

Quantum dynamics of Lorentzian spacetime foam

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A simple spacetime wormhole, which evolves classically from zero throat radius to a maximum value and recontracts, can be regarded as one possible mode of fluctuation in the microscopic “spacetime foam” first suggested by Wheeler. The dynamics of a particularly simple version of such a wormhole can be reduced to that of a single quantity, its throat radius; this wormhole thus provides a “minisuperspace model” for a mode of Lorentzian-signature foam. The classical equation of motion for the wormhole throat is obtained from the Einstein field equations and a suitable equation of state for the matter at the throat. Analysis of the quantum behavior of the hole then proceeds from an action corresponding to that equation of motion. The action obtained simply by calculating the scalar curvature of the hole spacetime yields a model with features like those of the relativistic free particle. In particular the Hamiltonian is nonlocal, and for the wormhole cannot even be given as a differential operator in closed form. Nonetheless the general solution of the Schrödinger equation for wormhole wave functions, i.e., the wave-function propagator, can be expressed as a path integral. Too complicated to perform exactly, this can yet be evaluated via a WKB approximation. The result indicates that the wormhole, classically stable, is quantum-mechanically unstable: A Feynman-Kac decomposition of the WKB propagator yields no spectrum of bound states. Although an initially localized wormhole wave function may oscillate for many classical expansion and recontraction periods, it must eventually leak to large radius values. The possibility of such a mode unstable against growth, combined with the observed absence of macroscopic wormholes, suggests that stability considerations may place constraints on the nature or even the existence of Planck-scale foamlike structure, at least of Lorentzian signature.

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I. INTRODUCTION

A quantum description of gravitation is one of the most eagerly sought goals of present-day physics. Approaches to such a description may be loosely classified as “grand schemes” and “small schemes.” The former are attempts at a comprehensive quantum theory of gravitation, such as supergravity or superstring theory, canonical quantization of general relativity with or without new variables, etc. The latter are attempts to analyze the quantum-gravitational physics of particular, simple systems, in hopes of understanding features expected to emerge from a complete, general theory. These small schemes include the extensive “minisuperspace” quantum-cosmology program [1], the numerous calculations of the effects of Planck-scale spacetime wormholes and “baby universes” on the constants of macroscopic physics [2], and the present work, in which we seek to probe the dynamics of such a Planck-scale “foam” structure itself. An outline of this investigation has been published previously [3]; here we present the calculations and results in detail.

The concept of “spacetime foam,” first suggested by Wheeler [4] some 35 years ago, has become standard lore in our quantum picture of gravitation: On scales of the Planck length, quantum fluctuations of the spacetime geometry become so dominant that spacetime takes on all manner of nontrivial topological structure, such as wormholes and handles; smooth, simply connected spacetime only emerges as a classical limit on larger scales. Yet in the intervening decades our understanding of quantum gravity still has not advanced sufficiently to prove or disprove this famous conjecture.

The difficulties inherent in such a proof, or in any detailed treatment of spacetime foam, are well known. Topological change in a Lorentzian manifold necessarily entails the complications of singularities or degeneracy of the geometry, or of closed timelike curves [5]. To circumvent these problems one may consider Euclidean manifolds instead [1]. Euclidean quantum gravity is physically distinct from Lorentzian, unlike ordinary quantum field theory in which the two formulations are equivalent via analytic continuation. Indeed on a quantum level it might be that spacetime is fundamentally Euclidean, with familiar Lorentzian spacetime only a classical limit attained far from the Planck regime. But Euclidean quantum gravity has its own formidable problems [6], including the failure of the Euclidean action to be positive definite or bounded below, the problem of inter-

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pretation, and the matter of recovering Lorentzian spacetime.

Both versions of spacetime foam continue to be subjects of great interest. Much recent work on Euclidean foam has explored its possible role in determining the fundamental constants of nature [2]. The suggestion that Lorentzian wormholes, extracted from microscopic foam and suitably enlarged, might conceivably be used for space or time travel [7] has inspired a great deal of recent and ongoing work on issues of causality and consistency in spacetime physics. Both sets of ideas have profound significance for fundamental physics, but both regard the spacetime foam itself as a given, neither probing the questions of its existence or structure substantially.

This work addresses the dynamics of Lorentzian spacetime foam, and the implications of that dynamics for its existence and nature. We begin by envisioning Lorentzian spacetime, filled on Planck scales with all manner of wormholes and other structures—modes of topological fluctuation—continually winking into existence, persisting for microscopic time periods, and pinching off. The moments of creation and disappearance of these structures, i.e., the actual points of topological change, may or may not require a Euclideanized treatment; our analysis does not attempt to deal with these points.

Classical wormhole geometries can be used to model such a picture. In general such wormholes, persisting for arbitrary lengths of time (in particular, sufficient to allow passage of matter or energy), must involve stress-energy distributions which violate the weak energy condition—negative energy density in some reference frames must appear somewhere in the wormholes' throats. Whether this suffices to rule out wormholes on macroscopic scales is unknown at present, but it is not expected to be a problem for microscopic structure, where quantum effects could readily give rise to such stress-energy [7,8]. Moreover, it does not appear to present a barrier to the growth of wormholes from Planck scales to large sizes, as the present work indicates. Indeed, classical wormhole models which grow from microscopic to macroscopic sizes via inflation have been constructed by Roman [9].

To make the analysis tractable we employ a brutally simplified wormhole model to represent one mode of spacetime-foam fluctuation. A class of wormholes can be constructed by excising the world tube of some surface from a $(3+1)$ -dimensional spacetime region and joining this to another such region, with like excision, at that surface [10]. The join, the throat of the resulting wormhole, contains a surface layer of stress-energy specified by the Einstein field equations expressed as junction conditions [11]. Here we take the external regions of such a wormhole to be flat, empty Minkowski space, the throat a sphere of time-varying radius [10]. Our model is thus an extreme version of a wormhole in which the spacetime curvature in the throat is much greater than that in the regions surrounding the mouths. Our simplifications reduce the geometric degrees of freedom of the spacetime to just one, to throat radius. Thus, our analysis is a minisuperspace model of a wormhole, analogous to the minisuperspace models used in quantum cosmology [1]. In fact, our model is simplified even fur-

ther than those: We treat the matter providing the stress energy on the wormhole throat via an equation of state, rather than as a separate dynamical field. The quantum-gravitational dynamics of the complete wormhole system becomes then the quantum mechanics of one variable, the throat radius. The result we seek is the evolution, in time as defined in the wormhole's flat exterior regions, of a wave function for that radius.

In the absence of a general theory of quantum gravity, there is no prescription from first principles for quantizing a system like this wormhole. The existence of an external time, in terms of which the evolution of the system is naturally to be described, makes unsuitable the usual time-independent Wheeler-deWitt equation [12]. Instead we choose to "solve the constraint" explicitly, i.e., to impose the Hamiltonian constraint classically, to reduce the phase space of the system. Combining the classical junction conditions at the wormhole throat—consisting of the Hamiltonian constraint plus a dynamical equation—with an equation of state for the matter on the throat yields an equation of motion for the throat radius alone. We then construct an action for the dynamics in the reduced phase space, i.e., depending only on the radius and its time derivatives, for which that equation of motion is the Euler-Lagrange equation. This can be done in a variety of ways, amenable to various forms of quantization. One such action, associated with gravitational dynamics in a straightforward way, is obtained from the scalar curvature of the wormhole spacetime. The wormhole quantum mechanics described by this action is treated in detail in this paper. Alternative actions and their implications for wormhole dynamics are examined in a subsequent paper [13].

The wormhole quantum mechanics which follows from the "curvature action" is in some respects similar to the Newton-Wigner first-quantized description of a relativistic free particle [14,15]. For both the Hamiltonian is nonpolynomial in the momentum, i.e., is a nonlocal operator in the coordinate representation. Indeed, for the wormhole the Hamiltonian is a rational function of "velocity," which is related to the canonical momentum via a transcendental equation which cannot be inverted in closed form. This unconventional form of the Hamiltonian makes it impossible to deduce the behavior of the system merely by examining the "potential" term; explicit solution for the evolution of the wave function is needed. Moreover, the corresponding Schrödinger equation for wormhole wave functions cannot be written explicitly as a differential equation. It can, however, be solved, at least formally: The wave-function propagator is obtained by using the action in a Feynman path integral. Exact evaluation of this integral is problematic; the action is complicated, and questions of the appropriate measure and skeletonization (factor-ordering problems), as well as the issue of the correct class of paths over which to integrate, arise. But the integral can be evaluated in a WKB approximation, as a sum of contributions from classical paths and small fluctuations about them. The measure and factor-ordering problems are thus avoided. It is crucial, though, to include all the appropriate paths. The analogy to the relativistic particle suggests that

spacelike as well as timelike and lightlike paths must be included in the integral [15,16]. Also, since negative throat radii are undefined, a suitable boundary condition must be imposed on the wave functions, i.e., on the propagator, at zero radius. Zero radius acts as a reflecting barrier, giving rise to classical paths—extrema of the action—which “bounce” there one or more times. (Such a boundary condition incorporates our neglect of any topological change at zero radius.) The WKB version of the propagator, then, consists of a sum of several terms, contributions of paths with different numbers of bounces; cancellation of terms effects the boundary condition at zero, just as for a particle in a half space. Explicit expressions for all the terms are obtained here. With these the evolution of a wave function is calculated via a convolution integral, evaluated numerically.

The results suggest that even with a matter equation of state chosen so that the classical evolution of the throat radius is bounded, the wormhole is quantum-mechanically unstable. This is not unprecedented—classically stable black holes, for example, are unstable to Hawking evaporation. In this case a Feynman-Kac decomposition of the wormhole propagator yields no spectrum of bound states. The calculated evolution of an initially localized wave function follows a classical trajectory for many expansion-recollapse-bounce cycles, but the wave function must eventually leak to arbitrarily large throat-radius values. Its behavior thus resembles that for a particle initially confined to a well with finite, though perhaps high and wide, walls, or an α particle in a heavy nucleus. This simple wormhole geometry appears to represent a mode of spacetime-foam structure unstable against growth to macroscopic size.

Though subject to the limitations of our model, viz., the simplifications and approximations we have applied, and the uncertainties associated with our quantization scheme, such a conclusion is potentially very significant. It indicates that Planck-scale gravitational physics might be constrained by observations on much larger scales: The apparent absence of any topological structure to spacetime on scales accessible to observations, ranging from particle physics to astronomy, could imply either that not all modes of topological fluctuation are possible or that microscopic (Lorentzian) spacetime foam does not exist altogether.

The wormhole model and its associated classical and quantum dynamics are presented in Sec. II below. Evaluation of the wormhole propagator in the WKB approximation is shown in detail in Sec. III. The implications of the result for wormhole wave-function evolution are discussed in Sec. IV. Conclusions and caveats follow in Sec. V. Units with $G=c=\hbar=1$ are used throughout. Sign conventions and general notation follow those of Misner, Thorne, and Wheeler [17].

II. MECHANICS OF THE WORMHOLE MODEL

A. Classical dynamics

A spherically symmetric “Minkowski wormhole” [10] provides a very simple model of a mode of topological

fluctuation in Lorentzian-signature spacetime foam. The classical geometry of such a wormhole is constructed by removing a ball of time-dependent radius $r=R(t)$ from two Minkowski-space regions and identifying the two boundary world tubes, making them the throat of the wormhole. The Einstein field equations are satisfied trivially in the flat, empty exterior regions. At the throat those equations take the form of the junction conditions [11]

$$S_j^i = \frac{1}{8\pi} [K_j^i - \delta_j^i K_m^m], \quad (2.1)$$

where S_j^i is the stress-energy tensor of a surface layer on the throat, and the right-hand side is the jump discontinuity in the throat extrinsic curvature K_j^i , minus its trace K_m^m . Here the nontrivial components of these equations are

$$S_{\lambda\lambda} = -\frac{1}{2\pi R} \frac{1}{(1-\dot{R}^2)^{1/2}} \quad (2.2a)$$

and

$$S_{\theta\theta} = \frac{1}{4\pi} \left[\frac{R}{(1-\dot{R}^2)^{1/2}} + \frac{R^2 \ddot{R}}{(1-\dot{R}^2)^{3/2}} \right], \quad (2.2b)$$

overdots denoting derivatives with respect to Minkowski-coordinate time t (in a frame “centered on the hole,” i.e., in which spherical symmetry is maintained). The throat coordinates here are proper time λ and polar angle θ ; proper and coordinate time are related via $d\lambda = (1-\dot{R}^2)^{1/2} dt$. Once a surface-layer equation of state, relating surface density $\sigma = S_{\lambda\lambda}$ and pressure $p = S_{\theta\theta}/R^2$, is specified, Eqs. (2.2) give the classical equation of motion for the wormhole. As is to be expected [7,8], Eq. (2.2a) indicates that negative energy densities appear.

For simplicity we specify the equation of state as a property of the model, rather than derive it from some more fundamental description of the surface layer. It could be chosen to make the equation of motion extremely simple. The choice $p = -\sigma/2$, for example, would imply $\ddot{R} = 0$. The dynamics of such a model is trivial. Classically the wormhole throat either sits at fixed radius or expands or contracts linearly. Its quantum behavior is the same as that of a free particle; a wave function initially concentrated about some R value will disperse to infinity. But this is not a model suitable for describing structures in spacetime foam, fluctuating into and out of existence. Rather we seek an equation of state such that the equation of motion describes expansion from zero radius to some maximum value and recollapse. The simple “dust” equation of state, $p=0$, implies the equation of motion $R\ddot{R} - \dot{R}^2 + 1 = 0$; the solutions of this are sine functions, with the desired behavior. This equation, however, is not suited to the action construction we use below. (It can be treated by other methods [13].) Instead we choose

$$p = -\sigma/4, \quad (2.3)$$

which yields the equation of motion

$$2R\ddot{R} - \dot{R}^2 + 1 = 0. \quad (2.4)$$

The corresponding classical trajectories are parabolic:

$$R_{\text{cl}}(t) = \frac{1}{\alpha} \left[1 - \frac{\alpha^2(t - t_0)^2}{4} \right], \quad (2.5)$$

where α and t_0 are constants. These solutions likewise behave as desired.

Thus, by restricting consideration to Minkowski-wormhole geometries, and combining the classical initial-value (Hamiltonian) constraint (2.2a) with the dynamical equation (2.2b) and the equation of state (2.3), we reduce the infinity of gravitational and matter degrees of freedom of the wormhole spacetime to one, the throat radius R . This is thus an extreme form of minisuperspace model.

The equation of motion (2.4) can be obtained as the Euler-Lagrange equation for extrema of the action

$$S = \int \left[R\dot{R} \ln \left| \frac{1+\dot{R}}{1-\dot{R}} \right| - 2R \right] dt. \quad (2.6)$$

This choice of reduced-phase-space action is motivated by the fact that it coincides with the Einstein-Hilbert action for the wormhole spacetime. The wormhole's scalar curvature \mathcal{R} is nonzero only on the throat:

$$\begin{aligned} \mathcal{R} &= 2[K_m^m]\delta(\rho) \\ &= \frac{4}{R^2(1-\dot{R}^2)^{1/2}} \left[R\dot{R} \ln \left| \frac{1+\dot{R}}{1-\dot{R}} \right| - 2R \right. \\ &\quad \left. - \frac{1}{2} \frac{d}{dt} \left[R^2 \ln \left| \frac{1+\dot{R}}{1-\dot{R}} \right| \right] \right] \delta(\rho), \end{aligned} \quad (2.7)$$

where ρ is proper distance normal to the throat, positive in one exterior region and negative in the other. The corresponding gravitational action—the integral of \mathcal{R} plus a surface term [18] eliminating the total-derivative term—is just Eq. (2.6).

The Hamiltonian dynamics of the wormhole, on its reduced phase space, follows immediately from the action. With Lagrangian L given by the integrand in Eq. (2.6), the canonical momentum is

$$P = \frac{\partial L}{\partial \dot{R}} = R \left[\ln \left| \frac{1+\dot{R}}{1-\dot{R}} \right| + \frac{2\dot{R}}{1-\dot{R}^2} \right]. \quad (2.8)$$

The Hamiltonian is

$$H = P\dot{R} - L = \frac{2R}{1-\dot{R}^2}. \quad (2.9)$$

The Hamiltonian cannot be expressed in terms of R and P in closed form, since the transcendental relation (2.8) between \dot{R} and P/R cannot be inverted explicitly.

It should be noted that the “energy” corresponding to this Hamiltonian is not to be identified with the Arnowitt-Deser-Misner (ADM) mass of the wormhole, which is zero by construction. To thus fix the “energy” would reduce the equation of motion to first order, and the classical initial-parameter space to one dimension,

making canonical quantization impossible. In fact, the Hamiltonian (2.9) corresponds to the classically conserved “mass” $2(2\pi R^{3/2}\sigma)^2$ appropriate to the equation of state (2.3). [The Einstein field equations imply the conservation law $d(4\pi R^2\sigma)/dt + pd(4\pi R^2)/dt = 0$; Eq. (2.3) then implies that $R^{3/2}\sigma$, hence H , are classically conserved.] This can take on various values for different wormhole states. In this respect our treatment contrasts with the analyses of Hájíček *et al.* [19] of the quantum mechanics of a collapsing dust shell. Those authors take the Hamiltonian to correspond to the ADM mass of the system, which may take on different values in different states. They treat the classically conserved proper mass of the shell as a fixed parameter.

Of course the wormhole action is not uniquely defined by the equation of motion specifying its extrema. A variety of actions, of other than Einstein-Hilbert form, can be constructed corresponding to Eq. (2.4)—or to equations obtained from other equations of state. Actions can be obtained which avoid some of the complexities encountered with the geometric action (2.6), e.g., the complicated kinetic term, and the transcendental relation (2.8). The construction of such actions and their implications for wormhole quantum mechanics are examined in another paper [13]. The rest of this paper treats the quantum dynamics obtained from the geometric action.

B. Quantum dynamics

In this minisuperspace model, then, the quantum wormhole is to be described by a wave function $\psi(R, t)$. With the Hamiltonian constraint (2.2a) imposed classically, i.e., incorporated into the reduction of the wormhole's phase space, no Wheeler-DeWitt equation imposes “time independence” of ψ [12]; instead its evolution in wormhole-exterior time t is generated by the Hamiltonian (2.9).

There is no hope of treating the corresponding Schrödinger equation directly: The transcendental nature of relation (2.8) and the form of H imply that the Hamiltonian contains all powers of the momentum P , or as an operator on ψ , of $\partial/\partial R$. It is therefore a nonlocal operator, similar to that for a relativistic particle, but rather more unwieldy [15]. Operator ordering in H is problematic as well. But the *solution* of the Schrödinger equation can be treated. The general solution, i.e., the time evolution of any wave function, can be given in terms of a propagator:

$$\psi(R, t) = \int G[R, t; R_0, 0] \psi_0(R_0) dR_0. \quad (2.10)$$

The propagator G is given formally by a Feynman path integral

$$G[R, t; R_0, 0] = \int_C e^{iS[R(t)]} \mathcal{D}[R(t)], \quad (2.11)$$

with C denoting the class of paths over which the integral is defined.

The appropriate class of paths is an important issue. The example of the relativistic particle [15], and more general quantum-gravitational considerations [16], suggest that all paths linking the initial and final points

should be included—spacelike as well as timelike and null. [Spacelike paths, with $|\dot{R}| > 1$, correspond to “negative-energy” contributions, in terms of the Hamiltonian (2.9).] Including spacelike paths implies that G need not vanish outside the “light cone” $|R - R_0| = t$, i.e., that it may admit “acausal propagation.”

Moreover, the wormhole geometry is only defined for non-negative R , hence only paths with $R(t) \geq 0$ should be included. This restriction can be implemented as for a particle confined to a half space, i.e., as if there were an infinite potential wall at $R = 0$. This implies the boundary condition $\psi(0, t) = 0$, hence that $G[R, t; R_0, 0]$ should vanish for R or R_0 zero. (More precisely, ψ must entail no current in the $-R$ direction at $R = 0$, so boundary conditions more general than this are possible. For simplicity we do not treat such here.) By imposing this condition we exclude consideration of topology-changing processes, i.e., wormhole creation or disappearance, at $R = 0$. The treatment of such processes might require a “second quantized” framework (actually “third quantized,” outside the restrictions of a minisuperspace model), rather than the “first quantized” (actually “second quantized”) description used here. Their bearing on our results is discussed further in Sec. V. Here the boundary condition on ψ and G gives the point $R = 0$ the role of a reflecting boundary for wormhole wave functions.

III. EVALUATION OF THE PROPAGATOR: WKB APPROXIMATION

Exact evaluation of the path integral (2.11) is out of reach: The integral, with action (2.6), is non-Gaussian; the operator-ordering problems involved in canonical quantization with the Hamiltonian (2.9) would reappear as ambiguities in the skeletonization of the path integral; the appropriate measure on the space of paths is unknown. But, over a suitable range of time intervals and throat-radius values, it should be possible to calculate the propagator via a WKB approximation. Then the path integral is taken to be dominated by the contributions of classical trajectories—local extrema of the action—and nearby paths. The approximate propagator takes the form

$$G^{(\text{WKB})}[R, t; R_0, 0] = \sum_{\text{classical paths}} \left[\frac{i}{2\pi} \frac{\partial^2 S[R_{\text{cl}}]}{\partial R \partial R_0} \right]^{1/2} e^{iS[R_{\text{cl}}]}. \quad (3.1)$$

The exponential factors arise from the classical paths R_{cl} between the initial and final values. The prefactors come from Gaussian integrals over small deviations from these, giving semiclassical corrections [20]. To that order $G^{(\text{WKB})}$ is independent of the choice of path-integral measure. Specifying the regime in which this approximation is accurate is problematic, since we have neither an exact solution for comparison nor even an explicit form of the Schrödinger equation. This question is considered further in Sec. V below.

The first step in evaluating $G^{(\text{WKB})}$, then, is finding the appropriate classical paths. More than one type of path

always contributes. There is exactly one parabolic trajectory of form (2.5), and taking only non-negative radius values, between any initial and final points $(R_0, 0)$ and (R, t) . Specifically, this trajectory has parameters

$$\alpha = \frac{8}{t^2} \left[\left(R_0 R + \frac{t^2}{4} \right)^{1/2} - \frac{R_0 + R}{2} \right] \quad (3.2a)$$

and

$$t_0 = \frac{t}{2} \frac{(R_0 R + t^2/4)^{1/2} - R_0}{(R_0 R + t^2/4)^{1/2} - (R_0 + R)/2}, \quad (3.2b)$$

and action

$$S^{(0)} = -\frac{3t}{2\alpha} \left[1 - \frac{\alpha^2}{12} (t^2 - 3tt_0 + 3t_0^2) \right] + \frac{R^2}{2} \ln \left| \frac{2 - \alpha(t - t_0)}{2 + \alpha(t - t_0)} \right| - \frac{R_0^2}{2} \ln \left| \frac{2 + \alpha t_0}{2 - \alpha t_0} \right|. \quad (3.3)$$

Because paths are restricted to the half-space of positive radius values, there are also classical trajectories which are piecewise of form (2.5), but which “bounce” one or more times at zero radius; extremizing the action determines the bounce times. Such a path which bounces just once, at time t_1 , has action

$$S^{(1)} = -\frac{1}{4} [t_1(t_1 + 2R_0) + (t - t_1)(t - t_1 + 2R)] + \frac{1}{2} \left[R^2 \ln \left| \frac{R}{t - t_1 - R} \right| - R_0^2 \ln \left| \frac{t_1 - R_0}{R_0} \right| \right]. \quad (3.4)$$

For the bounced path to contribute to the WKB propagator (3.1), this should be extremal with respect to variation of t_1 . That condition is

$$(t - t_1)^2(t_1 - R_0) - t_1^2(t - t_1 - R) = 0. \quad (3.5)$$

Each real solution of this equation with $t_1 \in [0, t]$ corresponds to a contributing trajectory. Since the left-hand side of Eq. (3.5) changes sign between $t_1 = 0$ and $t_1 = t$ there is always one such solution, and may be three. A classical path bouncing more than once, at times t_1, \dots, t_n , has action

$$S^{(n)} = -\frac{1}{4} \left[t_1(t_1 + 2R_0) + \sum_{j=2}^n (t_j - t_{j-1})^2 + (t - t_n)(t - t_n + 2R) \right] + \frac{1}{2} \left[R^2 \ln \left| \frac{R}{t - t_n - R} \right| - R_0^2 \ln \left| \frac{t_1 - R_0}{R_0} \right| \right]. \quad (3.6)$$

That this be extremal with respect to all bounce times is equivalent to the conditions

$$(n - 1)t_1^2 - (t_1 - R_0)(t_n - t_1) = 0, \quad (3.7a)$$

$$t_j = t_1 + \frac{j-1}{n-1}(t_n - t_1), \quad 2 \leq j \leq n-1, \quad (3.7b)$$

and

$$(n-1)(t-t_n)^2 - (t_n-t_1)(t-t_n-R) = 0. \quad (3.7c)$$

Any real solution to the coupled quadratic equations (3.7a) and (3.7c) with $0 \leq t_1 < t_n \leq t$ yields an n -bounce path contributing to $G^{(\text{WKB})}$. For a given n there may be zero, two, or four such. Conditions (3.5) and (3.7a)–(3.7c) are equivalent to requiring that the parameters α for all the parabolic segments of a bouncing path be equal, which is to say that the wormhole “bounces elastically.”

For any set of initial and final coordinates $(R_0, 0)$ and (R, t) there is a maximum number of bounces n_{\max} , beyond which there are no appropriate solutions to Eqs. (3.7a) and (3.7c) and no contributions to $G^{(\text{WKB})}$. For example, in the case $R_0 = R$, the four solutions of Eqs. (3.7a) and (3.7c) are

$$t_1^{(a)} = (t - \sqrt{t^2 - 4Rnt}) / (2n), \quad (3.8a)$$

$$t_n^{(a)} = [(2n-1)t - \sqrt{t^2 - 4Rnt}] / (2n),$$

$$t_1^{(b)} = (t + \sqrt{t^2 - 4Rnt}) / (2n), \quad (3.8b)$$

$$t_n^{(b)} = [(2n-1)t + \sqrt{t^2 - 4Rnt}] / (2n),$$

$$t_1^{(c)} = (t + 2R + \sqrt{t^2 - 4Rnt + 4R^2}) / [2(n+1)],$$

$$t_n^{(c)} = [(2n+1)t - 2R - \sqrt{t^2 - 4Rnt + 4R^2}] / [2(n+1)], \quad (3.8c)$$

and

$$t_1^{(d)} = (t + 2R - \sqrt{t^2 - 4Rnt + 4R^2}) / [2(n+1)],$$

$$t_n^{(d)} = [(2n+1)t - 2R + \sqrt{t^2 - 4Rnt + 4R^2}] / [2(n+1)]. \quad (3.8d)$$

For n between 2 and $t/(4R)$, if any, all four solutions are real and in the desired interval; four n -bounce paths contribute to $G^{(\text{WKB})}$. For any n between $t/(4R)$ and $(t^2 + 4R^2)/(4Rt)$ the a and b solutions are complex, and there are no corresponding classical paths, but the c and d solutions characterize two n -bounce paths contributing to the propagator. For n greater than $(t^2 + 4R^2)/(4Rt)$ there are no n -bounce classical paths. (The values $R_0 = R = 0$, for any t , are a limiting case, in which four n -bounce paths contribute for all n greater than 1.) It must be emphasized that these particular bounds for n , and the classification of the solutions a , b , c , and d , cannot be applied for arbitrary unequal R_0 and R , but there are always some bounds on the n values of contributing paths. This is because between bounces the paths are of form (2.5), for which there is a fixed ratio between maximum radius and the interval between zero crossings. For any t value, then, there is time for only so many bounces high enough to reach a given R_0 or R value.

The evaluation of $G^{(\text{WKB})}$ proceeds from these results. Expression (3.1) eventually takes the form

$$G^{(\text{WKB})}[R, t; R_0, 0] = \sum_{n=0}^{n_{\max}(R_0, R, t)} \sum_k P_k^{(n)} \left[\frac{P_k^{(n)} F_k^{(n)}}{4\pi i} \right]^{1/2} \exp(iS_k^{(n)}), \quad (3.9)$$

with n the number of bounces, as above, and k labeling the n -bounce paths, the range of k depending of course on n . The classical actions $S_k^{(n)}$ are given by Eqs. (3.3), (3.4), and (3.6). The functional determinants $F_k^{(n)}$ are

$$F^{(0)} = \frac{32/(at)}{4 + \alpha^2 t_0(t - t_0)}, \quad (3.10a)$$

with α and t_0 from Eqs. (3.2):

$$F^{(1)} = \frac{t_1^4 / (t_1 - R_0)^2}{t^2 + 2R_0 t - 6t t_1 + 6t_1^2 + 2(R - R_0)t_1}, \quad (3.10b)$$

with t_1 from Eq. (3.5):

$$F^{(n)} = \frac{t_1^2(t_n - t_1)/(t_1 - R_0)}{t[(4n^2 - 2n - 1)t_1 - (2n - 1)t_n - 2(n - 1)R_0] + R[t_n - (2n - 1)t_1] + 2n(t_1^2 + t_n^2) - 4n^2 t_1 t_n + R_0[(2n - 1)t_n - t_1]} \quad (3.10c)$$

for $n \geq 2$, with t_1 and t_n from Eqs. (3.7a) and (3.7c). The phase factors are $P_k^{(n)} = +1$ if $F_k^{(n)}$ has the same sign as $F^{(0)}$ and $P_k^{(n)} = -1$ if it has the opposite sign. This choice of phases ensures the boundary conditions $G^{(\text{WKB})}[0, t; R_0, 0] = 0$ and $G^{(\text{WKB})}[R, t; 0, 0] = 0$, hence the condition $\psi(0, t) = 0$ for any wormhole wave function, via the cancellation of contributions from paths differing by one bounce. For example, as R approaches zero, the contribution of the direct (zero-bounce) classical path becomes equal in magnitude and opposite in sign to that of

a one-bounce path¹ with bounce time approaching t . This phase prescription accords with that used, for exam-

¹The absolute value of the argument of the logarithm is taken in the action (2.6) and subsequent expressions to ensure that this cancellation still occurs when the direct path is just timelike and the one-bounce path just spacelike, i.e., for $R_0 \approx t$ and R approaching zero.

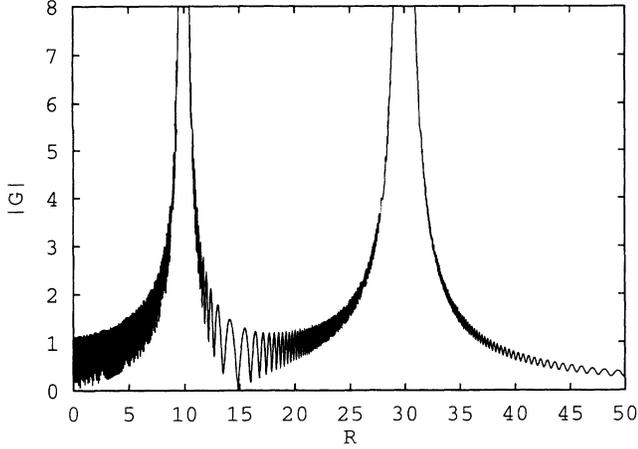


FIG. 1. Amplitude of the wormhole propagator $G^{(\text{WKB})}[R,t;R_0,0]$, for $R_0=10$ and $t=20$, all quantities in Planck units.

ple, for the simple problem of a particle confined to a half space [21].

Form (3.9) and the accompanying expressions show that even the approximate propagator for this simplified system is a complicated quantity. Some of its features are illustrated in Fig. 1. The propagator has singularities on the outgoing and ingoing or “bounced” light cones $R=R_0+t$, $R=R_0-t$, and $R=t-R_0$, respectively. (It has other singular points as well, described below.) As expected, given the inclusion of spacelike trajectories in the path integral as discussed in Sec. II, the propagator is nonvanishing outside the light cone. Thus, it admits acausal, i.e., superluminal, propagation. Overlying these gross features the propagator exhibits a great deal of high-frequency structure.

IV. QUANTUM EVOLUTION OF THE WORMHOLE

In principle, the propagator contains a complete description of the quantum dynamics of the wormhole. Analysis of the approximate form $G^{(\text{WKB})}$, carried out in

$$\int_{R-t-\Delta}^{R-t+\Delta} \left[\frac{F^{(0)}}{4\pi i} \right]^{1/2} \exp(iS^{(0)}) dR_0 = \left[\frac{t(2R-t)}{4\pi i} \right]^{1/2} \frac{4i}{t(2R-t)} \times \exp \left\{ \frac{i}{2} \left[t(2R-t) \ln \left[\frac{t(2R-t)}{R\Delta} \right] + (R-t)^2 \ln \left[\frac{R-t}{R} \right] \right] - \frac{3i}{4} t(2R-t) \right\} + \mathcal{O}(\Delta^2 \ln \Delta) \quad (4.1a)$$

on the outgoing light cone,

$$\int_{R+t-\Delta}^{R+t+\Delta} \left[\frac{F^{(0)}}{4\pi i} \right]^{1/2} \exp(iS^{(0)}) dR_0 = \left[\frac{t(2R+t)}{4\pi i} \right]^{1/2} \frac{4i}{t(2R+t)} \times \exp \left\{ \frac{i}{2} \left[t(2R+t) \ln \left[\frac{t(2R+t)}{R\Delta} \right] + (R+t)^2 \ln \left[\frac{R}{R+t} \right] \right] - \frac{3i}{4} t(2R+t) \right\} + \mathcal{O}(\Delta^2 \ln \Delta) \quad (4.1b)$$

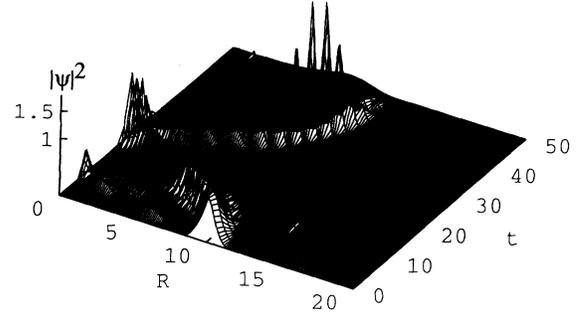


FIG. 2. Squared magnitude of a wormhole wave function $\psi(R,t)$ evolving via the propagator $G^{(\text{WKB})}$. The initial wave function is a real Gaussian, as described in the text. All quantities are in Planck units.

several stages, indicates that the wormhole is quantum-mechanically unstable against growth to large size.

Straightforward numerical evaluation of the propagation integral (2.10), using propagator $G^{(\text{WKB})}$ as given by Eq. (3.9) and a simple initial wave function, reveals behavior of the wormhole over short and intermediate time scales. Results of such calculations are displayed in Fig. 2. For this example the initial wave function is a localized “wave packet,”

$$\psi_0 = (2/\pi)^{1/4} \exp[-(R_0-10)^2],$$

all quantities in Planck units. For the purpose of numerical integration this is taken to be zero outside the interval $6 \leq R_0 \leq 14$. Although all terms in Eq. (3.9) have been given analytically, the numerical integration is nontrivial due to the existence of singular points in $G^{(\text{WKB})}$. The poles in the propagator on the outgoing, incoming, and bounced light cones are treated numerically by replacing the divergent term in $G^{(\text{WKB})}$, in a small interval (denoted Δ) in R_0 about the light cone, by its average over that interval. This is determined from an expansion in R_0 of the appropriate term about the light cone. In particular, we obtain

on the ingoing light cone, and

$$\int_{t-R-\Delta}^{t-R+\Delta} P^{(1)} \left[\frac{P^{(1)} F^{(1)}}{4\pi i} \right]^{1/2} \exp(iS^{(1)}) dR_0 = - \left[\frac{R^2 + (t-R)^2}{4\pi i} \right]^{1/2} \frac{4i}{R^2 + (t-R)^2} \\ \times \exp \left\{ \frac{i}{2} \left[R^2 \ln \left[\frac{R^2 + (t-R)^2}{R\Delta} \right] - (t-R)^2 \ln \left[\frac{(t-R)\Delta}{R^2 + (t-R)^2} \right] \right] \right. \\ \left. - \frac{3i}{4} [R^2 + (t-R)^2] \right\} + O(\Delta^2 \ln \Delta) \quad (4.1c)$$

on the bounced light cone (where only one single-bounce, and no multiple-bounce, classical path contributes). In the indicated intervals, then, these singular terms in the propagator are replaced by $1/(2\Delta)$ times the right-hand sides of these expressions. We have checked that although the integrals (4.1) depend on Δ , the final integrals (2.10) are independent of Δ over a range of values from 10^{-4} to 10^{-1} Planck lengths. The WKB propagator has additional singularities, caustics, where new bouncing paths appear, i.e., where Eq. (3.5) or Eqs. (3.7a) and (3.7c) have coincident roots. These are avoided by evaluating Eq. (2.10) at t values which are transcendental to machine accuracy; rational values of R and R_0 are taken so the t values at which caustics would occur are algebraic or rational. Figure 2, then, shows the resulting wave function $\psi(R, t)$. Its most prominent feature is that the wave packet follows an essentially classical bouncing trajectory. Some rapid spreading of the packet appears early on, but this behavior does not seem to recur at later times. Numerical difficulties hinder the reliable evaluation of ψ

at t values larger than those shown: Accurate numerical integration becomes problematic as the propagator develops ever higher-frequency behavior, and the problem of accurately solving Eqs. (3.5) and (3.7a) and (3.7c) with large coefficients makes even calculating the propagator difficult. To determine the long-term behavior and stability of the wormhole a different sort of analysis is required.

The late-time behavior of the propagator can be examined analytically in a special, but very useful, case. For arbitrary R_0 , R , and t the explicit form of $G^{(\text{WKB})}$ is intractable—the general expressions for the bounce times t_1 and t_n appearing in the $S^{(n)}$ and $F^{(n)}$ are too unwieldy even for computer manipulation. But, for $R_0 = R$, solutions (3.8) yield manageable results, and nearby R_0 and R values can be handled perturbatively. The terms in the sum (3.9) are then labeled a , b , c , and d , corresponding to those solutions. In the limit $t \gg R, R_0$ with $R = R_0 + \varepsilon$ and $|\varepsilon| \ll R_0$, the functional determinants and classical actions in those terms are

$$F_a^{(n)} = F_b^{(n)} = -\frac{1}{n} \left[1 + \frac{4nR_0}{t} + \frac{16n^2R_0^2}{t^2} + O \left[\frac{n^3R_0^3}{t^3} \right] \right] + O \left[\frac{\varepsilon}{R_0} \right], \quad (4.2a)$$

$$S_a^{(n)} = -\frac{t^2}{4n} - \left[\frac{t}{2nR_0} + \ln \left[\frac{t}{nR_0} \right] - 1 - \frac{3nR_0}{t} + O \left[\frac{n^2R_0^2}{t^2} \right] \right] R_0\varepsilon + O \left[\frac{t}{nR_0} \varepsilon^2 \right], \quad (4.2b)$$

and

$$S_b^{(n)} = -\frac{t^2}{4n} + \left[\frac{t}{2nR_0} + \ln \left[\frac{t}{nR_0} \right] - 1 - \frac{3nR_0}{t} + O \left[\frac{n^2R_0^2}{t^2} \right] \right] R_0\varepsilon + O \left[\frac{t}{nR_0} \varepsilon^2 \right], \quad (4.2c)$$

for n less than $t/(4R_0)$, and

$$F_c^{(n)} = \frac{1}{n+1} \left[1 + \frac{4(n+1)R_0}{t} + \frac{2(n+1)(8n+3)R_0^2}{t^2} + O \left[\frac{n^3R_0^3}{t^3} \right] \right] + O \left[\frac{\varepsilon}{R_0} \right], \quad (4.2d)$$

$$S_c^{(n)} = -\frac{t^2}{4(n+1)} - \frac{R_0 t}{n+1} - R_0^2 \ln \left[\frac{t}{(n+1)R_0} \right] + \frac{R_0^2}{2} \frac{n-1}{n+1} + \frac{2nR_0^3}{t} + O \left[\frac{n^2R_0^4}{t^2} \right] \\ - \left[\frac{t}{2(n+1)R_0} + \ln \left[\frac{t}{(n+1)R_0} \right] - \frac{n}{n+1} - \frac{3nR_0}{t} + O \left[\frac{n^2R_0^2}{t^2} \right] \right] R_0\varepsilon + O \left[\frac{t}{(n+1)R_0} \varepsilon^2 \right], \quad (4.2e)$$

$$F_d^{(n)} = \frac{1}{n-1} \left[1 + \frac{4(n-1)R_0}{t} + \frac{2(n-1)(8n-3)R_0^2}{t^2} + O\left(\frac{n^3 R_0^3}{t^3}\right) \right] + O\left(\frac{\varepsilon}{R_0}\right), \quad (4.2f)$$

$$S_d^{(n)} = -\frac{t^2}{4(n-1)} + \frac{R_0 t}{n-1} + R_0^2 \ln \left[\frac{t}{(n-1)R_0} \right] - \frac{R_0^2}{2} \frac{n+1}{n-1} - \frac{2nR_0^3}{t} + O\left(\frac{n^2 R_0^4}{t^2}\right) \\ + \left[\frac{t}{2(n-1)R_0} + \ln \left[\frac{t}{(n+1)R_0} \right] - \frac{n}{n-1} - \frac{3nR_0}{t} + O\left(\frac{n^2 R_0^2}{t^2}\right) \right] R_0 \varepsilon + O\left(\frac{t}{(n+1)R_0} \varepsilon^2\right), \quad (4.2g)$$

for n less than $(t^2 + 4R_0^2)/(4R_0 t)$. The $n=0$ term is a c term, given by Eqs. (4.2d) and (4.2e), and the $n=1$ terms are a , b , and c terms given by Eqs. (4.2a)–(4.2e). The order- ε terms in the actions are obtained from the relation $\partial S_{\text{cl}}/\partial R = P(t)$ and Eq. (2.8) for the momentum P , evaluated on the appropriate classical path. Terms of order ε and higher in the functional determinants give rise to subdominant contributions and are not needed here.

The above expansions in inverse powers of t do not converge for $n = t/(4R_0)$. Indeed, the a , b , and c terms in the propagator are singular for these parameter values, at which the roots (3.8a)–(3.8c) coincide. But the contribution of these singular terms to the evolution of a wave function can still be estimated. At fixed R_0 , such a singularity or “caustic point” occurs at intervals in t of $4R_0$, the period of a classical bouncing trajectory with maximum radius value R_0 . The singular contribution is thus associated with the classically bouncing peak seen in Fig. 2. Unlike the similar singularity in a harmonic-oscillator propagator [22], however, this singularity in $G^{(\text{WKB})}$ does not correspond to a δ function. Rather, in the immediate vicinity of the caustic, i.e., for ε small compared to t^{-2} (in Planck units), the c term has functional determinant

$$F_c^{(n)} \sim \frac{-8R_0}{3t^{1/3}\varepsilon^{2/3}} + \dots \quad (4.3a)$$

and action

$$S_c^{(n)} \sim -R_0 t - \frac{R_0 t^{1/3}}{2} \varepsilon^{2/3} + \dots \quad (4.3b)$$

The contribution to a wave function from this vicinity is therefore of order $t^{-3/2}$, not unity, as a δ function would yield. However, the form of the action (4.3b) suggests that a significant contribution from this term may arise from a region of width $t^{-1/2}$ in ε . Such a contribution might be of order unity, as Fig. 2 suggests, but the expansions used in Eqs. (4.3) do not suffice to calculate it precisely. The a and b terms do not, in fact, exist for nonzero ε about the caustic point; they give no contribution here. Another set of caustic points occurs where the c and d roots coincide. Here again the a and b terms do not contribute, those roots being complex, while the c and d terms are similar in form to those given by Eqs. (4.3).

Hence it is easiest to examine the behavior of $G^{(\text{WKB})}$ for parameter values such that no caustics occur at integer n values. Then the expansions (4.2) give approxi-

mate values for all terms. The actions vary slowest with ε , i.e., with R_0 , for the largest- n terms, so it is these which determine the width of the region in integral (2.10) which gives a significant contribution to a wave function. That width corresponds roughly to one oscillation of the slowest-varying terms. For example, the integral of $G^{(\text{WKB})}[R, t; R - \varepsilon, 0]$, with terms given by Eqs. (4.2), over the interval $\varepsilon \in [-\pi/R, +\pi/R]$, is shown in Fig. 3 for fixed R and various t values. These results suggest that a wave function at fixed radius value continues to oscillate even for quite large t values, as might be expected from a sum of terms with rapidly varying phases. Thus, they still give no clear picture of the very-long-term behavior of the wormhole.

But telling features of that behavior can be extracted from the propagator as approximated by Eqs. (4.2). Did the Hamiltonian (2.9) describe a bound system, its ground-state energy (or, for a free system, the bottom of its continuous spectrum) would be given by the Feynman-Kac [23] limit

$$E_0 = - \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln G[R, -i\tau; R, 0] \quad (4.4)$$

of the propagator in imaginary time. This can be evaluated via Eqs. (4.2). The propagator is still a sum of terms with rapidly varying phases with respect to increasing τ , owing to the t^2 terms in the actions. Consequently, it os-

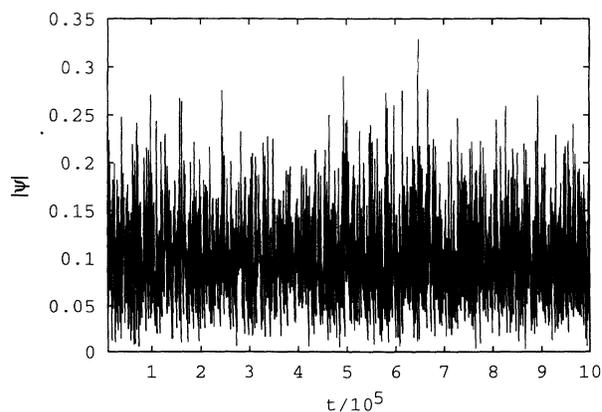


FIG. 3. Magnitude of the dominant contribution ψ to a wormhole wave function taken to be initially unity near the radius $R_0 = 10$ Planck lengths, evaluated at the same radius after 10^4 – 10^6 Planck times.

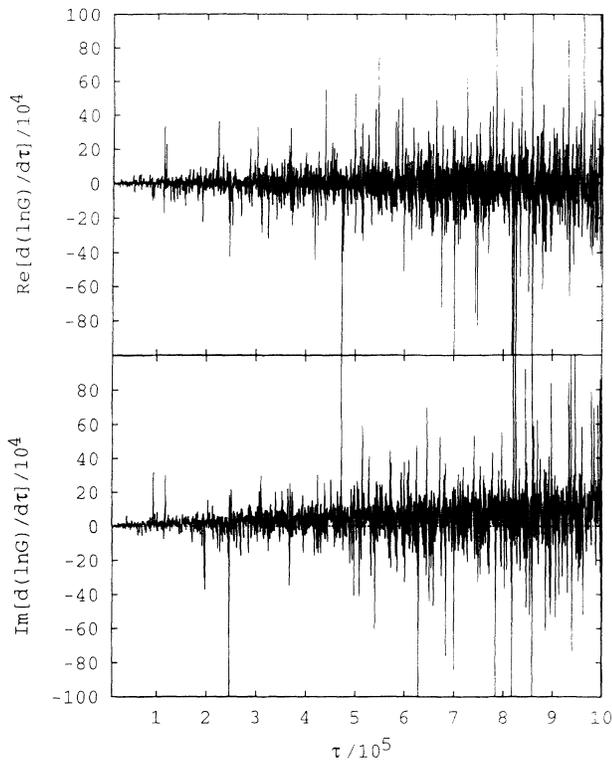


FIG. 4. Behavior of the WKB propagator at large imaginary time τ , here in units of the Planck time. The initial and final radii are fixed at 10 Planck lengths. Shown are the real and imaginary parts of the τ derivative of $\ln G$. If the Feynman-Kac ground-state energy [Eq. (4.4) of the text] existed, the real part would approach that value and the imaginary part would tend to zero at large τ .

cillates forever; the limit on the right side of Eq. (4.4) does not exist. This is illustrated in Fig. 4. This means that the quantum wormhole possesses no spectrum of bound states. The expectation value of R cannot be confined in the late-time limit. As for a system with an inverted potential diverging to negative infinity, or a metastable system such as a particle confined by finite walls, the wormhole wave function must eventually run or “leak” to arbitrarily large radius values—the wormhole is unstable against eventual growth to large size. The previous results suggest that it is the slow leaking behavior which characterizes these wormholes. They appear reminiscent of α particles in a nucleus, oscillating perhaps millions of times before escaping to infinity.

V. IMPLICATIONS AND LIMITATIONS

Spherically symmetric Minkowski wormholes [10] provide a simple model of a mode of topological fluctuation in Lorentzian spacetime foam, a mode apparently unstable against growth to macroscopic size. The quantum-gravitational dynamics of these wormholes is reduced to the quantum mechanics of a single variable, the throat radius, by describing the matter at the wormhole throat with a suitable equation of state and imposing the Hamil-

tonian constraint at the classical level to reduce the phase space of the system. The corresponding reduced action is used in a Feynman path integral to obtain the propagator for wormhole wave functions; this is evaluated in the WKB approximation. The result indicates that these wormholes have no bound quantum states. Though their throat radii are classically bounded, i.e., they are classically stable, they will nonetheless grow to large size by quantum “diffusion.”

Many systems exhibit similar behavior. For a particle with the familiar quadratic kinetic term in the action, the form of the potential determines whether such diffusion or spreading occurs: A potential well with walls or barriers which fall off at large distances will allow a classically bound particle to leak out via quantum tunneling (as in the case of α decay), while one which increases monotonically with distance will not. For these wormholes, with more complicated action (2.6), so simple an analysis is not possible. The more involved examination of the wormholes’ quantum dynamics described here is needed to see that spreading of the wave function to large radii will take place.

Our result suggests that dynamics, and stability considerations in particular, may be of great importance in understanding the quantum nature of gravitation. A definitive demonstration of the existence of an unstable mode of fluctuation in spacetime foam would have profound implications. Since a macroscopic structure of wormholes is not observed, i.e., spacetime appears to be smooth and topologically trivial on all scales accessible to laboratory physics, it would imply the existence of a mechanism for suppressing such a mode, or even the absence of (Lorentzian-signature) spacetime foam altogether. Of course, the present work is far from such a definitive demonstration; it serves to point out lines along which these matters should be studied further.

The most fundamental limitation of our calculation is the restriction of the gravitational degrees of freedom to those of the spherically symmetric Minkowski wormhole, i.e., the use of a “minisuperspace model” for topological structure. In fact, our model is even more restricted than the usual minisuperspace models [1], since the matter in the hole is treated not as a dynamical field but by the use of an equation of state. Moreover, we use the particular equation of state (2.3), to simplify the calculations; other possible choices are considered in Ref. [13].

We analyze our constrained model by quantization in the reduced phase space, as described above. In the absence of a general framework for quantum-gravity calculations, this method seems best suited to the problem. It does differ markedly, though, from the Wheeler-deWitt approach [12].

Furthermore, we use the particular reduced action (2.6). Other forms corresponding to the classical equation (2.4) are possible; the effect of this choice on the results will be examined elsewhere [13].

Moreover, our choice of action implies a choice of Hamiltonian fundamentally different from that used in similar calculations [19], corresponding here to a classically conserved “proper mass” rather than the ADM mass of the wormhole. This choice raises some interest-

ing questions. For example, since the Hamiltonian (2.9) is time independent, its expectation value is conserved. Its form appears to suggest, then, that the expectation value of the throat radius R should remain bounded. It might be expected, however, that any initially localized wave function must contain both positive- and negative-“energy” ($|\dot{R}| > 1$) components; this is a well-known feature of, e.g., the quantum mechanics of a relativistic particle [14,15]. The interplay of these components could account for the eventual spreading to large radius values. More detailed calculations of the wormholes’ quantum dynamics should clarify this.

We employ the WKB approximation to evaluate the wormhole propagator. WKB calculations of quantum instabilities in classically stable systems, such as tunneling, are well known. Here it is more difficult to be precise about the accuracy of the approximation. It is certainly to be expected to be valid in the late-time limit $t \gg R, R_0$, in which the instability is manifest. But the accuracy of the numerical evolution of a wave function, as shown in Fig. 2, is harder to establish. Lacking any exact solution for comparison, we have tested the accuracy of the calculation via the composition relation

$$G^{(\text{WKB})}[R, t; R_0, 0] = \int G^{(\text{WKB})}[R, t; R_1, t_1] G^{(\text{WKB})}[R_1, t_1; R_0, 0] dR_1, \quad (5.1)$$

with $0 < t_1 < t$. It can be shown analytically that this should hold if the WKB approximation is strictly valid at the intermediate time t_1 . We found, however, that for some t_1 values relation (5.1) is not well satisfied. This may indicate inaccuracy of the WKB approximation, or may be due to numerical difficulties associated with the integration, given the rapidly varying phase of the propagator.

A final limitation, of fundamental significance, is our implementation of the restriction that throat radii are non-negative. Here we do this as for a particle in a half space, leading to the boundary condition $\psi(0, t) = 0$. Other implementations might be used, the most general condition being only that the wave function ψ entails no current in the $-R$ direction at $R = 0$. Our condition

eliminates from consideration any processes such as wormhole creation or disconnection at $R = 0$. Including these processes would drastically alter the physics of the model—essentially, from the quantum mechanics of one variable to quantum field theory—and would require a formalism for describing the topology changes. However, it should not directly affect the existence of the instability suggested by our results. At issue is the stability of the spacetime foam, of which an individual wormhole is just one fluctuation. Certainly for any particular wormhole, the probability of growth to large size is affected strongly by the inclusion or exclusion of topology change. Indeed, since the time scale implied by our results for the wave function to leak to large radii is much longer than that for a classical bounce, the chance that a specific wormhole grows large should be much smaller than that it pinches off and disappears at zero radius, if that is allowed with more than an extremely small probability. But given the possibility of topology change, the form should contain an equilibrium population of holes fluctuating into and out of existence. If it is possible for a hole to grow large, this population will eventually give rise to some large holes. Again the analogy may be drawn to the α decay of a heavy nucleus: α particles continually form and disperse within the nucleus, on a time scale typically much shorter than that of the decay; the instability represented by the tunneling of an α particle out of the nucleus remains.

The more sophisticated analyses needed to probe the quantum dynamics of spacetime beyond the restrictions and limitations of these calculations present a considerable challenge. Our results suggest, however, that this is an aspect of quantum gravity theory well worth such consideration.

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- [1] See, e.g., J. J. Halliwell, *Int. J. Mod. Phys. A* **5**, 2473 (1990); in *Quantum Cosmology and Baby Universes*, Proceedings of the 7th Jerusalem Winter School, Jerusalem, Israel, 1990, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991), Vol. 7, pp. 159–243.
- [2] S. W. Hawking, *Phys. Rev. D* **37**, 904 (1988); S. B. Giddings and A. Strominger, *Nucl. Phys.* **B307**, 857 (1988); S. Coleman, *ibid.* **B307**, 867 (1988); **B310**, 643 (1988); also papers in Ref. [1].
- [3] I. H. Redmount and W.-M. Suen, *Phys. Rev. D* **47**, R2163 (1993).
- [4] J. A. Wheeler, *Ann. Phys. (N.Y.)* **2**, 604 (1957); *Geometro-*

- dynamics* (Academic, New York, 1962), pp. 71–83.
- [5] R. P. Geroch, *J. Math. Phys.* **8**, 782 (1967); P. Yodzis, *Commun. Math. Phys.* **26**, 39 (1972).
- [6] See, e.g., W. Unruh, *Phys. Rev. D* **40**, 1053 (1989), and references cited therein.
- [7] M. S. Morris, K. S. Thorne, and U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988); V. P. Frolov and I. D. Novikov, *Phys. Rev. D* **42**, 1057 (1990); J. Friedman, M. S. Morris, I. D. Novikov, F. Echeverria, G. Klinkhammer, K. S. Thorne, and U. Yurtsever, *ibid.* **42**, 1915 (1990); K. S. Thorne, in *Nonlinear Problems in Relativity and Cosmology*, Proceedings of the 6th Florida Workshop in Nonlinear Astronomy, Gainesville, Florida, 1990, edited by J. R.

- Buchler, S. L. Detweiler, and J. R. Ipser (New York Academy of Sciences, New York, 1991), Vol. 631, pp. 182–193; S. W. Hawking, in *Proceedings of the 6th Marcel Grossmann Meeting on General Relativity*, Kyoto, Japan, 1991, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1992), Part A, pp. 3–13.
- [8] T. A. Roman, *Phys. Rev. D* **33**, 3526 (1986); **37**, 546 (1988).
- [9] T. A. Roman, *Phys. Rev. D* **47**, 1370 (1993).
- [10] M. Visser, *Phys. Rev. D* **39**, 3182 (1989); **41**, 1116 (1990); *Nucl. Phys.* **B328**, 203 (1989).
- [11] W. Israel, *Nuovo Cimento B* **44**, 1 (1966); **48**, 463(E) (1967); C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 551–555; S. K. Blau, E. I. Guendelman, and A. H. Guth, *Phys. Rev. D* **35**, 1747 (1987).
- [12] M. Visser, *Phys. Lett. B* **242**, 24 (1990); *Phys. Rev. D* **43**, 402 (1991).
- [13] I. H. Redmount, W.-M. Suen, and K. Young (in preparation).
- [14] T. Newton and E. Wigner, *Rev. Mod. Phys.* **21**, 400 (1949); see also S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row and Peterson, Evanston, 1961), pp. 54–62.
- [15] I. H. Redmount and W.-M. Suen, *Int. J. Mod. Phys. A* **8**, 1629 (1993).
- [16] J. B. Hartle, *Phys. Rev. D* **38**, 2985 (1988).
- [17] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), frontispiece.
- [18] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984), pp. 453–459.
- [19] P. Hájíček, *Commun. Math. Phys.* **150**, 545 (1992); P. Hájíček, B. S. Kay, and K. V. Kuchař, *Phys. Rev. D* **46**, 5439 (1992).
- [20] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981), pp. 92–96.
- [21] Schulman [20], pp. 40 and 41.
- [22] Schulman [20], pp. 32–39, and 118.
- [23] Schulman [20], pp. 42–44.

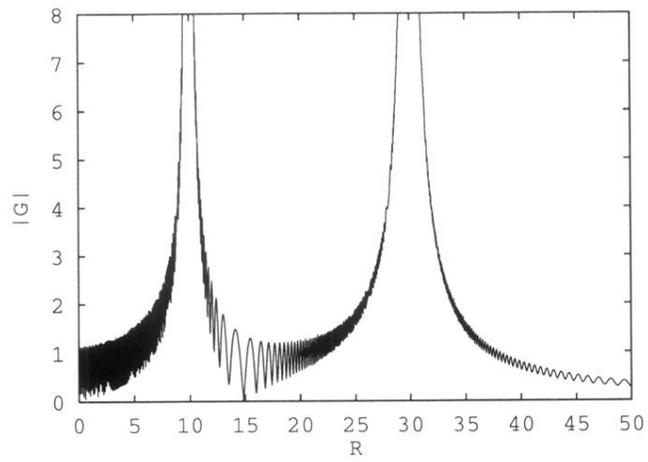


FIG. 1. Amplitude of the wormhole propagator $G^{(\text{WKB})}[R,t;R_0,0]$, for $R_0=10$ and $t=20$, all quantities in Planck units.

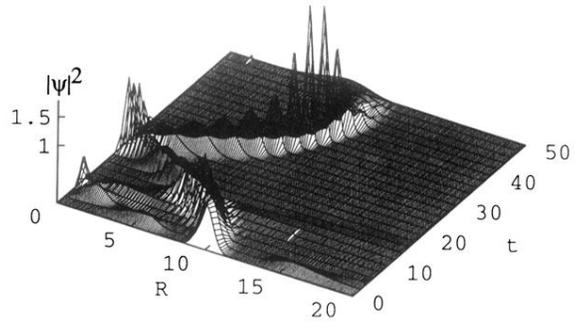


FIG. 2. Squared magnitude of a wormhole wave function $\psi(R,t)$ evolving via the propagator $G^{(\text{WKB})}$. The initial wave function is a real Gaussian, as described in the text. All quantities are in Planck units.