us to calculate some of the thermodynamic quantities associated to black holes. In particular, we are interested in the entropy-area relationship. Since the metric is known to be spherically symmetric and static, it is easy to employ standard Euclidean techniques to obtain the temperature associated to the black hole and an expression for its entropy [13]. However the same results are obtained with a more simple minded thermodynamical treatment [11]. We prefer here to follow this last method.

With the argument due to Hawking, which is based on a semiclassical calculation of quantum effects near the horizon, one can associate with a black hole a temperature $T = \kappa/(2\pi)$ where κ is the surface gravity of the black hole's horizon. Note that for our metric the horizon is at a radial coordinate r_H given by

$$r_H = M + \sqrt{M^2 - Q^2}. \tag{4.1}$$

The surface gravity of the horizon $\kappa(r_H)$ is then easily computed

$$\kappa(r_{\rm H}) = -\frac{1}{2}\frac{\partial g_{tt}/\partial r}{\sqrt{-g_{tt}g_{rr}}} = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}. \tag{4.2}$$

We notice here that $\kappa(r_{\rm H})$ has the same value as for the RN black hole of general relativity. Consequently the same is true for the temperature.

This coincidence should not create the wrong impression that everything regarding the thermodynamics-geometry relationship for our black hole is the same as for general relativity. On the one hand the coincidence itself is probably accidental in our approximation and does not survive in the exact solution. On the other hand, even to our approximation, the other important geometric quantity of the black hole, namely, its area A, does not have the same expression as in the RN solution and in our solution given by Eq. (3.7). In fact, the horizon area of our quadratic gravity black hole is given by

$$A = 4\pi r_H^2 \left(1 - \frac{2Q^2}{m^2 r_H^4}\right). \tag{4.3}$$

By replacing into this expression the $r_{\rm H}$ of Eq. (4.1) and solving the resulting expression for the mass M of the black hole we obtain the fundamental relation of black hole thermodynamics:

$$M = \frac{\left[\frac{A}{8\pi} + \sqrt{\left(\frac{A}{8\pi}\right)^2 + \frac{2Q^2}{m^2}} + Q^2\right]}{2\left[\frac{A}{8\pi} + \sqrt{\left(\frac{A}{8\pi}\right)^2 + \frac{2Q^2}{m^2}}\right]^{1/2}}. \tag{4.4}$$

By differentiation of this equation we obtain the first law of black hole thermodynamics:

$$dM = \frac{\partial M}{\partial A}\Big|_Q dA + \frac{\partial M}{\partial Q}\Big|_A dQ := T_H dS + \Phi_H dQ, \tag{4.5}$$

where S and Φ represent, respectively, the entropy and the electric potential at the horizon of the black hole.

Now the task is the proper identification of the quantities T_H, S, Φ_H and Q in (4.5) with the black hole parameters. The derivatives appearing in (4.5) can be explicitly obtained using Eq. (4.4). The temperature T_H is naturally identified with the Hawking temperature $T = \kappa(r_H)/(2\pi)$ that we have calculated above. Similarly Q is the total electric charge of the black hole, i.e., the same quantity that appears in our quadratic black hole metric. On the other hand, the electromagnetic potential Φ_H on the horizon can be computed by direct use of Maxwell equations

$$\Phi_H = \int_{r_H}^{\infty} \frac{Q}{r^2}\left(1 - \frac{Q^2}{m_1^2 r^4}\right) dr = \frac{Q}{r_H} + O\left(\frac{Q^3}{m_1^2 r^5}\right). \tag{4.6}$$

Thus by inserting Eqs. (4.2) and (4.6) into Eq. (4.5) we can, upon integration, identify the black hole entropy S as

$$S \approx \frac{A}{4} + \frac{8\pi^2 Q^2}{m_1^2 A} \approx \pi r_{\rm H}^2. \tag{4.7}$$

Here we observe that extra terms appear when we consider S as a function of A. Thus in general, the simple relationship $S = \frac{1}{4}A$ no longer holds in quadratic gravitational Lagrangian theories. Notice, however, that in terms of the black hole parameters M and Q, the entropy S has the same value as in RN black hole. The same is true, as we have seen above, for T_H and Φ_H and also for other derived quantities as, e.g., the heat capacity, since

$$C_Q = T\frac{\partial S}{\partial T}\Big|_Q = C_Q^{\rm RN}. \tag{4.8}$$

Let us remark once again that this nice coincidence probably happens only in our approximation and will not survive in the exact quadratic charged black hole solution.

V. EXTENSION TO THE ROTATING CASE

In this section we will extend our calculations to the case of rotating charged black holes in the quadratic theory (1). We will work in the approximation where we can use the field equations (2.5)–(2.7). We will use the same methods that we used in the previous sections for the static case. The metric can now be written in the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2. \tag{5.1}$$

As in the static case we can almost immediately solve the approximate field equations. Indeed, $\widehat{g}_{\mu\nu}$ will be the known general relativistic Kerr-Newman rotating black hole metric. The scalar field χ will not get excited, since the source is the traceless electromagnetic stress energy momentum tensor. Therefore, we are left with the field equations for the massive scalar field $\psi_{\mu\nu}$, which are to be solved in the background of the KN metric. Then with the same arguments as in static case we find that the solution for $\psi_{\mu\nu}$ is approximately given by

$$\psi_{\mu\nu} \approx -\frac{16\pi}{m_1^2} T_{\mu\nu}^{\rm (KN)}, \tag{5.2}$$

where $T_{\mu\nu}^{\rm (KN)}$ is the energy tensor of the Kerr-Newman black hole. Finally the metric in the quadratic theory is given by

$$g_{\mu\nu} \approx \widehat{g}_{\mu\nu} + \psi_{\mu\nu}. \tag{5.3}$$

Using the metric tensor and the $T_{\mu\nu}^{\rm (KN)}$ tensor of the Kerr-Newman black hole in Boyer-Lindquist coordinates [14], we obtain the metric $g_{\mu\nu}$:

$$\begin{aligned} g_{tt} &= -\frac{\Delta I_{(+)}}{\rho^2} + \frac{a^2\sin^2\theta}{\rho^2} I_{(-)}, \\ g_{t\phi} &= [\Delta I_{(+)} - (\rho^2 + a^2) I_{(-)}]\frac{a\sin^2\theta}{\rho^2}, \\ g_{\phi\phi} &= [(r^2+a^2)^2 I_{(-)} - \Delta a^2 \sin^2\theta I_{(+)}]\frac{\sin^2\theta}{\rho^2}, \\ g_{\theta\theta} &= \rho^2 I_{(-)}, \\ g_{rr} &= \frac{\rho^2}{\Delta} I_{(+)}, \end{aligned} \tag{5.4}$$

where

$$I_{(\pm)} = \left(1 \pm \frac{2Q^2}{m_1^2\rho^4}\right), \tag{5.5}$$

and $\rho := r^2+a^2\cos^2\theta, \Delta := r^2-2Mr+a^2+Q^2$. Here a is the usual angular momentum per total mass parameter.

As in the static case we shall take $|Q| \ll M$ to be able to discuss the thermodynamic behavior of the solution. Having the metric it is now straightforward to calculate the radius of the event horizon r_H, the surface gravity $\kappa(r_H)$ and the horizon's angular velocity Ω_H. All of them turn out to have the general relativity values: namely,

$$r_H = M + \sqrt{M^2 - Q^2 - a^2}, \tag{5.6}$$

$$\kappa(r_H) = \frac{\sqrt{M^2-a^2-Q^2}}{r_H^2+a^2}, \tag{5.7}$$

$$\Omega_H = \frac{a}{r_H^2+a^2}. \tag{5.8}$$

Straightforward is also the calculation for the area A of the event horizon:

$$A = 4\pi\left\{(r_H^2+a^2) + \frac{Q^2}{m_1^2 r_H^2}\left[1 + \frac{(r_H^2+a^2)}{a r_H}\arctan\left(\frac{a}{r_H}\right)\right]\right\}. \tag{5.9}$$

Note that, as in the static case, this is different from the general relativity result. From this equation we can obtain M as a function of A, Q, and J.

The first law of black hole thermodynamics now reads

$$\begin{aligned} dM &= \left.\frac{\partial M}{\partial A}\right|_{Q,J} dA + \left.\frac{\partial M}{\partial Q}\right|_{A,J} dQ + \left.\frac{\partial M}{\partial J}\right|_{A,Q} dJ \\ &:= T_H dS + \Phi_H dQ + \Omega_H dJ, \end{aligned} \tag{5.10}$$

where the new quantities not present in the static case are the horizon's angular velocity Ω_H and black hole's angular momentum J. The partial derivatives in (5.10) can be explicitly calculated, and the T_H, Q, Φ_H are naturally identified as in the static case. Hence we are left with a differential expression for the calculation of the entropy S in terms of the parameters A, J, and Q. Upon integration of this expression we find

$$S(A,J,Q) = \frac{A}{4} + \frac{8\pi^2Q^2}{m_1^2 A} + \frac{8(4\pi)^4Q^2J^2}{3m_1^2A^3} + \cdots . \tag{5.11}$$

We observe that in general the entropy will not be a function only of the area of black hole, but also of the other parameters that characterize it.

It is interesting to notice that also in the rotational case, the actual value of the entropy, that is the one in terms of M, Q, J, is the same as in general relativity, since $S = 4\pi r_H^2$ still holds in our approximation. But probably this is not true in the exact solution.

VI. SUMMARY AND DISCUSSION

We have looked for solutions representing electrically charged, static and stationary black holes in the theory with the quadratic gravitational Lagrangian of Eq. (1.1). The gravitational field equations, which involved fourth-order derivatives for the metric, have been reduced to a set of field equations with second-order derivatives for a massless tensorial field together with a massive scalar and a massive tensorial field. Solving approximately these equations we have seen that the modifications from the general relativistic results come only from the massive tensorial field. The scalar field does not contribute, since its source is the trace of the matter stress energy momentum tensor, which in our case is the traceless electromagnetic energy tensor. This is in fact an exact result, since, as we have directly checked, the Reissner-Nordström solution is an exact solution of the initial quadratic gravitational field equations only if the massive tensorial field is missing [$\alpha \neq 0, \beta = 0$ in Eq. (1.1)].

Computing several thermodynamic quantities for our black hole solutions we have found that these quantities retain, in our approximation, their general relativistic value. This coincidence will quite probably not survive in the exact solution. However, it is interesting to note here that for generalized charged black holes in other theories usually one finds corrections in the temperature and other thermodynamic quantities, see Ref. [15] for a discussion in the context of string theory and Ref. [16] in terms of Kaluza-Klein theories.

Finally, for our black hole solutions we have found that their entropy does not follow the simple area law $S = A/4$. Such a behavior is known for theories that contain nonlinear curvature terms in their gravitational Lagrangian and have dimensionality higher than four. The respective investigations have been mainly performed with the Lovelock Lagrangian [5]. To have in this theory nontrivial results in four dimensions one should have an effective nontrivial coupling of the curvature to some other field, see, e.g., Ref. [16] in the context of Kaluza-Klein theories. As we have shown in this paper, we can have in four dimensions a violation of the area law just by including a minimally coupled electromagnetic field in the quadratic gravitational Lagrangian of Eq. (1.1).

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APPENDIX

The field equations of the quadratic theory in the approximation where Ricci squared terms are neglected, are given by Eq. (2.3).

Let us now consider the metric transformation

$$g_{ab} \to \widehat{g}_{ab} = g_{ab} - \chi g_{ab} - \psi_{ab}, \tag{A1}$$

with χ and ψ_{ab} as scalar and tensorial field perturbation, that is with $|\chi| \ll 1$ and $\psi_{ab} \ll g_{ab}$. Hereafter, quantities with carets will refer to the $\widehat{g}_{ab}$ metric.

For the purposes of the paper we will keep only linear terms in χ and ψ_{ab} fields. Then the transformation in Eq. (A1) implies that the Einstein tensor $\widehat{G}_{ab}$ is related to G_{ab} with

$$\widehat{G}_{ab} = G_{ab} + \frac{1}{2}\Box\psi_{ab} + \frac{1}{2}(\psi_p{}^p{}_{;ab} - \psi_{pa;b}{}^{;p} - \psi_{pb;a}{}^{;p}) + \frac{1}{2}g_{ab}(\psi_{pq}{}^{;pq} - \Box\psi_p{}^p) + \chi_{;ab} - (\Box\chi)g_{ab} + O(\psi^2,\chi^2). \quad \text{(A2)}$$

Here the derivative operators, being without carets, are with respect to the g_{ab} metric.

Substituting Eq. (2.3) in Eq. (A2) we obtain (to linear order)

$$\widehat{G}_{ab} \approx 8\pi T_{ab} - \left(2\alpha + \frac{\beta}{2}\right)g_{ab}\Box R + (2\alpha+\beta)R_{;ab} - \beta\Box R_{ab} - 2\beta R_{pq}R_a{}^p{}_b{}^q$$
$$+\frac{1}{2}\Box\psi_{ab} + \frac{1}{2}(\psi_p{}^p{}_{;ab} - \psi_{pa;b}{}^{;p} - \psi_{pb;a}{}^{;p}) + \frac{1}{2}g_{ab}(\psi_{pq}{}^{;pq} - \Box\psi_p{}^p) + \chi_{;ab} - (\Box\chi)g_{ab} + O(\psi^2,\chi^2). \quad \text{(A3)}$$

Our aim now is to decrease the order of metric derivatives that appear in the field equations (A3). This can be done by selecting the tensorial field ψ_{ab} and the scalar field χ in such a way that the terms on the right-hand side of Eq. (A3), which contain higher order than second metric derivatives cancel out. Obviously, with ψ_{ab} we can cancel the term $\Box R_{ab}$. For this let us choose

$$\psi_{ab} = 2\beta R_{ab} + \lambda R g_{ab}, \quad \text{(A4)}$$

where λ is some constant. Then it follows

$$\widehat{G}_{ab} = 8\pi T_{ab} - (2\alpha+\beta+\lambda)g_{ab}\Box R + (2\alpha+\beta+\lambda)R_{;ab} + \chi_{;ab} - g_{ab}\Box\chi. \quad \text{(A5)}$$

In deriving Eq. (A5) we have made use of the Bianchi identity $R_{ab}{}^{;b} = \frac{1}{2}R_{;a}$.

Now with the obvious choice

$$\chi = -(2\alpha+\beta+\lambda)R, \quad \text{(A6)}$$

we obtain from Eq. (A5) that

$$\widehat{G}_{ab} \approx 8\pi T_{ab}, \quad \text{(A7)}$$

which is indeed an equation with only second derivatives in the metric $\widehat{g}_{ab}$. Note that the constant λ remains unspecified and can be chosen arbitrarily. The field equations for χ can be derived from the trace of Eq. (2.3):

$$(6\alpha+2\beta)\Box R - R = 8\pi T. \quad \text{(A8)}$$

With the choice of λ,

$$\lambda = -\frac{\beta}{3}, \quad \text{(A9)}$$

and using Eq. (A6) we can write Eq. (A8) as

$$\Box\chi - m_0^2\chi = -\frac{8\pi}{3}T, \qquad m_0^{-2} = 6\alpha + 2\beta. \quad \text{(A10)}$$

The field ψ_{ab} should satisfy a set of equations resulting from the Bianchi identity

$$\left(\psi_{ab} - \frac{\beta+\lambda}{2(\beta+2\lambda)}\psi_p{}^p g_{ab}\right)^{;b} = 0. \quad \text{(A11)}$$

The field equations for ψ_{ab} can be obtained from Eq. (2.3), and Eq. (A4)

$$(\Box - m_1^2)\psi_{ab} = 16\pi T_{ab} + (\lambda - 4\alpha - \beta)\Box R g_{ab} + \left(\frac{\lambda}{\beta}+1\right)R g_{ab} + 2(2\alpha+\beta)R_{;ab}. \quad \text{(A12)}$$

With the choice (A9) and using Eqs. (A6), (A10) and (A4) we can write Eq. (A12) as

$$(\Box - m_1^2)\psi_{ab} - (1 - \frac{m_1^2}{m_0^2}\psi_p{}^p)_{;ab} = 16\pi(T_{ab} - \frac{1}{3}T g_{ab}). \quad \text{(A13)}$$

Thus our initial field Eqs. (2.3) are equivalent to the system of Eqs. (A7), (A10), and (A13). In these equations the $\Box$ operator is with respect to the metric g_{ab}, but it can be also taken with respect to $\widehat{g}_{ab}$ in our approximation, where we keep only linear terms in χ, ψ_{ab}. This is the system of field equations used in the body of the paper.

Note here that Eq. (A10) is not independent, since it is just the trace of Eq. (A13). From the physical point of view, however, it is interesting, since it clearly shows that, at least in our approximations, the massive scalar sector of the quadratic theory is missing whenever the trace of the stress-energy-momentum tensor of matter fields T is zero. The case of weak gravitational limit is contained in our results, and one may compare with the results of Teyssandier [9], which however are only valid in a conveniently chosen coordinate system.

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