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# Perturbative method to solve fourth-order gravity field equations

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We develop a method for solving the field equations of a quadratic gravitational theory coupled to matter. The quadratic terms are written as a function of the matter stress tensor and its derivatives in such a way as to have, order by order, a set of Einstein field equations with an effective $T_{\mu\nu}$. We study the cosmological scenario recovering the de Sitter exact solution, and the first order (in the coupling constants $\alpha$ and $\beta$ appearing in the gravitational Lagrangian) solution to the gauge cosmic string metric and the charged black hole. For this last solution we discuss the consequences on the thermodynamics of black holes, and, in particular, the entropy-area relation which gets additional terms to the usual $A/4$ value.

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## I. INTRODUCTION

Higher derivative gravitational theories have been proposed at the classical level as an extension of Einstein's theory in an attempt to unify other fields with gravity [1] and to avoid the cosmological singularity [2]. Quadratic curvature counterterms also appear in the renormalization of the one-loop semiclassical approximation [3]. The pure quadratic theory is renormalizable [4] and asymptotically free [5] although it has problems with unitarity [4]. Higher order gravitational theories can lead naturally to inflation [6,7] and arise as the low energy limit of string theory [8].

For the sake of definiteness we deal with the Lagrangian formulation of quadratic theories:

$$S = S_G + S_m = \int d^4x\sqrt{-g}\left\{-2\Lambda + R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + k\mathcal{L}_m\right\} , \quad (1)$$

where we have dropped the $R^2_{\mu\nu\lambda\rho}$ term by use of the Gauss-Bonnet invariant in four dimensions [9].

The field equations derived by extremizing the action $S$ are given by (we use the sign conventions of Ref. [10])

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H_{\mu\nu} + \beta I_{\mu\nu} = kT_{\mu\nu} \doteq -\frac{2k}{\sqrt{-g}}\frac{\partial S_m}{\partial g^{\mu\nu}} , \quad (2)$$

where we have chosen units such that $c = 1$ and $k = 16\pi G$, and where

$$H_{\mu\nu} = -2R_{;\mu\nu} + 2g_{\mu\nu}\,\Box R - \tfrac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu} \quad (3)$$

and

$$I_{\mu\nu} = -2R^{\alpha}_{\mu\ ;\nu\alpha} + \Box R_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}\,\Box R + 2R_{\mu}{}^{\alpha}R_{\alpha\nu} - \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} . \quad (4)$$

Note that the trace of Eq. (2) takes the simple form

$$2(3\alpha+\beta)\,\Box R - R + 4\Lambda = \tfrac{1}{2}kT . \quad (5)$$

In Sec. II of this paper we develop a method to find a metric solution of the classical field equations of higher order gravitational theories with sources by successive perturbations around a solution to Einstein gravity, which will represent for us the zeroth order. In Sec. III we apply this method to find first-order solutions in the coupling constants $\alpha$ and $\beta$ for the straight gauge cosmic string and the charged black hole. For this latter solution the associated thermodynamics is studied and the corrections to the Bekenstein-Hawking temperature and entropy are discussed. We end the paper with some further discussion on this first-order solution.

## II. PERTURBATIVE SOLUTION

From Eq. (5) we can see that $R$ satisfies a massive scalar wave equation. For this field to have a real mass we impose

$$3\alpha + \beta \geq 0, \quad \beta \leq 0, \quad (6)$$

where the last inequality comes from linearizing Eq. (2) and asking a real mass for the massive spin-two field $\psi_{\mu\nu}$ related to $R_{\mu\nu}$ (see Ref. [11]). Conditions (6) are called the no-tachyon constraints.

The coupling constants $\alpha$ and $\beta$ have to be at most of the atomic scale, since otherwise they could have observable effects in, for instance, the solar system or binary pulsars. In Ref. [12] it was found that $\alpha < 10^{15}l^2_{\rm Pl}$ by requiring that the inflationary period be of sufficient du-

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ration (where $l_{\rm Pl}$ is of the order of $10^{-33}$ cm). A similar bound for $\beta$ can thus be obtained.

We will then consider only small curvatures in our method, such that

$$\alpha|R| \ll 1, \quad |\beta R_{\mu\nu}| \ll 1 . \tag{7}$$

Thus, we can obtain perturbative solutions to the field equations in a power series of $\alpha$ and $\beta$. We will take as the starting metric $g^{(0)}_{\mu\nu}$ a solution of Einstein equations and then, systematically, successive orders of the coupling constant $\alpha$ and $\beta$.

To *zeroth order*,

$$R_{\mu\nu}(g^{(0)}_{\mu\nu}) - \tfrac{1}{2}R^{(0)}g^{(0)}_{\mu\nu} + \Lambda g^{(0)}_{\mu\nu} = kT_{\mu\nu}(g^{(0)}_{\mu\nu}) , \tag{8}$$

with trace

$$R^{(0)} = 4\Lambda - kT^{(0)}. \tag{9}$$

Once we solve for the Einstein metric $g^{(0)}_{\mu\nu}$, we pursue the iteration to *first order*:

$$R^{(1)}_{\mu\nu} - \tfrac{1}{2}R^{(1)}g^{(1)}_{\mu\nu} + \Lambda g^{(1)}_{\mu\nu} = kT_{\mu\nu}(g^{(1)}_{\mu\nu}) - \alpha H_{\mu\nu}(g^{(0)}_{\mu\nu}) - \beta I_{\mu\nu}(g^{(0)}_{\mu\nu}) , \tag{10}$$

where to first order in $\alpha$ and $\beta$ it is enough to consider in Eq. (10) $H^{(0)}_{\mu\nu}$ and $I^{(0)}_{\mu\nu}$.

By use of Eq. (8) the right-hand side of Eq. (10) can be written as

$$T^{(1)\rm eff}_{\mu\nu} = T^{(1)}_{\mu\nu} - \frac{\alpha}{k}H_{\mu\nu}(g^{(0)}_{\mu\nu}, T^{(0)}_{\mu\nu}) - \frac{\beta}{k}I_{\mu\nu}(g^{(0)}_{\mu\nu}, T^{(0)}_{\mu\nu}) , \tag{11}$$

where

$$\begin{aligned}\frac{1}{k}H^{(0)}_{\mu\nu} &\doteq \frac{1}{k}H_{\mu\nu}(g^{(0)}_{\mu\nu}, T^{(0)}_{\mu\nu}) \\ &= 2T^{(0)}_{;\mu\nu} - 2g^{(0)}_{\mu\nu}\,\Box\, T^{(0)} + \frac{k}{2}g^{(0)}_{\mu\nu}(T^{(0)})^2 \\ &\quad -2kT^{(0)}T^{(0)}_{\mu\nu} + 8\Lambda T^{(0)}_{\mu\nu} - 2\Lambda T^{(0)}g^{(0)}_{\mu\nu}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{1}{k}I^{(0)}_{\mu\nu} &\doteq \frac{1}{k}I_{\mu\nu}(g^{(0)}_{\mu\nu}, T^{(0)}_{\mu\nu}) \\ &= T^{(0)}_{;\mu\nu} - 2T^{(0)\alpha}{}_{\mu;\nu\alpha} + \Box T^{(0)}_{\mu\nu} - g^{(0)}_{\mu\nu}\,\Box\, T^{(0)} + 4\Lambda T^{(0)}_{\mu\nu} - \Lambda T^{(0)}g_{\mu\nu} + \frac{k}{2}g^{(0)}_{\mu\nu}(T^{(0)})^2 \\ &\quad -2kT^{(0)}T^{(0)}_{\mu\nu} + 2kT^{(0)\alpha}{}_{\mu}T^{(0)}_{\alpha\nu} - \frac{k}{2}g^{(0)}_{\mu\nu}T^{(0)\alpha\beta}T^{(0)}_{\alpha\beta} .\end{aligned} \tag{13}$$

Thus $T^{(1)\rm eff}_{\mu\nu}$ plays the role of an effective energy-momentum tensor in an Einsteinian field equation for $g^{(1)}_{\mu\nu}$. It is easy to see that $T^{\rm eff}_{\mu\nu;}{}^{\nu} = 0$, i.e., satisfies a conservation law (with respect to $g^{(1)}_{\mu\nu}$). $T^{\rm eff}_{\mu\nu}$ also inherits the symmetry properties of $T_{\mu\nu}$. These properties will hold to every order of the approximation.

We can now generalize to the *n*th-*order* approximation

$$\begin{aligned}R^{(n)}_{\mu\nu} - \tfrac{1}{2}R^{(n)}g^{(n)}_{\mu\nu} + \Lambda g^{(n)}_{\mu\nu} &= kT^{{\rm eff}(n)}_{\mu\nu} \\ &= kT_{\mu\nu}(g^{(n)}_{\mu\nu}) - \alpha H_{\mu\nu}(g^{(n-1)}_{\mu\nu}, T^{(n-1)}_{\mu\nu}) - \beta I_{\mu\nu}(g^{(n-1)}_{\mu\nu}, T^{(n-1)}_{\mu\nu}) .\end{aligned} \tag{14}$$

Expressions (11)–(13) greatly simplify when $T_{\mu\nu}$ is diagonal and its components depend essentially on only one coordinate; let us call it $r$. This will be the case in the applications we will deal with in the next section (we will also take $\Lambda = 0$ for simplicity). Thus, in this case,

$$\frac{1}{k}H^{(0)}_{\mu\mu} = 2\left\{T_{,rr}\delta^r_\mu - \Gamma^r_{\mu\mu}T_{,r} - g_{\mu\mu}\left[g^{rr}T_{,rr} - g^{\alpha\alpha}\Gamma^r_{\alpha\alpha}T_{,r} - \frac{k}{4}T^2 + kTT^\mu_\mu\right]\right\} \tag{15}$$

and

$$\begin{aligned}\frac{1}{k}I^{(0)}_{\mu\mu} &= T_{,rr}\delta^r_\mu - \Gamma^r_{\mu\mu}T_{,r} - 2\left[\left(T^r_{r\ ,rr} + \Gamma^\alpha_{r\alpha}(T^r_{r\ ,r} - T^\alpha_{\alpha\ ,r})\right)\delta^r_\mu - \Gamma^r_{\mu\mu}T^\mu_{\mu\ ,r}\right] \\ &\quad -g_{\mu\mu}\left[2kTT^\mu_\mu - g^{rr}T_\mu{}^\mu{}_{,rr} + g^{\alpha\alpha}\Gamma^r_{\alpha\alpha}(T^\mu_{\mu\ ,r} - T_{,r}) + g^{rr}T_{,rr} - \frac{k}{2}T^2 - 2k(T_\mu{}^\mu)^2 + \frac{k}{2}T^{\alpha\alpha}T_{\alpha\alpha}\right] ,\end{aligned} \tag{16}$$

where the metric dependence in this expressions is with respect to the zeroth order, i.e., the solution of the usual Einstein equations, Eq. (8). In Eqs. (15) and (16) the sum is over the $\alpha$ index but not over $\mu$.

## III. FIRST-ORDER SOLUTIONS

We are now ready to compute the different metric solutions of the fourth-order field equations. Three main astrophysical scenarios where gravity plays an important role can be studied: cosmology, topological defects, and black holes.

### A. De Sitter universe

It is a solution of the vacuum Einstein equations with a cosmological constant. This solution is useful for describing an inflationary phase in the very early Universe where higher order gravity terms are presumably worth taking into account.

In our approximation, for $T_{\mu\nu}=0$, we have that the de Sitter metric is a solution of the fourth-order field equations order by order [see Eq. (14)], without restrictions for the value of $\Lambda$. Thus, it is an *exact* solution as can be directly verified from the field equations [13]. In fact, in general, vacuum solutions to Einstein equations (even with cosmological constant) are solutions to the quadratic theory (the converse, in general, is not true).

The Robertson-Walker metric can be also replaced in Eqs. (10) and (11) to find the corrections to the general relativistic results. However, this approximation breaks down near the cosmological singularity and we already dispose of some exact solutions (see Ref. [14]).

### B. Gauge cosmic strings

Our approximation is especially suitable for dealing with the gravitational field of topological defects since while its effects are of relevance they are relatively small compared to the Planck scale; thus a first-order approximation in $\alpha$ and $\beta$ should provide acceptable results.

Topological defects are expected to be formed during phase transitions driven by the evolving (and cooling down) early Universe whenever the manifold of equivalent vacua after spontaneous symmetry breaking is not shrinkable to a point. We can model this process by studying an $n$-internal component scalar field with a Mexican-hat-like effective potential with an associated energy scale of symmetry breaking, $\eta^2$, of order $10^{-6}$ for the grand unified theory (GUT) scale [15] and coupling constant $\lambda$. Depending on the dimension of the vacua manifold the topological defects can be domain walls, cosmic strings, or monopoles and, depending on whether the (internal) symmetry that breaks down is a local (or gauge) or a global one, the formed topological defects will be localized in a tiny core of size $r_c\sim 1/\eta\sqrt{\lambda}$ or infinitely extended [17].

To see explicitly the effects of this higher derivative theory of gravity we chose one particular topological defect. Since in the next section we will deal with a spherically symmetric system (a charged black hole) we will deal here with the local (or gauge) straight cosmic string that possesses cylindrical symmetry. The other topological defects can be treated in an analogous way (see Ref. [11] for a thorough account of them).

A straight, static, cylindrically symmetric local string lying along the $z$ axis can be characterized by the energy-momentum tensor (for $r\gg r_c$)

$$T_t{}^t=T_z{}^z=-\frac{\eta^2\delta(r)}{2\pi r\sqrt{B(r)}}\quad,\quad T_r{}^r=-T_\theta{}^\theta=0\ . \qquad (17)$$

For a generic metric of the form

$$ds^2=A(r)(-dt^2+dz^2)+dr^2+r^2B(r)d\theta^2\ , \qquad (18)$$

the exact metric, the solution of Einstein equations is given by [16].

$$A(r)=1,\quad B(r)=B_0=\left(1-k\frac{\eta^2}{4}\right)^2\ . \qquad (19)$$

By plugging the generic metric (18) into Eqs. (10) and (11) we can now write the solution to our first-order Einstein equation with effective source $T^{\text{eff}}_{\mu\nu}$:

$$A(r)=c_1+k\int^r r''dr''\int^{r''}\frac{dr'}{r'}\left[T^{\text{eff}\ \theta}_{\theta}-T^{\text{eff}\ r}_{r}\right] \qquad (20)$$

and

$$B(r)=c_2-\frac{2kB_0}{r}\int^r dr''\left\{\int^{r''}dr'r'\left[T^{\text{eff}\ \theta}_{\theta}-\frac{1}{2}T\right.\right.$$
$$\left.\left.-(3\alpha+\beta)\,\Box\,T\right]+A(r'')\right\}\ , \qquad (21)$$

where $c_1$ and $c_2$ are arbitrary constants to be conveniently chosen in such a way to recover general relativistic results.

Explicitly replacing here the form of the gauge cosmic string stress tensor given by Eq. (17) and upon integration and regularization of $\delta$ squared terms, we obtain the metric components up to linear terms in $\alpha$ and $\beta$:

$$A(r)=1+(2\alpha+\beta)kT\ , \qquad (22)$$

$$B(r)=\left(1-k\frac{\eta^2}{4}\right)^2-4\alpha k\left(1-k\frac{\eta^2}{2}\right)^2T$$
$$-\frac{\left(\alpha+\frac{\beta}{2}\right)k^2\eta^4}{(2\pi)^4}\frac{1}{r^2}\ . \qquad (23)$$

We observe here that the corrections to the general relativistic metric due to including quadratic terms in the curvature in the gravitational Lagrangian can be classified in two types. The terms in $A(r)$ and $B(r)$ proportional to the trace of the matter stress tensor $T$, give localized contributions. They are different from zero only in the core of the gauge string and vanish for $r>r_c$. This is essentially what was found in Ref. [11], where we restricted the analysis to linearized terms in the curvature and only considered its higher derivatives. It is precisely the additional terms not considered in Ref. [11], i.e., quadratic in the stress tensor, that generate the nonlocalized term, proportional to $r^{-2}$ appearing in $B(r)$. The structure of this term is such that it is linear in the coupling constants $\alpha$ and $\beta$ (due to our approximation), and if one considers they have an associated radius $r_1$, the dependence $(r_1/r)^2$ is the only extended possible one not divergent as $r\to\infty$. On the other hand the factor $k^2\eta^4$ already appears in processes such as particle production by the formation of cosmic strings [18] and global monopoles [19]. We see that due to the quadratic gravity terms, the space outside a straight local cosmic string is no longer flat as in general relativity, but curved with curvature terms typically going as $r^{-4}$. This dependence also appears when one considers the renormalized energy-momentum tensor due to vacuum polarization [20–22].

### C. Charged black holes

We shall now study spherically symmetric solutions to the quadratic field equations in the first-order approximation Eqs. (10) and (11), which represent charged black holes; starting from the general relativistic solution, i.e., the Reissner-Nordström metric,

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 \ , \tag{24}$$

where $-g_{tt}(r) = g_{rr}(r)^{-1} = (1 - 2M/r + Q^2/r^2)$ and $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$.

The nonvanishing components of the energy-momentum tensor are [10]

$$T_t^t = T_r^r = -T_\vartheta^\vartheta = -T_\varphi^\varphi = -\frac{Q^2}{r^4} \ . \tag{25}$$

The exact metric, solution of the Einstein equations with source, can be written as [23] [in the Schwarzschild gauge, Eq. (24)]

$$g_{rr}^{-1}(r) = 1 - \frac{2M}{r} + \frac{1}{r}\int_\infty^r \tilde{r}^2 T_t^{\text{eff}\ t} d\tilde{r} \ , \tag{26}$$

$$g_{tt}(r) = -g_{rr}(r)^{-1} \exp\left\{\int_\infty^r (T_r^r - T_t^t)^{\text{eff}} \tilde{r} g_{rr}(\tilde{r}) d\tilde{r}\right\} \ . \tag{27}$$

By use of Eq. (25) to compute the $T_{\mu\nu}^{\text{eff}}$ [Eq. (11)] we find the first-order corrections to the energy-momentum tensor

$$\Delta T_r^{\text{eff}\ r} = \tfrac{1}{3}\Delta T_t^{\text{eff}\ t} = -\tfrac{1}{2}\Delta T_\vartheta^{\text{eff}\ \vartheta} = -\tfrac{1}{2}\Delta T_\varphi^{\text{eff}\ \varphi}$$
$$= \frac{4\beta Q^2}{kr^6}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \ , \tag{28}$$

and replacing it into Eqs. (26) and (27) we obtain

$$g_{rr}(r)^{-1} \simeq 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{12\beta Q^2}{r^4}\left(\frac{1}{3} - \frac{M}{2r} + \frac{Q^2}{5r^2}\right) \ , \tag{29}$$

$$-g_{tt}(r) \simeq g_{rr}(r)^{-1} e^{2\beta Q^2/r^4}$$
$$\simeq 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{2\beta Q^2}{r^4}\left(1 - \frac{M}{r} + \frac{Q^2}{5r^2}\right) \ . \tag{30}$$

We observe here that the $\alpha$ coupling constant does not appear. This is due to the fact that the trace of the electromagnetic energy momentum is zero. In fact, Whitt [24] has shown that for the quadratic theories coupled only to the $\alpha$ term [i.e., $\beta = 0$ in Eq. (2)] there exist a "no hair" theorem stating that the only black hole solution (with spherical symmetry), must be the Reissner-Nordström family. This can be directly seen from our method, since order by order the $\alpha$ contributions vanish, thus leaving us with the Reissner-Nordström solution. This is not the case, of course, when one allows the $\beta$ terms to be different from zero, and expressions Eq. (30) represent the $O(\beta)$ generalized Reissner-Nordström metric.

Direct inspection of metric (30) allows us to see the improved result that generates the perturbative method presented in this paper with respect to the "linearized" approach of Ref. [25]. There, only terms up to $O(r^{-4})$ have been considered.

The validity of this metric will be assured if the condition Eq. (7) holds. In our case, this takes the form $-\beta Q^2/r^4 \ll 1$. Thus, Eq. (30) will be a good approximation to the (even extremely) charged black hole solutions in quadratic theories if

$$r_H \gg \sqrt{-\beta} \ , \tag{31}$$

where $r_H$ is the radial coordinate of the event horizon (in Schwarzschild's gauge).

The geodesic motion acquires an additional small repulsive term in the effective potential since for the no-tachyon constraint $\beta < 0$. This provides, in principle, a way of detecting the physical effects produced by the higher order corrections to the gravitational Lagrangian.

The radial coordinate of the horizon can be computed directly from making vanish $g_{tt}(r_H)$ given by Eq. (30):

$$r_H = r_+ + \beta\frac{Q^2}{r_+^3}\frac{\left(1 - \frac{3Q^2}{5r_+^2}\right)}{\left(1 - \frac{Q^2}{r_+^2}\right)} \ , \quad r_+ = M + \sqrt{M^2 - Q^2} \ . \tag{32}$$

We observe that the quadratic black hole shrinks with respect to the corresponding general relativistic one.

The extreme black hole will be now reached with a maximal charge lower than the general relativistic one ($\beta < 0$):

$$Q_{\max}^2 = M^2 + \tfrac{2}{5}\beta \ ; \tag{33}$$

thus $r_H$ given by Eq. (32) will remain always bounded. In fact [26], $r_H^{\min} = M + \beta/5M$ .

The horizon area $A_H$ will then be given by

$$A_H = 4\pi r_+^2\left[1 + \frac{2\beta Q^2}{r_+^4}\frac{\left(1 - \frac{3Q^2}{5r_+^2}\right)}{\left(1 - \frac{Q^2}{r_+^2}\right)}\right] \ , \tag{34}$$

which is smaller than in the general relativistic case.

The electric potential on the horizon can be independently computed by use of Maxwell equations in curved spacetime Eq. (30):

$$\Phi_H(r_+) = \int_{r_H}^\infty \frac{Q dr}{r^2\sqrt{-g_{tt}g_{rr}}}$$
$$= \frac{Q}{r_+}\left[1 - \frac{2\beta Q^2}{5r_+^4}\left(\frac{3 - 2Q^2/r_+^2}{1 - Q^2/r_+^2}\right)\right] \ . \tag{35}$$

Let us now turn to the thermodynamical properties of black holes and see how they are modified by the quadratic corrections.

We can easily compute the Bekenstein-Hawking temperature from the surface gravity $\kappa$:

$$T_H = \frac{\kappa}{2\pi} = -\frac{1}{4\pi}\frac{g'_{tt}}{\sqrt{-g_{tt}g_{rr}}}\bigg|_{r=r_H} . \qquad (36)$$

Thus,

$$T_H = \frac{1}{4\pi r_+}\left(1-\frac{Q^2}{r_+^2}\right) + \frac{\beta Q^4}{5\pi r_+^7}\left(\frac{2-Q^2/r_+^2}{1-Q^2/r_+^2}\right) . \qquad (37)$$

Since $\beta < 0$, we observe that the effect of the quadratic gravitational theory corrections will be that of decreasing the black hole radiation temperature with respect to the general relativistic value (with the same $M$ and $Q$), leaving thus out open the possibility of switching off black hole evaporation and leaving behind a charged remnant with a mass of the order of the Planck mass.

By inverting Eq. (34) for $M$ as a function of $A_H$ and $Q$. We obtain the fundamental relation of black hole thermodynamics. Differentiation of this equation produces the first law

$$dM = \frac{\partial M}{\partial A}\bigg|_Q dA + \frac{\partial M}{\partial Q}\bigg|_A dQ = T_H dS + \Phi_H dQ \qquad (38)$$

that can be used to obtain the entropy of the quadratic black hole, $S$.

Since

$$\frac{\partial M}{\partial A}\bigg|_Q = \frac{1}{16\pi r_+}\left(1-\frac{Q^2}{r_+^2}\right) \times\left[1+\frac{2\beta Q^2}{r_+^4}\frac{\left(1-\frac{6}{5}\frac{Q^2}{r_+^2}+\frac{3}{5}\frac{Q^4}{r_+^4}\right)}{\left(1-\frac{Q^2}{r_+^2}\right)^2}\right] \qquad (39)$$

and

$$\frac{\partial M}{\partial Q}\bigg|_A = \frac{Q}{r_+} - \beta\frac{Q}{r_+^3}\frac{\left(1-\frac{6}{5}\frac{Q^2}{r_+^2}+\frac{3}{5}\frac{Q^4}{r_+^4}\right)}{\left(1-\frac{Q^2}{r_+^2}\right)} , \qquad (40)$$

we can obtain the entropy of the black hole given $\Phi_H$ and $T_H$ by Eqs. (35)–(37),

$$S = \frac{A}{4} - \frac{8\pi^2\beta Q^2}{A} + \cdots . \qquad (41)$$

We observe here that the entropy of the charged black hole is not simply one-quarter of the area, but in general will be a more complicated function of $A$. The effect of the corrections linear in $\beta$ will then be that of increasing the gravitational entropy by an amount proportional to $Q^2/A$. We can also see that the other parameters characterizing the black hole, such as the charge $Q$ (and the angular momentum if we had considered rotation (see Ref. [25])) enter explicitly in the equation defining the entropy.

## IV. DISCUSSION

In this paper we have presented a perturbative approach to find solutions to the classical field equations of a quadratic (in the curvature) theory of gravitation. Since this theory can be considered a generalization of general relativity, we start the solutions from the general relativistic metric and then add corrections (presumably small) of successive order in the coupling constants $\alpha$ and $\beta$. It is worth to stress here that our method does not obtain all the possible solutions to the quadratic theories, but only those expandable around a metric, solution of the Einstein equations, in powers of the coupling constants $\alpha$ and $\beta$. We have thus analyzed the three possible scenarios of application, cosmology, topological defects, and black holes, and obtained the first-order corrections to the general relativistic results. Our method also allows us to make a systematic study of the higher order corrections by means of symbolic computing programs [26].

The corrected metric (23) may be of relevance to study the evolution of gauge strings in the early Universe for both the structure formation scenario where the $r^{-2}$ term in $B(r)$ could play an important role and for the collision simulations where the short-range contributions may change the predictions of a string network. A detailed study of these effects might provide a link between observation and the $\alpha$ and $\beta$ parameters.

The charged black hole metric in quadratic theories gets only modified by the $\beta$ coupling. The Reissner-Nordström metric is no longer a solution to the problem and one must study a different solution [Eq. (30)]. The thermodynamics of these charged quadratic black holes is different from that of a Reissner-Nordström black hole. The Bekenstein-Hawking temperature as well as the other thermodynamical parameters acquire corrections with respect to its general relativistic values (in Ref. [25] they coincide with those of general relativity) as one would expect for a different theory of gravitation (see Ref. [27] and Ref. [28] for a discussion in the context of string and Kaluza-Klein theories, respectively).

It is also interesting to remark that for the black hole solution, the relation between entropy and area is no longer the simple $S = \frac{1}{4}A$, but that given by Eq. (41). We stress that this modification of the entropy-area relation has been obtained by considering a four-dimensional quadratic gravitational theory minimally coupled to the electromagnetic field. We also expect that in an exact black hole solution this simple equation will break down leaving its place to a more fundamental relation [29,30].

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