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Quantum groups, gravity, and the generalized uncertainty principle

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We investigate the relationship between the generalized uncertainty principle in quantum gravity and the quantum deformation of the Poincaré algebra. We find that a deformed Newton-Wigner position operator and the generators of spatial translations and rotations of the deformed Poincaré algebra obey a deformed Heisenberg algebra from which the generalized uncertainty principle follows. The result indicates that in the κ-deformed Poincaré algebra a minimal observable length emerges naturally.

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I. INTRODUCTION

There are many indications that in quantum gravity there might exist a minimal observable distance on the order of the Planck length. The emergence of a minimal length is usually considered a *dynamical* phenomenon, related to the fact that at the Planck scale there are violent fluctuations of the metric and even topology changes, as in Wheeler space-time foam [1, 2]. In the context of string theories, the emergence of a minimal measurable distance is nicely encoded in a generalized uncertainty principle [3–9]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha G \Delta p, \tag{1}$$

where α is a constant. (We have written explicitly $\hbar$ and G and we have set $c = 1$.) Equation (1) has been obtained from the study of string collisions at Planckian energies, so again it has a *dynamical* origin (although for strings the dynamical and kinematical aspects are strongly correlated). The purpose of this paper is to investigate whether it is possible to understand Eq. (1) at a purely *kinematical* level, independently of any specific dynamical theory.

Our original motivation for this investigation is the fact that in Ref. [10] we obtained Eq. (1) without considering strings, but rather discussing a Gedanken experiment in which the radius of the apparent horizon of a black hole is measured. In this context the generalized uncertainty principle is rediscovered using only very general and model-independent considerations, which would presumably be fulfilled by any candidate quantum theory of gravitation. As a matter of fact, the only physical input is the existence of Hawking radiation [11] emitted by black holes. This fact suggests to look for a mathematical structure, which reproduces Eq. (1) in a natural way. In Ref. [12] we have indeed found that a suitable algebraic structure exists, and it is given by the deformed Heisenberg algebra

$$[X_i, X_j] = -\frac{\hbar^2}{4\kappa^2} i\epsilon_{ijk} J_k, \tag{2}$$

$$[X_i, P_j] = i\hbar\delta_{ij} \left(1 + \frac{\mathbf{P}^2 + m^2}{4\kappa^2}\right)^{1/2}. \tag{3}$$

The commutation relations of the angular momentum J_i with the coordinates X_i and the momenta P_i, as well as between themselves, are the standard ones, $[X_i, J_j] = i\epsilon_{ijk}X_k$, etc. The deformation parameter κ has dimensions of mass, and in the limit $\kappa \to \infty$ the undeformed Heisenberg algebra is recovered. In the following we will identify κ with the Planck mass, times a numerical constant.[1] The algebra defined by Eqs. (2) and (3) is well defined, in the sense that the Jacobi identities are satisfied. Moreover, we found in [12] that the requirement that Jacobi identities are satisfied is so restrictive that, within rather reasonable assumptions, this is the unique possible deformation of the Heisenberg algebra, when the deformation parameter is dimensionful.

From Eq. (3) the generalized uncertainty principle follows:

$$\Delta x_i \Delta p_j \geq \frac{\hbar}{2}\delta_{ij} \left\langle \left(1 + \frac{\mathbf{P}^2 + m^2}{4\kappa^2}\right)^{1/2} \right\rangle. \tag{4}$$

(Here and in the following we denote operators by capital letters and their expectation values with small case letters.) Expanding the square root at lowest order and using $\langle \mathbf{P}^2 \rangle = \mathbf{p}^2 + (\Delta p)^2$ we find

$$\Delta x_i \Delta p_j \geq \frac{\hbar}{2}\delta_{ij} \left(1 + \frac{\mathbf{p}^2 + m^2 + (\Delta p)^2}{8\kappa^2}\right). \tag{5}$$

[1] A comment on the conventions is in order. In the above formulas the deformation parameter always appears in the combination 2κ. Because of this, in [12] we have rescaled κ by a factor of 2. In this paper we do not perform such a rescaling, so that the parameter that we call κ here agrees with the one used in the literature on the κ-deformed Poincaré algebra.

Thus, in the regime $\mathbf{p}^2 + m^2 \ll \kappa^2, \Delta p \lesssim \kappa$ we recover Eq. (1). Instead, in the asymptotic regime $\mathbf{p}^2 \sim (\Delta p)^2 \gg \kappa^2$, Eq. (4) gives

$$\Delta x \geq \text{const} \times \frac{\hbar}{\kappa}\,. \tag{6}$$

The purpose of this paper is to illustrate the relationship between the generalized uncertainty principle and the quantum deformation of the Poincaré algebra. This investigation can be useful because, on the one hand, we can find a kinematic framework in which Eqs. (2) and (3) are satisfied. Independently of whether this specific framework will be relevant or not for quantum gravity, we can expect to gain a better understanding of the physical meaning of the deformed Heisenberg algebra and of the generalized uncertainty principle. On the other hand, such an investigation can be interesting from the point of view of quantum groups [13–18], since it indicates that in such a structure a minimum length is automatically built in (at least when the deformation parameter is dimensionful).

The problem of finding a quantum deformation of the Poincaré group has received much attention recently, and different approaches have been developed. An important line of research is concerned with defining the differential calculus on quantum groups [18–21]; then one can define curvatures through Cartan's equation, and try to construct a q generalization of Einstein action.

A second approach consists in looking for a deformation of the algebra, rather than of the group [22–26]. A very interesting technique, which has been used in this context is the contraction procedure first introduced in [27]. One first considers the q deformation of the anti–de Sitter algebra, U(O(3,2)). This can be done with the standard Drinfeld-Jimbo method [13, 14], and introduces a dimensionless deformation parameter q. Then one sends to infinity the de Sitter radius R, while $q \to 1$ in such a way that $R \ln q \to \kappa^{-1}$, fixed. One therefore recovers a deformation of the $d = 4$ Poincaré algebra, which depends on a *dimensionful* parameter κ. In this way a fundamental length enters the theory.

For our purposes, what is needed is the knowledge of the deformed algebra. We will consider the deformed Poincaré algebra given in [25]; however, our line of reasoning is more general, and could be adapted to different deformations, as long as they introduce a *dimensionful* parameter.

In the following, a fundamental role is played by the Newton-Wigner position operator [28]. In the undeformed case it represents the relativistic position operator of a particle. The main concern of this paper will be to find a proper generalization of this operator to the deformed case. The plan of the paper is as follows. In Sec. II we recall the main results concerning the quantum Poincaré algebra, which will be useful in the following. In Sec. III we discuss the generalization of the Newton-Wigner position operator to the deformed case and in Sec. IV we discuss our results.

II. THE QUANTUM POINCARÉ ALGEBRA

We now briefly recall the main properties of the κ-deformed Poincaré algebra given in [25] (see also [22–24]). All commutators are the same as in the usual Poincaré algebra, except for the boost–boost and boost–3-momentum commutators:

$$[K_i, K_j] = -i\epsilon_{ijk}\left(J_k \cosh\frac{P_0}{\kappa} - \frac{1}{4\kappa^2}P_k \mathbf{P}\cdot\mathbf{J}\right)\,, \tag{7}$$

$$[P_i, K_j] = -i\delta_{ij}\kappa \sinh\frac{P_0}{\kappa}\,. \tag{8}$$

Here P_μ, J_i, K_i are the deformed four-momentum, angular momentum, and boost generators, respectively. In the limit $\kappa \to \infty$ the standard commutators are recovered. The first Casimir operator is [24, 25]

$$C_1 = \mathbf{P}^2 - \left(2\kappa \sinh\frac{P_0}{2\kappa}\right)^2\,, \tag{9}$$

so that the dispersion relation reads

$$\left(2\kappa \sinh\frac{P_0}{2\kappa}\right)^2 = m^2 + \mathbf{P}^2\,. \tag{10}$$

To study unitary representations, one considers the Hilbert space with a positive definite scalar product invariant under κ-deformed Poincaré transformations,

$$\begin{aligned}(\phi,\psi) &= \int \frac{d^4p}{(2\pi)^4}\,\theta(p_0)2\pi\delta\left(\mathbf{p}^2 + m^2 - 4\kappa^2\sinh^2\frac{p_0}{2\kappa}\right)\phi^*(p)\psi(p) \\ &= \int \frac{d^3p}{(2\pi)^3 2\kappa\sinh(p_0/\kappa)}\phi^*(\mathbf{p})\psi(\mathbf{p})\,, \end{aligned} \tag{11}$$

where in the last line p_0 has become a notation for the positive solution of Eq. (10). This scalar product has the correct limit for $\kappa \to \infty$. Limiting ourselves to the spin zero case, so that the term $\mathbf{P}\cdot\mathbf{J}$ in Eq. (7) does not contribute,[2] the representation of the generators of the deformed Poincaré algebra on this Hilbert space reads

$$\begin{aligned} P_\mu &= p_\mu, \\ J_i &= -i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}, \\ K_i &= i\kappa\sinh(\frac{p_0}{\kappa})\frac{\partial}{\partial p_i}\,. \end{aligned} \tag{12}$$

These operators are Hermitian with respect to the scalar product (11).

III. THE NEWTON-WIGNER POSITION OPERATOR

A. The undeformed case

We now wish to represent the relativistic position operator on our Hilbert space. Let us first recall how this is done in the undeformed case. The concept of relativistic position operator was first introduced in a fundamental paper by Newton and Wigner [28], and further discussed by Wightman [29] and Mackey [30]. (For a pedagogical discussion see also [31].) For a massive particle, one considers the Hilbert space of functions with the invariant scalar product

$$(\phi,\psi) = \int \frac{d^3p}{(2\pi)^3 2p_0}\phi^*(\mathbf{p})\psi(\mathbf{p})\,, \tag{13}$$

where p_0 is the positive solution of $p^2 = m^2$. On this space, the generators of the Poincaré group have the well-known realization (limiting ourselves again to the spin zero case)

$$\begin{aligned} P_\mu &= p_\mu, \\ J_i &= -i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}, \\ K_i &= ip_0\frac{\partial}{\partial p_i}\,. \end{aligned} \tag{14}$$

The representation of the Newton-Wigner position operator $Q_i^{(NW)}$ is

$$Q_i^{(NW)} = i\hbar\left(\frac{\partial}{\partial p_i} - \frac{p_i}{2p_0^2}\right)\,. \tag{15}$$

It satisfies the commutation relations

$$\left[Q_i^{(NW)}, Q_j^{(NW)}\right] = 0, \tag{16}$$

$$\left[Q_i^{(NW)}, P_j\right] = i\hbar\delta_{ij}, \tag{17}$$

$$\left[Q_i^{(NW)}, J_j\right] = i\epsilon_{ijk}Q_k^{(NW)}\,. \tag{18}$$

The second term on the right-hand side of Eq. (15) is chosen in such a way that $Q_i^{(NW)}$ is Hermitian with the scalar product given in Eq. (13). In the nonrelativistic limit $\mathbf{Q}^{(NW)} = \hbar\mathbf{K}/m$. The time derivative of $Q^{(NW)}$ in the Heisenberg representation is

$$\frac{d}{dt}Q_i^{(NW)} = \frac{i}{\hbar}\left[P_0, Q_i^{(NW)}\right] = \frac{p_i}{p_0}\,, \tag{19}$$

where the relation $\partial p_0/\partial p_i = p_i/p_0$ has been used, since we are working on-mass shell. Therefore, the time derivative of $Q_i^{(NW)}$ is actually the relativistic velocity of the particle, which is a necessary requirement if we want to identify it with the position operator.

B. The deformed case

We must now find an operator $Q_i^{(\kappa)}$, which generalizes the Newton-Wigner position operator to the κ-deformed case. Two necessary requirements are, first, that it should reduce to $Q_i^{(NW)}$ in the $\kappa \to \infty$ limit and, second, that it should be Hermitian with respect to the scalar product given in Eq. (11). It is also natural to ask that the commutation relations with $\mathbf{J}$ are not modified, so that it remains a vector under space rotations. Then the operator must be of the general form

$$Q_i^{(\kappa)} = i\hbar\left[A\left(\frac{p_0}{\kappa}\right)\frac{\partial}{\partial p_i} - B\left(\frac{p_0}{\kappa}\right)\frac{p_i}{2p_0^2}\right]\,, \tag{20}$$

with $A(0) = B(0) = 1$. The hermiticity condition gives immediately

$$B\left(\frac{p_0}{\kappa}\right) = \frac{p_0^2}{\kappa\sinh p_0/\kappa}\left[\frac{1}{\kappa}\coth\left(\frac{p_0}{\kappa}\right)A\left(\frac{p_0}{\kappa}\right) - \frac{dA}{dp_0}\right]\,. \tag{21}$$

In terms of the function A one computes the following commutators,

$$\left[Q_i^{(\kappa)}, Q_j^{(\kappa)}\right] = -\frac{\hbar^2 A}{\kappa\sinh(p_0/\kappa)}\frac{dA}{dp_0}i\epsilon_{ijk}J_k, \tag{22}$$

$$\left[Q_i^{(\kappa)}, P_j\right] = i\hbar A\delta_{ij}\,. \tag{23}$$

It would be tempting at this stage to set $A(p_0/\kappa) = 1$; the corresponding operator

$$Q_i^{(\kappa)} = i\hbar\left(\frac{\partial}{\partial p_i} - \frac{\cosh(p_0/\kappa)}{2\kappa^2\sinh^2(p_0/\kappa)}p_i\right)\,, \tag{24}$$

is Hermitian with respect to the scalar product (11) and satisfies

$$\left[Q_i^{(\kappa)}, Q_j^{(\kappa)}\right] = 0, \qquad \left[Q_i^{(\kappa)}, P_j\right] = i\delta_{ij}\,. \tag{25}$$

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[2]In Ref. [12] a strong restriction on the possible forms of the deformed algebra was obtained requiring that the Jacobi identities are satisfied independently of whether $\mathbf{P}\cdot\mathbf{J} = 0$ or not. Limiting ourselves to the case $\mathbf{P}\cdot\mathbf{J} = 0$ we might in principle introduce some extra solution. We will see, however, that our final result is the same as the one found in [12].

Before interpreting it as the generalization of the position operator to the κ-deformed case, we must, however, check if its time derivative is the velocity of the particle.

A priori, we do not know how to define the velocity in terms of energy and momentum in the deformed case. In principle, the relation $\mathbf{p} = p_0\mathbf{v}$ can be modified. If we take the time derivative of the operator $Q^{(\kappa)}$ given in Eq. (24) we find

$$\dot{Q}_i^{(\kappa)} = \frac{i}{\hbar}\left[P_0, Q_i^{(\kappa)}\right] = \frac{p_i}{\kappa\sinh(p_0/\kappa)} = \frac{p_i}{\sqrt{m^2+\mathbf{p}^2}\cosh(p_0/2\kappa)}\,, \quad (26)$$

where we have used the dispersion relation (10). We see that, for a particle with mass m, $|\dot{Q}^{(\kappa)}|$ is bounded by

$$|\dot{Q}_i^{(\kappa)}| \le \frac{1}{\cosh[p_0/(2\kappa)]} < \left(1+\frac{m^2}{4\kappa^2}\right)^{-1/2}\,, \quad (27)$$

instead of being allowed to vary between zero and one as we expect for a velocity. Furthermore, as a function of p_0, it reaches a maximum value smaller than one, and then decreases exponentially for large p_0. Even if our understanding of physics at the Planck scale is limited, such a behavior seems rather nonsensical, and suggests that one cannot identify the right-hand side of Eq. (26) with the velocity of a particle. In turn, this means that the operator $Q_i^{(\kappa)}$, which satisfies the undeformed commutation relations (25), cannot represent the relativistic position operator in the κ-deformed theory.

We therefore need a criterium, which allows us to identify the velocity operator. We suggest that the proper deformation of the relation between momentum, velocity, and energy is

$$p_i = 2\kappa\sinh\left(\frac{p_0}{2\kappa}\right)v_i\,. \quad (28)$$

This assumption is rather natural, since it just amounts to the replacement $p_0 \to 2\kappa\sinh(p_0/2\kappa)$, which is the same that takes place in the Casimir operator. The relation (28) has the correct undeformed limit, and v is allowed to vary between zero and one and is a monotonic function of energy. In fact, eliminating $\sinh(p_0/2\kappa)$ with the use of the dispersion relation, one finds

$$p_i = \gamma m v_i, \qquad \gamma = (1-v^2)^{-1/2}\,, \quad (29)$$

so that this classical relation is not deformed.

It is easy to see that, if we require the time derivative of the position operator to be $p_i/(2\kappa\sinh\frac{p_0}{2\kappa})$, we get

$$A\left(\frac{p_0}{\kappa}\right) = \cosh\frac{p_0}{2\kappa}\,. \quad (30)$$

The function B then follows from Eq. (21). We are therefore lead to propose the following generalization of the Newton-Wigner position operator, which we denote X_i:

$$X_i = i\hbar\cosh\frac{p_0}{2\kappa}\left(\frac{\partial}{\partial p_i} - \frac{p_i}{8\kappa^2\sinh^2(p_0/2\kappa)}\right)\,, \quad (31)$$

which is Hermitian with respect to the scalar product (11), has the classical commutation relations with J_i and satisfies $\dot{X} = v$, with v defined by Eq. (29). It is now straightforward to compute the $[X, X]$ and $[X, P]$ commutators:

$$[X_i, X_j] = -\frac{\hbar^2}{4\kappa^2}i\epsilon_{ijk}J_k\,, \quad (32)$$

$$[X_i, P_j] = i\hbar\delta_{ij}\cosh\frac{P_0}{2\kappa}\,. \quad (33)$$

Using the dispersion relation, Eq. (10), we see that this is just the algebra given in Eqs. (2) and (3). Note that Eq. (3) is written in a form that is independent of the specific dispersion relation. In the κ-Poincaré algebra it takes the form (33), but we can as well consider Eqs. (2,3) within the standard Poincaré group, and then $\mathbf{P}^2 + m^2 = E^2$.

The result that we have obtained does not come out as a surprise, since we have shown in Ref. [12] that the κ deformation of the Heisenberg algebra is (essentially[3]) unique.

The fact that $[X_i, X_j]$ is nonzero is consistent with the spirit of noncommutative geometry [32], which is at the basis of the quantum group approach to physics at the Planck scale [33]. The noncommutativity shows up only at length scales on the order of the Planck length. It is also important to observe that the deformed Heisenberg algebra ties the generalized uncertainty principle with noncommutativity of space-time at very short distances.

The deformation constant κ can be estimated if we assume that the uncertainty principle obtained from quantum groups at lowest order in $\Delta p/\kappa$ and $E \ll \kappa$, which for $i = j$ reads

$$\Delta x\Delta p \ge \frac{\hbar}{2}\left(1+\frac{(\Delta p)^2}{8\kappa^2}+\cdots\right)\,, \quad (34)$$

agrees with the one found in string theory, which reads (apart from numerical constants of order one)

$$\Delta x\Delta p \ge \frac{1}{2}[\hbar + \alpha'(\Delta p)^2]\,. \quad (35)$$

Here α' is the inverse string tension, $\alpha' = \lambda_s^2/(2\hbar)$, and λ_s is the quantization constant of string theory; its relation to the Planck length $L_{\rm Pl}$ is somewhat model dependent. In heterotic string theory $L_{\rm Pl} = \alpha_{\rm GUT}\lambda_s/4$. In this case, therefore, the comparison suggests

$$\kappa \sim \frac{1}{8}\alpha_{\rm GUT}M_{\rm Pl} \sim \left(10^{-2}-10^{-3}\right)M_{\rm Pl}\,. \quad (36)$$

[3]In Ref. [12] we also found a second possible solution of the Jacobi identities, of the form $[X_i, P_j] = i\hbar\delta_{ij}\{1-(\mathbf{P}^2+m^2)/(4\kappa^2)\}^{1/2}$. It is easy to see that one obtains this algebra using the κ-deformed Poincaré algebra suggested in Refs. [22, 24] instead of the one suggested in Ref. [25], since the former can be obtained from the latter with the formal replacement $\kappa \to i\kappa$.

IV. DISCUSSION

We have found that the κ-deformed Poincaré algebra provides an explicit realization of the κ-deformed Heisenberg algebra, Eqs. (2) and (3), once X is identified with a suitably deformed Newton-Wigner position operator. Other definitions of the deformed Newton-Wigner position operator are possible (a different definition is suggested in [34]). This ambiguity is due to the fact that the definition of velocity in terms of energy and momentum in the κ-deformed theory is not fixed *a priori*, the only requirement being that it should reduce to the classical relation as $\kappa \to \infty$. Our definition is dictated by the choice that the relation $p = \gamma mv$ is not deformed, see Eq. (29).

A very relevant feature of the κ-deformed Heisenberg algebra is the fact that it is not compatible with exact Lorentz invariance at the Planck scale, as it is clear from the fact that it implies the existence of a minimal *spatial* length— a concept that is obviously non-Lorentz-invariant. The fact that at the Planck scale Lorentz boost should saturate has been suggested recently by Susskind [35]. The κ-Poincaré group provides an explicit example of a kinematical framework in which Lorentz transformations are modified. In this case Lorentz invariance is broken by the parameter κ. Note also that there is no κ-deformed Lorentz subalgebra of the κ-Poincaré algebra, since the boost-boost commutator involves the momentum. This explicit example also shows clearly how a fundamental length can emerge at a purely kinematical level.

The fact that quantum groups can provide the kinematical framework of physical systems was already realized in [36, 37]. The authors of this very interesting work consider the propagation of phonons in a harmonic crystal in 1+1 dimensions, and discover that the $d = 2$ κ-deformed Poincaré algebra is its kinematical symmetry. Our approach could be considered complementary to theirs, since we rather start with a κ-deformed algebra and discover that a minimal length emerges.

The emergence of a minimal length obviously has important consequences also concerning the possibility that quantum groups provide a natural ultraviolet cutoff mechanism for quantum field theory [38].

Finally, we note that Eq. (4) agrees with Eq. (1) only at lowest order in $\Delta p/\kappa$. In particular, asymptotically Eq. (4) gives $\Delta x \geq \hbar/\kappa$, while Eq. (1) gives $\Delta x \geq \hbar\Delta p/\kappa^2$. It is easy to see why the arguments presented in [10] fail in the region $\Delta p \gg \kappa$. In our Gedanken experiment Δp was on the order of the energy of the particle used to probe the black hole; and we cannot treat semiclassically a particle with super-Planckian energy. It would be interesting to see if higher-order terms in $\Delta p/\kappa$ can be obtained in the string theoretic derivation of Eq. (1).

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