Quantization of the Bianchi type-IX model in supergravity with a cosmological constant

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Diagonal Bianchi type-IX models are studied in the quantum theory of $N=1$ supergravity with a cosmological constant. It is shown, by imposing the supersymmetry and Lorentz quantum constraints, that there are no physical quantum states in this model. The $k=+1$ Friedmann model in supergravity with a cosmological constant does admit quantum states. However, the Bianchi type-IX model provides a better guide to the behavior of a generic state, since more gravitino modes are available to be excited. These results indicate that there may be no physical quantum states in the full theory of $N=1$ supergravity with a nonzero cosmological constant.

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I. INTRODUCTION

Recently a number of quantum cosmological models have been studied in which the action is that of supergravity, with possible additional coupling to supermatter [1–11]. It is sufficient, in finding a physical state, to solve the Lorentz and supersymmetry constraints of the theory [12,13]. Because of the anticommutation relations $[S_A,\tilde{S}_{A'}]_+\sim\mathcal{H}_{AA'}$, the supersymmetry constraints $S_A\Psi=0,\bar{S}_{A'}\Psi=0$ on a physical wave function Ψ imply the Hamiltonian constraint $\mathcal{H}_{AA'}\Psi=0$ [12,13].

In the case of the Bianchi type-I model in $N=1$ supergravity with a cosmological constant $\Lambda=0$ [8], only two quantum states appear. Using the factor ordering of [8], one state is $h^{1/4}$ in the bosonic sector, where $h=\det h_{ij}$ is the determinant of the three-metric, and the other state is $h^{-1/4}$ in the sector filled with fermions. In the case of Bianchi type IX with $\Lambda=0$, there are again two states, of the form $\exp(\pm I/\hbar)$ where I is a certain Euclidean action, one in the empty and one in the filled fermionic sector [9,14]. When the usual choice of spinors constant in the standard basis is made for the gravitino field, the bosonic state $\exp(-I/\hbar)$ is the wormhole state [9,15]. With a different choice, one obtains the Hartle-Hawking state [14,16]. Similar states were found for $N=1$ supergravity in the more general minisuperspace models of class A [10]. [Supersymmetry (as well as other considerations) forbids minisuperspace models of class B.] It was also found in the general theory of quantized $N=1$ supergravity with $\Lambda=0$ that there are two bosonic states of the form $\exp(-I/\hbar)$, where I is the wormhole or the Hartle-Hawking classical action [17]. (There are also many other bosonic states.) There are also two states of the form $\exp(I/\hbar)$ in the filled sector.

It is of interest to extend these results by studying more general locally supersymmetric actions, initially in Bianchi models. Possibly the simplest such generalization is the addition of a cosmological constant in $N=1$ supergravity [18]. It was found that in the Bianchi type-I case there are no physical states for $N=1$ supergravity with a Λ term [11]. The Bianchi type-IX model with a Λ term and with $N=1$ supersymmetry in one dimension was studied by Graham [4]. Here we treat the Bianchi type-IX model with a Λ term with the full $N=4$ supersymmetry in one dimension. We shall see that there are again no physical quantum states. The calculations are described in Sec. II. We also treat briefly in Sec. III the spherical $k=+1$ Friedmann model, and find that there is a two-parameter family of solutions of the quantum constraints with a Λ term. Nevertheless, as will be seen, the Bianchi type-IX model provides a better guide to the generic result, since more spin-$\frac{3}{2}$ modes are available to be excited in the Bianchi type-IX model, while the form of the fermionic fields needed for supersymmetry in the $k=+1$ Friedmann model is very restrictive [6]. Section IV contains the Conclusion.

II. QUANTUM STATES FOR THE BIANCHI TYPE-IX MODEL WITH A Λ TERM

Using two-component spinors [6,13], the action [18] is

$$S=\int d^4x\,[(2\kappa^2)^{-1}(\det e)(R-3g^2)\ .\\ +\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^{A'}{}_{\mu}e_{AA'\nu}D_{\rho}\psi^{A}{}_{\sigma}+\text{H.c.})\\ -\tfrac{1}{2}g(\det e)(\psi^{A}{}_{\mu}e_{AB'}{}^{\mu}e_{B}{}^{B'\nu}\psi^{B}{}_{\nu}+\text{H.c.})]\ . \tag{2.1}$$

Here the tetrad is $e^{a}{}_{\mu}$ or equivalently $e^{AA'}{}_{\mu}$. The gravitino field $(\psi^{A}{}_{\mu},\bar{\psi}^{A'}{}_{\mu})$ is an odd (anticommuting) Grassmann quantity. The scalar curvature R and the covariant derivative D_ρ include torsion. We define $\kappa^2=8\pi$. Here g is a constant, and the cosmological constant is $\Lambda=\frac{3}{2}g^2$.

There are two possible approaches to the quantization of this model. One possibility is to substitute the Bianchi type-IX ansatz for the geometry $e^{AA'}{}_{\mu}$ and gravitino field $(\psi^{A}{}_{\mu},\bar{\psi}^{A'}{}_{\mu})$ into the action (2.1). The components $\psi^{A}{}_{\mu}e^{BB'\mu}$ and $\bar{\psi}^{A'}{}_{\mu}e^{BB'\mu}$ are required to be spatially con-

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stant with respect to the standard triad [19] on the Bianchi type-IX three-sphere. One finds that, in order for the form of the ansatz to be left invariant by one-dimensional local supersymmetry transformations, possibly corrected by coordinate and Lorentz transformations [6], one must study the general nondiagonal Bianchi type-IX model [19]. The reduced action could then be computed, leading to the Hamiltonian and supersymmetry constraints. Finally the supersymmetry constraints could be imposed on physical wave functions. They would be complicated because of the number of parameters needed to describe the off-diagonal model.

The other alternative, taken here, is to apply the supersymmetry constraints of the general theory at a diagonal Bianchi type-IX geometry [9]. This is valid since the supersymmetry constraints are of first order in bosonic derivatives, and give expressions such as $\delta\Psi/\delta h_{im}(x)$ in terms of known quantities and Ψ. These equations can be evaluated at a diagonal Bianchi type-IX geometry, parametrized by three radii A, B, C. One multiplies (e.g.) by $\delta h_{im}(x)=\partial h_{im}/\partial A$ and integrates $\int d^3x(\ )$ to obtain an equation for $\partial\Psi/\partial A$ in terms of known quantities. The need to consider off-diagonal metrics is thereby avoided.

The general classical supersymmetry constraints are, with the help of [13], seen to be

$$\bar{S}_{A'}=gh^{1/2}e^{A}{}_{A'}{}^{i}n_{AB'}\bar{\psi}^{B'}{}_{i} +\epsilon^{ijk}e_{AA'i}{}^{3s}D_j\psi^{A}{}_{k}+\tfrac{1}{2}i\kappa^2\psi^{A}{}_{i}p_{AA'}{}^{i} , \tag{2.2}$$

and the conjugate S_A. Here $n^{AA'}$ is the spinor version of the unit future-pointing normal n^μ to the surface $t=$const. It is a function of the $e^{AA'}{}_{i}$, defined by

$$n^{AA'}e_{AA'i}=0,\quad n^{AA'}n_{AA'}=1 . \tag{2.3}$$

In Eq. (2.2), $p_{AA'}{}^{i}$ is the momentum conjugate to $e^{AA'}{}_{i}$. The expression ${}^{3s}D_j$ denotes the three-dimensional covariant derivative without torsion. Since the components of $\psi^{A}{}_{k}$ are taken to be constant in the Bianchi type-IX basis, one can replace ${}^{3s}D_j\psi^{A}{}_{k}$ by $\omega^{A}{}_{Bj}\psi^{B}{}_{k}$, where $\omega^{A}{}_{Bj}$ gives the torsion-free connection [13].

The corresponding quantum constraints read, with the help of [13],

$$\bar{S}_{A'}\Psi=-i\hbar g h^{1/2}e^{A}{}_{A'}{}^{i}n_{AB'}D^{BB'}{}_{ji}\left[h^{1/2}\frac{\partial\Psi}{\partial\psi^{B}{}_{j}}\right] +\epsilon^{ijk}e_{AA'i}\omega^{A}{}_{Bj}\psi^{B}{}_{k}\Psi-\frac{1}{2}\hbar\kappa^2\psi^{A}{}_{i}\frac{\delta\Psi}{\delta e^{AA'}{}_{i}}=0 , \tag{2.4}$$

$$S_A\Psi=gh^{1/2}e_{A}{}^{A'i}n_{BA'}\psi^{B}{}_{i}\Psi-i\hbar\omega_{A}{}^{B}{}_{i}\left[h^{1/2}\frac{\partial\Psi}{\partial\psi^{B}{}_{i}}\right] +\frac{1}{2}i\hbar^2\kappa^2 D^{BA'}{}_{ji}\left[h^{1/2}\frac{\partial}{\partial\psi^{B}{}_{j}}\right]\frac{\delta\Psi}{\delta e^{AA'}{}_{i}}=0 . \tag{2.5}$$

Here

$$D^{BA'}{}_{ji}=-2ih^{-1/2}e^{BB'}{}_{i}e_{CB'j}n^{CA'} , \tag{2.6}$$

and $\partial/\partial\psi^{B}{}_{j}$ denotes the left derivative [20]. We have made the replacement $\delta\Psi/\delta\psi^{B}{}_{j}\rightarrow h^{1/2}\partial\Psi/\partial\psi^{B}{}_{j}$. The $h^{1/2}$ factor ensures that each term has the correct weight in the equations. One can also check that this replacement gives the correct supersymmetry constraints in the $k=+1$ Friedmann model (without Λ term), where the model was quantized using the alternative approach via a supersymmetric ansatz [6].

In addition to the supersymmetry constraints, a physical state Ψ must also obey the Lorentz constraints

$$J^{AB}\Psi=0,\quad \bar{J}^{A'B'}\Psi=0 . \tag{2.7}$$

These imply that Ψ is formed from the three-metric h_{ij} and from scalar invariants in the gravitino field. To specify this, note the decomposition [11] of $\psi^{A}{}_{BB'}=e_{BB'}{}^{i}\psi^{A}{}_{i}$:

$$\psi_{ABB'}=-2n^{C}{}_{B'}\gamma_{ABC} +\tfrac{2}{3}(\beta_A n_{BB'}+\beta_B n_{AB'})-2\epsilon_{AB}n^{C}{}_{B'}\beta_C , \tag{2.8}$$

where $\gamma_{ABC}=\gamma_{(ABC)}$ is totally symmetric and ϵ_{AB} is the alternating spinor. The general Lorentz-invariant wave function is a polynomial of the sixth degree in Grassmann variables:

$$\Psi(e^{AA'}{}_{i},\psi^{A}{}_{i})=\Psi_0(h_{ij})+(\beta_A\beta^A)\Psi_{21}(h_{ij})+(\gamma_{ABC}\gamma^{ABC})\Psi_{22}(h_{ij}) +(\beta_A\beta^A)(\gamma_{BCD}\gamma^{BCD})\Psi_{41}(h_{ij})+(\gamma_{ABC}\gamma^{ABC})^2\Psi_{42}(h_{ij})+(\beta_A\beta^A)(\gamma_{BCD}\gamma^{BCD})^2\Psi_6(h_{ij}) . \tag{2.9}$$

As described in [11], any other Lorentz-invariant fermionic polynomials can be written in terms of these.

We now proceed to solve the supersymmetry and Lorentz constraints. The diagonal Bianchi type-IX three-metric is given in terms of the three radii A,B,C by

$$h_{ij}=A^2E^{1}{}_{i}E^{1}{}_{j}+B^2E^{2}{}_{i}E^{2}{}_{j}+C^2E^{3}{}_{i}E^{3}{}_{j} , \tag{2.10}$$

where $E^{1}{}_{i},E^{2}{}_{i},E^{3}{}_{i}$ are a basis of unit left-invariant one-forms on the three-sphere [19]. In the calculation, we shall repeatedly need the expression, formed from the connection:

$$\omega_{ABi}n^{A}{}_{B'}e^{BB'j}=\frac{i}{4}\left[\frac{C}{AB}+\frac{B}{CA}-\frac{A}{BC}\right]E^{1}{}_{i}E^{1j} +\frac{i}{4}\left[\frac{A}{BC}+\frac{C}{AB}-\frac{B}{CA}\right]E^{2}{}_{i}E^{2j} +\frac{i}{4}\left[\frac{B}{CA}+\frac{A}{BC}-\frac{C}{AB}\right]E^{3}{}_{i}E^{3j} . \tag{2.11}$$

This can be derived from the expressions for $\omega^{AB}{}_{i}$, given in [9,13].

First consider the $\bar{S}_{A'}\Psi=0$ constraint at the level ψ^1 in powers of fermions. One obtains

$$\frac{3}{16}\hbar g h^{1/2} e_{BA'}{}^{i}\psi^{B}{}_{i}\Psi_{21}+\epsilon^{jki}e_{AA'j}\omega^{A}{}_{Bk}\psi^{B}{}_{i}\Psi_{0} +\hbar\kappa^{2}e_{BA'j}\psi^{B}{}_{i}\frac{\delta\Psi_{0}}{\delta h_{ij}}=0 . \quad (2.12)$$

Since this holds for all $\psi^{B}{}_{i}$, one can conclude

$$\frac{3}{16}\hbar g h^{1/2} e_{BA'}{}^{i}\Psi_{21}+\epsilon^{jki}e_{AA'j}\omega^{A}{}_{Bk}\Psi_{0} +\hbar\kappa^{2}e_{BA'j}\frac{\delta\Psi_{0}}{\delta h_{ij}}=0 . \quad (2.13)$$

Now multiply this equation by $e^{BA'm}$, giving

$$-\frac{3}{16}\hbar g h^{im} h^{1/2}\Psi_{21}+\epsilon^{jki}e_{AA'j}e^{BA'm}\omega^{A}{}_{Bk}\Psi_{0} -\hbar\kappa^{2}\frac{\delta\Psi_{0}}{\delta h_{im}}=0 . \quad (2.14)$$

The second term can be simplified using [6]

$$e_{AA'j}e^{BA'}{}_{m}=-\frac{1}{2}h_{jm}\epsilon_{A}{}^{B}+i\epsilon_{jmn}h^{1/2}n_{AA'}e^{BA'n} . \quad (2.15)$$

One then notes, as above, that by taking a variation among the Bianchi type-IX metrics, such as

$$\delta h_{ij}=\frac{\partial h_{ij}}{\partial A}\delta A=2AE^{1}{}_{i}E^{1}{}_{j}\delta A , \quad (2.16)$$

multiplying by $\delta\Psi_0/\delta h_{ij}$ and integrating over the three-geometry, one obtains $\partial\Psi_0/\partial A$. Putting this information together one obtains the constraint

$$\hbar\kappa^{2}\frac{\partial\Psi_{0}}{\partial A}+16\pi^{2}A\Psi_{0}+6\pi^{2}\hbar g BC\Psi_{21}=0 , \quad (2.17)$$

and two others given by cyclic permutation of ABC.

Next we consider the $S_A\Psi=0$ constraint at order ψ^1. One uses the relations $\partial(\beta_A\beta^A)/\partial\psi^{B}{}_{i}=-n_{A}{}^{B'}e_{BB'}{}^{i}\beta^{A}$ and $\partial(\gamma_{ADC}\gamma^{ADC})/\partial\psi^{B}{}_{i}=-2\gamma_{BDC}n^{CC'}e^{D}{}_{C'}{}^{i}$, and writes out β^A and γ_{BDC} in terms of $e^{EE'}{}_{j}$ and $\psi^{E}{}_{j}$. Proceeding by analogy with the previous calculation above, one again "divides out" by $\psi^{B}{}_{j}$ to obtain

$$gh^{1/2}e_{A}{}^{A'j}n_{BA'}\Psi_{0}-\frac{1}{4}i\hbar\omega_{A}{}^{C}{}_{i}h^{1/2}e_{CB'}{}^{i}e_{B}{}^{B'j}\Psi_{21}-\left[\frac{1}{3}i\hbar\omega_{ABi}h^{1/2}e_{DA'}{}^{j}e^{DA'i}+\frac{2}{3}i\hbar\omega_{A}{}^{E}{}_{i}h^{1/2}e_{EA'}{}^{j}e_{B}{}^{A'i}\right]\Psi_{22}$$
$$+\frac{1}{4}\hbar^{2}\kappa^{2}e^{C}{}_{B'i}n_{C}{}^{A'}e_{B}{}^{B'j}e_{AA'm}\frac{\delta\Psi_{21}}{\delta h_{im}}-2\hbar^{2}\kappa^{2}\left[-\frac{2}{3}\delta_{i}{}^{j}n_{B}{}^{A'}+\frac{1}{3}e_{B}{}^{C'}{}_{i}n^{CA'}e_{CC'}{}^{j}\right.$$
$$\left.-\frac{1}{6}n_{C}{}^{B'}e^{CA'}{}_{i}e_{BB'}{}^{j}-\frac{1}{6}n_{B}{}^{B'}e^{CA'}{}_{i}e_{CB'}{}^{j}\right]e_{AA'm}\frac{\delta\Psi_{22}}{\delta h_{im}}=0 . \quad (2.18)$$

One replaces the free spinor indices AB by the spatial index n on multiplying by $n^{A}{}_{D'}e^{BD'n}$, giving

$$-\frac{1}{2}gh^{1/2}h^{jn}\Psi_{0}+\frac{1}{8}i\hbar h^{1/2}(h^{ij}\omega_{ABi}n^{A}{}_{B'}e^{BB'n}-h^{in}\omega_{ABi}n^{A}{}_{B'}e^{BB'j}+h^{jn}\omega_{ABi}n^{A}{}_{B'}e^{BB'i})\Psi_{21}$$
$$+\frac{1}{3}i\hbar h^{1/2}(2h^{ij}\omega_{ABi}n^{A}{}_{B'}e^{BB'n}+h^{in}\omega_{ABi}n^{A}{}_{B'}e^{BB'j}-h^{jn}\omega_{ABi}n^{A}{}_{B'}e^{BB'i})\Psi_{22}$$
$$+\frac{1}{16}\hbar^{2}\kappa^{2}(\delta_{i}{}^{j}\delta_{m}{}^{n}-\delta_{i}{}^{n}\delta_{m}{}^{j}+h_{im}h^{nj})\frac{\delta\Psi_{21}}{\delta h_{im}}-\frac{1}{3}\hbar^{2}\kappa^{2}(2\delta_{i}{}^{j}\delta_{m}{}^{n}+\delta_{i}{}^{n}\delta_{m}{}^{j}-h_{im}h^{jn})\frac{\delta\Psi_{22}}{\delta h_{im}}=0 . \quad (2.19)$$

Multiplying by different choices $\delta h_{im}=(\partial h_{im}/\partial A)\delta A$, etc., and integrating over the manifold, one finds the constraints

$$\frac{1}{16}\hbar^{2}\kappa^{2}A^{-1}\left[A\frac{\partial\Psi_{21}}{\partial A}+B\frac{\partial\Psi_{21}}{\partial B}+C\frac{\partial\Psi_{21}}{\partial C}\right]-\frac{1}{3}\hbar\kappa^{2}\left[3\frac{\partial\Psi_{22}}{\partial A}-A^{-1}\left[A\frac{\partial\Psi_{22}}{\partial A}+B\frac{\partial\Psi_{22}}{\partial B}+C\frac{\partial\Psi_{22}}{\partial C}\right]\right]$$
$$-16\pi^{2}gBC\Psi_{0}-\pi^{2}\hbar BC\left[\frac{A}{BC}+\frac{B}{CA}+\frac{C}{AB}\right]\Psi_{21}+\frac{1}{3}(16\pi^{2})\hbar BC\left[\frac{2A}{BC}-\frac{B}{CA}-\frac{C}{AB}\right]\Psi_{22}=0 , \quad (2.20)$$

and two more equations given by cyclic permutation of ABC.

Now consider the $\bar{S}_{A'}\Psi=0$ constraint at order ψ^3. It will turn out that we need go no further than this. The constraint can be written as

$$\frac{1}{2}\hbar g h^{1/2}e^{B}{}_{A'j}n_{C}{}^{B'}e_{BB'}{}^{j}\beta_{C}(\gamma_{DEF}\gamma^{DEF})\Psi_{41}+\epsilon^{ijk}e_{AA'i}\omega^{A}{}_{Bj}\psi^{B}{}_{k}[(\beta_{C}\beta^{C})\Psi_{21}+(\gamma_{CDE}\gamma^{CDE})\Psi_{22}]$$
$$-\frac{1}{2}\hbar^{2}\kappa^{2}\psi^{A}{}_{i}\left[(\beta_{C}\beta^{C})\frac{\delta\Psi_{21}}{\delta e^{AA'}{}_{i}}+(\gamma_{CDE}\gamma^{CDE})\frac{\delta\Psi_{22}}{\delta e^{AA'}{}_{i}}\right]=0 . \quad (2.21)$$

The terms $\psi^{B}{}_{k}$ and $\psi^{A}{}_{i}$ in the last two lines can be rewritten in terms of β_A and γ_{FGH}, using Eq. (2.8). Then one can set separately to zero the coefficient of β^C ($\gamma_{DEF}\gamma^{DEF}$), the symmetrized coefficient of γ_{DEF} ($\beta_C\beta^C$), and the symmetrized coefficient of γ_{FGH} ($\gamma_{CDE}\gamma^{CDE}$). These three equations give

$$\frac{3}{4}\hbar g h^{1/2} n^C{}_{A'}\Psi_{41} - \frac{8}{3}\epsilon^{ijk} e_{AA'i}\omega^A{}_{Bj} n^B{}_{C'} e^{CC'}{}_k \Psi_{22} + \frac{4}{3}\hbar\kappa^2 n^A{}_{B'} e^{CB'}{}_i \frac{\delta\Psi_{22}}{\delta e^{AA'}{}_i} = 0 , \quad (2.22)$$

$$2\epsilon^{ijk} e_{AA'i}\omega^A{}_{Bj} n^D{}_{B'} e^{CB'}{}_k \Psi_{21} - \hbar\kappa^2 n^D{}_{B'} e^{CB'}{}_i \frac{\delta\Psi_{21}}{\delta e^{BA'}{}_i} + (BCD \to CDB) + (BCD \to DBC) = 0 , \quad (2.23)$$

and Eq. (2.23) with Ψ_{21} replaced by Ψ_{22}. Contracting Eq. (2.22) with $n_C{}^{A'}$ and integrating over the three-surface gives

$$\frac{3}{4}(16\pi^2)\hbar g ABC\Psi_{41} + \frac{2}{3}(16\pi^2)(A^2+B^2+C^2)\Psi_{22} + \frac{2}{3}\hbar\kappa^2 \left[A\frac{\partial\Psi_{22}}{\partial A} + B\frac{\partial\Psi_{22}}{\partial B} + C\frac{\partial\Psi_{22}}{\partial C} \right] = 0 . \quad (2.24)$$

Contracting Eq. (2.23) with $e^{BA'l} n_{CC'} e_D{}^{C'n}$, multiplying by $\delta h_{ln} = (\partial h_{ln}/\partial A)\delta A$ and integrating gives

$$3\hbar\kappa^2 \frac{\partial\Psi_{21}}{\partial A} - \hbar\kappa^2 A^{-1} \left[A\frac{\partial\Psi_{21}}{\partial A} + B\frac{\partial\Psi_{21}}{\partial B} + C\frac{\partial\Psi_{21}}{\partial C} \right] - 16\pi^2 BC \left[\frac{C}{AB} + \frac{B}{CA} - 2\frac{A}{BC} \right] \Psi_{21} = 0 , \quad (2.25)$$

and two more equations given by permuting ABC cyclically. The equation (2.25) also holds with Ψ_{21} replaced by Ψ_{22}.

There is a duality between wave functions $\Psi(e^{AA'}{}_i, \psi^A{}_i)$ and wave functions $\tilde{\Psi}(e^{AA'}{}_i, \tilde{\psi}^{A'}{}_i)$, given by a fermionic Fourier transform [13]. The S_A and $\bar{S}_{A'}$ operators interchange roles under this transformation, and the roles of Ψ_0 and Ψ_6, Ψ_{21} and Ψ_{42}, and Ψ_{22} and Ψ_{41} are interchanged. We shall proceed by showing that Ψ_{22}, Ψ_{21}, and Ψ_0 must vanish for $g \neq 0$ (or $\Lambda \neq 0$), and hence by the duality the entire wave function must be zero.

Consider first the equation (2.25) and its permutations for Ψ_{21} and Ψ_{22}. One can check that these are equivalent to

$$\hbar\kappa^2 \left[A\frac{\partial\Psi_{21}}{\partial A} - B\frac{\partial\Psi_{21}}{\partial B} \right] = 16\pi^2(B^2 - A^2)\Psi_{21} \quad (2.26)$$

and cyclic permutations. One can then integrate Eq. (2.26) along a characteristic $AB = \text{const}$, $C = \text{const}$, using the parametric description $A = w_1 e^{\tau}, B = w_2 e^{-\tau}$, to obtain

$$\Psi_{21} = h_1(AB, C)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}(A^2+B^2) \right] . \quad (2.27)$$

The general solution of

$$\hbar\kappa^2 \left[B\frac{\partial\Psi_{21}}{\partial B} - C\frac{\partial\Psi_{21}}{\partial C} \right] = 16\pi^2(C^2 - B^2)\Psi_{21} \quad (2.28)$$

is similarly

$$\Psi_{21} = h_2(BC, A)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}(B^2+C^2) \right] . \quad (2.29)$$

Equations (2.27) and (2.29) are only consistent if Ψ_{21} has the form

$$\Psi_{21} = F(ABC)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}(A^2+B^2+C^2) \right] . \quad (2.30)$$

Similarly

$$\Psi_{22} = G(ABC)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}(A^2+B^2+C^2) \right] . \quad (2.31)$$

Substituting Eqs. (2.30) and (2.31) into Eq. (2.20), one obtains

$$\begin{aligned} 16\pi^2 g\Psi_0 = &-2\pi^2\hbar(ABC)^{-1}(A^2+B^2+C^2)(\exp)F \\ &+ \frac{3}{16}\hbar^2\kappa^2(\exp)F' \\ &+ \frac{2}{3}(16\pi^2)\hbar(ABC)^{-1}(2A^2-B^2-C^2)(\exp)G \end{aligned} \quad (2.32)$$

and cyclically, where

$$\exp = \exp\left[-\frac{8\pi^2}{\hbar^2\kappa^2}(A^2+B^2+C^2) \right] . \quad (2.33)$$

Now Ψ_0 should be invariant under permutations of A, B, C. Hence $G = 0$, i.e.,

$$\Psi_{22} = 0 . \quad (2.34)$$

The equation (2.32) and its cyclic permutations, with $\Psi_{22} = 0$, must be solved consistently with Eq. (2.17) and its cyclic permutations. Eliminating Ψ_0, one finds

$$\begin{aligned} &\frac{3\hbar^3\kappa^4}{16(16\pi^2 g)}F'' - \frac{\hbar^2\kappa^2}{8g}\frac{A^2+B^2+C^2}{ABC}F' + 6\pi^2\hbar g F \\ &\quad - \frac{\hbar^2\kappa^2}{4g}\frac{1}{B^2C^2}F + \frac{\hbar^2\kappa^2}{8g}\frac{A^2+B^2+C^2}{(ABC)^2}F = 0 , \end{aligned} \quad (2.35)$$

and cyclic permutations. Since $F = F(ABC)$ is invariant under permutations, the $(BC)^{-2}F$ term and its permutations imply $F = 0$. Thus

$$\Psi_{21} = 0 . \quad (2.36)$$

Hence, using Eq. (2.32),

$$\Psi_0 = 0 .$$

Then we can argue using the duality mentioned earlier, to conclude that

$$\Psi_{41}=\Psi_{42}=\Psi_6=0 \ . \tag{2.37}$$

Hence there are no physical quantum states obeying the constraint equations in the diagonal Bianchi type-IX model. This result will be discussed further in the Conclusion.

This shows that the Chern-Simons semiclassical wave function of Sano and Shiraishi [21] for $N=1$ supergravity with a Λ term can only be an approximate, and not an exact state in the Bianchi type-IX model. If it were exact, then one could make a Fourier transformation from the Ashtekar variables used in [21] to the variables A,B,C used here, to find a nontrivial solution.

III. THE $k=+1$ FRIEDMANN MODEL WITH A Λ TERM

The $k=+1$ Friedmann model without a Λ term has been discussed in [2,6]. There are two linearly independent physical quantum states. One is bosonic and corresponds to the wormhole state [15]; the other is at quadratic order in fermions and corresponds to the Hartle-Hawking state [16]. In the Friedmann model with a Λ term, the coupling between the different fermionic levels "mixes up" this pattern [4].

In the Friedmann model, the wave function has the form [6]

$$\Psi=\Psi_0(A)+(\beta_C\beta^C)\Psi_2(A) \ . \tag{3.1}$$

As part of the ansatz of [6], one requires $\psi^A{}_i=e^{AA'}{}_i\tilde{\psi}_{A'}$ and $\tilde{\psi}^{A'}{}_i=e^{AA'}{}_i\psi_A$; this is in order that the form of the one-dimensional ansatz should be preserved under one-dimensional local supersymmetry, suitably modified by local coordinate and Lorentz transformations. Thus the gravitino field is truncated to spin $\frac{1}{2}$. Note that $\beta^A=\frac{3}{4}n^{AA'}\tilde{\psi}_{A'}$.

One then proceeds as in Sec. II to derive the consequences of the $\bar{S}_{A'}\Psi=0$ and $S_A\Psi=0$ constrints at level ψ^1, by writing down the general expression for a constraint and then evaluating it at a Friedmann geometry. Note that it is not equivalent to set $A=B=C$ in Eqs. (2.17) and (2.20); the coefficients in the constraint equations are different. One then obtains

$$\hbar\kappa^2\frac{d\Psi_0}{dA}+48\pi^2A\Psi_0+18\pi^2\hbar g A^2\Psi_2=0 \tag{3.2}$$

and

$$\hbar^2\kappa^2\frac{d\Psi_2}{dA}-48\pi^2\hbar A\Psi_2-256\pi^2 g A^2\Psi_0=0 \ . \tag{3.3}$$

These give second-order equations: for example,

$$A\frac{d^2\Psi_0}{dA^2}-2\frac{d\Psi_0}{dA}+\left[-\frac{48\pi^2}{\hbar\kappa^2}A-\frac{(48)^2\pi^4}{\hbar^2\kappa^4}A^3+\frac{9\times 512\pi^4 g^2}{\hbar^2\kappa^4}A^5\right]\Psi_0=0 \ . \tag{3.4}$$

This has a regular singular point at $A=0$, with indices $\lambda=0$ and 3. There are two independent solutions, of the form

$$\Psi_0=a_0+a_2A^2+a_4A^4+\cdots \ , \tag{3.5}$$

$$\Psi_0=A^3(b_0+b_2A^2+b_4A^4+\cdots) \ ,$$

convergent for all A. They obey complicated recurrence relations, where (e.g.) a_6 is related to a_4, a_2, and a_0.

One can look for asymptotic solutions of the type $\Psi_0\sim(B_0+\hbar B_1+\hbar^2B_2+\cdots)\exp(-I/\hbar)$, and finds

$$I=\pm\frac{\pi^2}{g^2}(1-2g^2A^2)^{3/2} \ , \tag{3.6}$$

for $2g^2A^2<1$. The minus sign in I corresponds to taking the action of the classical Riemannian solution filling in smoothly inside the three-sphere, namely, a portion of the four-sphere S^4 of constant positive curvature. This gives the Hartle-Hawking state [16]. For $A^2>(1/2g^2)$, the Riemannian solution joins onto the Lorentzian solution [22]

$$\Psi\sim\cos\left\{\hbar^{-1}\left[\frac{\pi^2(2g^2A^2-1)^{3/2}}{g^2}-\frac{\pi}{4}\right]\right\} \ , \tag{3.7}$$

which describes de Sitter space-time.

IV. CONCLUSION

We have seen here that there are no physical quantum states for $N=1$ supergravity with a Λ term, in the diagonal Bianchi type-IX model. The same result was found for nondiagonal Bianchi type-I models in [11]. The physical states found in Sec. III for the $k=+1$ Friedmann model, where the degrees of freedom carried by the gravitino field are β_A, disappear when the further fermionic degrees of freedom γ_{ABC} of the Bianchi type-IX model are included.

One could also study this from the point of view of perturbation theory about the $k=+1$ Friedmann model. As well as the usual gravitational harmonics [23], gravitino harmonics can be used [24]. For example, the Bianchi type-IX model with radii A,B,C close together describes a particular type of "gravitational wave" distortion of the Friedmann model; similarly for the γ_{ABC} of the Bianchi type-IX model, which describes a particular "gravitino wave" distortion. Quite generally, in perturbation theory [23,25] one expects to find a wave function which is a product of the background wave function $\Psi(A)$ times an infinite product of wave functions ψ_n (perturbations) where n labels the harmonics. And one further expects that the perturbation wave function corresponding to the Bianchi type-IX modes must be zero, by a perturbative version of the argument of Sec. II. (It will be interesting to investigate this.) Hence the complete perturbative wave function should be zero; then physical states would be forbidden for a generic model of the gravitational and gravitino fields with a Λ term. This suggests that the full theory of $N=1$ supergravity with a nonzero Λ term should have no physical states.

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