Metric of a rotating, charged, magnetized, deformed mass

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An exact asymptotically flat five-parameter solution of the Einstein-Maxwell equations generalizing the well-known Kerr-Newman metric is obtained in an explicit form. In addition to the independent parameters of mass, angular momentum, and electric charge it also contains two other arbitrary parameters associated with the magnetic dipole and mass-quadrupole moments of the source. An important peculiar feature of this solution that distinguishes it from other solutions for a magnetized rotating mass recently discussed in the literature is its symmetry with respect to the equatorial plane of the source in the general case.

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Recently several exact asymptotically flat solutions of the Einstein-Maxwell equations able to describe the exterior gravitational fields of magnetized stationary massive sources have been obtained [1] with the aid of the method [2] based on the use of the integral equation whose nucleus is constructed from the Ernst potentials [3] defined at the symmetry axis and extended to the complex plane of an analytical parameter. Although all the solutions from Ref. [1] contain arbitrary parameters of mass, angular momentum, and magnetic dipole moment, their multipole structures are different due to different distributions of higher multipole moments determined by the particular form of each solution at the symmetry axis. It should be mentioned that these solutions are not symmetric with respect to the equatorial plane since, in addition to the magnetic dipole moment, they also contain the magnetic quadrupole component which should not appear in the metrics possessing the reflection symmetry. This fact, being non-negative for the physical interpretation of the solutions obtained (as is known [4], real astrophysical objects such as neutron stars can have the magnetic quadrupole moment), still puts forward the problem of finding the solutions for a magnetized rotating mass which would be symmetric about the equatorial plane of the source because such solutions are expected to have a simpler explicit form, and are, thus, more suitable for the modeling of the exterior fields of stationary magnetized astrophysical objects.

The aim of our paper is to consider the simplest possible five-parameter solution of the Einstein-Maxwell equations able to describe the exterior field of a rotating, charged, magnetized, deformed source which would possess the reflection symmetry in the general case. The new solution generalizes the well-known Kerr-Newman metric [5] for a charged rotating mass and contains two additional arbitrary parameters associated with the magnetic dipole and mass-quadrupole moments of the source.

The reported solution is defined at the symmetry axis by two complex Ernst potentials of the form

$$\mathcal{E}(\rho=0,z)\equiv e(z)=\frac{z(z-m-ia)+b}{z(z+m-ia)+b}\,,$$
$$\Phi(\rho=0,z)\equiv F(z)=\frac{qz+ic}{z(z+m-ia)+b}\,, \tag{1}$$

where ρ and z are the Weyl-Papapetrou cylindrical coordinates, and a, b, c, m, and q are arbitrary real constants. With $b=c=0$ from (1) follow the expressions for the Ernst potentials of the Kerr-Newman solution taken at the symmetry axis ($\rho=0$), so that formulas (1) contain naturally and in the simplest possible way two new physically very important parameters describing the mass-quadrupole and magnetic dipole moments of the source (b and c, respectively), the resulting stationary electrovac solution not losing, as will be seen later on, the property of being symmetric with respect to the equatorial plane ($z=0$).

We shall now use the method [2] to construct the complex potentials $\mathcal{E}$ and Φ which satisfy the Ernst equations [3], and whose behavior at the symmetry axis is defined by (1). Recall that the required potentials can be obtained from the integrals

$$\mathcal{E}=\frac{1}{\pi}\int_{-1}^{1}\frac{\mu(\sigma)e(\xi)d\sigma}{\sqrt{1-\sigma^2}},\quad \Phi=\frac{1}{\pi}\int_{-1}^{1}\frac{\mu(\sigma)F(\xi)d\sigma}{\sqrt{1-\sigma^2}}\,, \tag{2}$$

where $e(\xi)$ and $F(\xi)$ are the local holomorphic continuations of the functions $e(z)$ and $F(z)$ to the complex plane; $\xi\equiv z+i\rho\sigma,\sigma\in[-1,1]$, and the unknown function $\mu(\sigma)$ satisfies the integral equation

$$\fint_{-1}^{1} \frac{[e(\xi)+\tilde{e}(\eta)+2F(\xi)\tilde{F}(\eta)]\mu(\sigma)d\sigma}{(\xi-\eta)\sqrt{1-\sigma^2}}=0 \quad (3)$$

and the normalizing condition

$$\int_{-1}^{1} \frac{\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}}=\pi \; . \quad (4)$$

In (3), $\tilde{e}(\eta)\equiv[e(\eta^*)]^*$, $\eta\equiv z+i\rho\tau$, $\tau\in[-1,1]$, and $\fint$ denotes the principal value of the integral.

The form in which one has to search the function $\mu(\sigma)$ depends on the roots of the algebraic equation

$$e(\xi)+\tilde{e}(\xi)+2F(\xi)\tilde{F}(\xi)=0 \quad (5)$$

and, in our case, as follows from (1), Eq. (5) is the biquadratic one, i.e.,

$$\xi^4+(a^2+q^2+2b-m^2)\xi^2+b^2+c^2=0 \quad (6)$$

with the roots

$$\alpha_1=-\alpha_2=(\kappa_+ +\kappa_-)/2, \quad \alpha_3=-\alpha_4=(\kappa_+ -\kappa_-)/2 \; ,$$
$$\kappa_\pm\equiv[m^2-a^2-q^2+2(\pm d-b)]^{1/2}, \quad d\equiv(b^2+c^2)^{1/2} \; . \quad (7)$$

The existence of simple roots of Eq. (5) is a remarkable distinctive feature of our choice of the functions $e(z)$ and $f(z)$ possessing five arbitrary parameters in the form (1) (in the known magnetized generalization of the Kerr-Newman solution from Ref. [1] an analogous algebraic equation of the fourth order has very complicated roots which are not given explicitly in that paper). It also should be mentioned that the absence of all odd terms of ξ in Eq. (6) is a sufficient condition for the solution we are looking for to be symmetric with respect to the equatorial plane, but we leave the discussion of the general questions related to the construction of the solutions with the reflection symmetry for the future [6].

Thus, the expression for $\mu(\sigma)$ must have the form [2]

$$\mu(\sigma)=A_0+\sum_{n=1}^{4} A_n(\xi-\alpha_n)^{-1} \; , \quad (8)$$

where the coefficients A_0 and A_n depending on ρ and z can be found from Eqs. (3) and (4) (they enter into the latter equations as parameters).

The substitution of (8) into (3) and (4) leads, after the evaluation of the integrals and equation of the coefficients at the independent powers of η to zero, to the following linear system of five algebraic equations for the determination of A_0 and A_n:

$$A_0+\sum_{n=1}^{4}\frac{A_n}{r_n}=1 \; ,$$
$$A_0+\sum_{n=1}^{4}\frac{A_n}{\beta_+-\alpha_n}=0 \; ,$$
$$A_0+\sum_{n=1}^{4}\frac{A_n}{\beta_--\alpha_n}=0 \; , \quad (9)$$
$$\sum_{n=1}^{4}\frac{c_nA_n}{(\alpha_nm_n+b)r_n}=0 \; ,$$
$$\sum_{n=1}^{4}\frac{d_nA_n}{(\alpha_nm_n+b)r_n}=0 \; , \quad r_n\equiv\sqrt{\rho^2+(z-\alpha_n)^2} \; ,$$

where we have introduced

$$c_n\equiv[m\beta_+^*(\alpha_nm_n+b)-(q\beta_+^*-ic)(q\alpha_n+ic)]/(\beta_+^*-\alpha_n) \; ,$$
$$d_n\equiv[m\beta_-^*(\alpha_nm_n+b)-(q\beta_-^*-ic)(q\alpha_n+ic)]/(\beta_-^*-\alpha_n) \; , \quad (10)$$
$$m_n\equiv m+\alpha_n-ia \; ,$$

$\beta_\pm$ being the poles of the denominator of the function $e(\xi)$ (an asterisk denotes the complex conjugation),

$$\beta_\pm\equiv\tfrac{1}{2}[-m+ia\pm\sqrt{(m-ia)^2-4b}\,] \; . \quad (11)$$

The solution of the system (9) has the form

$$A_0=\frac{A}{A+B}, \quad A_n=\frac{(\alpha_nm_n+b)r_nN_n}{A+B}, \quad n=1,2,3,4 \; ,$$
$$A\equiv(\alpha_1-\alpha_2)c_{(43)}r_1r_2+(\alpha_1-\alpha_3)c_{(24)}r_1r_3+(\alpha_2-\alpha_3)c_{(41)}r_2r_3$$
$$+(\alpha_1-\alpha_4)c_{(32)}r_1r_4+(\alpha_4-\alpha_2)c_{(31)}r_2r_4+(\alpha_4-\alpha_3)c_{(12)}r_3r_4 \; ,$$
$$B\equiv m\{[\alpha_2c_{(34)}+\alpha_3c_{(42)}+\alpha_4c_{(23)}]r_1+[\alpha_1c_{(43)}+\alpha_3c_{(14)}+\alpha_4c_{(31)}]r_2$$
$$+[\alpha_1c_{(24)}+\alpha_2c_{(41)}+\alpha_4c_{(12)}]r_3+[\alpha_1c_{(32)}+\alpha_2c_{(13)}+\alpha_3c_{(21)}]r_4\} \; , \quad (12)$$
$$N_1\equiv c_{(43)}r_2+c_{(24)}r_3+c_{(32)}r_4, \quad N_2\equiv c_{(34)}r_1+c_{(41)}r_3+c_{(13)}r_4 \; ,$$
$$N_3\equiv c_{(42)}r_1+c_{(14)}r_2+c_{(21)}r_4, \quad N_4\equiv c_{(23)}r_1+c_{(31)}r_2+c_{(12)}r_3, \quad c_{kl}\equiv c_kd_l-c_ld_k \; .$$

Further, we find from (2) and (8) the form of the potentials $\mathcal{E}$ and Φ in terms of A_0 and A_n,

$$\mathcal{E}=1-\sum_{n=1}^{4}\frac{2m\alpha_n A_n}{(\alpha_n m_n+b)r_n},\quad \Phi=\sum_{n=1}^{4}\frac{(q\alpha_n+ic)A_n}{(\alpha_n m_n+b)r_n} \tag{13}$$

whence, accounting for (12), we find the final expressions for $\mathcal{E}$ and Φ:

$$\begin{aligned}&\mathcal{E}=\frac{A-B}{A+B},\quad \Phi=-\frac{C}{A+B},\\ &C\equiv[(q\alpha_2+ic)c_{(34)}+(q\alpha_3+ic)c_{(42)}+(q\alpha_4+ic)c_{(23)}]r_1+[(q\alpha_1+ic)c_{(43)}+(q\alpha_3+ic)c_{(14)}+(q\alpha_4+ic)c_{(31)}]r_2\\ &\quad+[(q\alpha_1+ic)c_{(24)}+(q\alpha_2+ic)c_{(41)}+(q\alpha_4+ic)c_{(12)}]r_3+[(q\alpha_1+ic)c_{(32)}+(q\alpha_2+ic)c_{(13)}+(q\alpha_3+ic)c_{(21)}]r_4\ .\end{aligned} \tag{14}$$

Formulas (14) and (12) define the desired solution of the Ernst equations whose behavior at the symmetry axis is given by (1).

The potentials $\mathcal{E}$ and Φ from (14) can now be used to obtain the corresponding metric functions f, γ, and ω entering into the Papapetrou stationary axisymmetric line element [7]:

$$ds^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\varphi^2]-f(dt-\omega\, d\varphi)^2\ . \tag{15}$$

The function f is simply $\mathrm{Re}\mathcal{E}+\Phi\Phi^*$ [3], while the discussion of the derivation of the functions γ and ω can be found in Refs. [8,9]. Below we give the final expressions for f, γ, and ω, which turn out to have the form

$$\begin{aligned}&f=\frac{AA^*-BB^*+CC^*}{(A+B)(A^*+B^*)},\quad e^{2\gamma}=\frac{AA^*-BB^*+CC^*}{K_0r_1r_2r_3r_4},\\ &\omega=-f^{-1}\left\{i\left[-m+\sum_{n=1}^{4}\left[1-\frac{z-\alpha_n}{r_n}\right]A_n\right]-\sum_{n=1}^{4}\frac{m\alpha_nB_n^*}{(\alpha_nm_n^*+b)r_n}+\left[iq+\sum_{n=1}^{4}\frac{(q\alpha_n-ic)B_n^*}{(\alpha_nm_n^*+b)r_n}\right]\Phi\right\},\end{aligned} \tag{16}$$

where K_0 is the constant which can be found from the regularity condition $\gamma|_{\rho=0}=0$, and B_n satisfy the system of linear algebraic equations which follow from the system (9) by changing the right-hand sides of Eqs. (9), respectively, to

$$iz,\quad i\beta_+,\quad i\beta_-,\quad -im\beta_+^*,\quad -im\beta_-^* \tag{17}$$

and substituting B_0 instead of A_0, and B_n instead of A_n. The form of B_n is

$$\begin{aligned}&B_n=\frac{i(\alpha_nm_n+b)r_nL_n}{A+B},\quad n=1,2,3,4\ ,\\[1ex]
&L_1\equiv m[(\alpha_3-\alpha_2)b_4r_2r_3+(\alpha_2-\alpha_4)b_3r_2r_4+(\alpha_4-\alpha_3)b_2r_3r_4]\\
&\quad+\{m[(\alpha_3m_3+b)b_4-(\alpha_4m_4+b)b_3]+c_{(43)}(z+m_2)\}r_2\\
&\quad-\{m[(\alpha_2m_2+b)b_4-(\alpha_4m_4+b)b_2]+c_{(42)}(z+m_3)\}r_3\\
&\quad+\{m[(\alpha_2m_2+b)b_3-(\alpha_3m_3+b)b_2]+c_{(32)}(z+m_4)\}r_4\\
&\quad+(\alpha_2m_2+b)c_{(43)}+(\alpha_3m_3+b)c_{(24)}+(\alpha_4m_4+b)c_{(32)}\ ,\\[1ex]
&L_2\equiv m[(\alpha_1-\alpha_3)b_4r_1r_3+(\alpha_4-\alpha_1)b_3r_1r_4+(\alpha_3-\alpha_4)b_1r_3r_4]\\
&\quad-\{m[(\alpha_3m_3+b)b_4-(\alpha_4m_4+b)b_3]+c_{(43)}(z+m_1)\}r_1\\
&\quad+\{m[(\alpha_1m_1+b)b_4-(\alpha_4m_4+b)b_1]+c_{(41)}(z+m_3)\}r_3\\
&\quad-\{m[(\alpha_1m_1+b)b_3-(\alpha_3m_3+b)b_1]+c_{(31)}(z+m_4)\}r_4\\
&\quad+(\alpha_1m_1+b)c_{(34)}+(\alpha_3m_3+b)c_{(41)}+(\alpha_4m_4+b)c_{(13)}\ ,\\[1ex]
&L_3\equiv m[(\alpha_2-\alpha_1)b_4r_1r_2+(\alpha_1-\alpha_4)b_2r_1r_4+(\alpha_4-\alpha_2)b_1r_2r_4]\\
&\quad+\{m[(\alpha_2m_2+b)b_4-(\alpha_4m_4+b)b_2]+c_{(42)}(z+m_1)\}r_1\\
&\quad-\{m[(\alpha_1m_1+b)b_4-(\alpha_4m_4+b)b_1]+c_{(41)}(z+m_2)\}r_2\\
&\quad+\{m[(\alpha_1m_1+b)b_2-(\alpha_2m_2+b)b_1]+c_{(21)}(z+m_4)\}r_4\\
&\quad+(\alpha_1m_1+b)c_{(42)}+(\alpha_2m_2+b)c_{(14)}+(\alpha_4m_4+b)c_{(21)}\ ,\end{aligned} \tag{18}$$

$$L_4 \equiv m[(\alpha_1-\alpha_2)b_3r_1r_2+(\alpha_3-\alpha_1)b_2r_1r_3+(\alpha_2-\alpha_3)b_1r_2r_3]$$
$$-\{m[(\alpha_2m_2+b)b_3-(\alpha_3m_3+b)b_2]+c_{(32)}(z+m_1)\}r_1$$
$$+\{m[(\alpha_1m_1+b)b_3-(\alpha_3m_3+b)b_1]+c_{(31)}(z+m_3)\}r_2$$
$$-\{m[(\alpha_1m_1+b)b_2-(\alpha_2m_2+b)b_1]+c_{(21)}(z+m_3)\}r_3$$
$$+(\alpha_1m_1+b)c_{(23)}+(\alpha_2m_2+b)c_{(31)}+(\alpha_3m_3+b)c_{(12)} ,$$
$$b_n \equiv \beta_-^* c_n - \beta_+^* d_n ,$$

and one can see that the expressions for L_n are more complicated than the respective expressions for N_n in (12).

Therefore, formulas (16), (18), (14), and (12) fully determine the new stationary electrovacuum metric describing the exterior gravitational field of an axisymmetric rotating, charged, magnetized, massive source which is symmetric with respect to the equatorial plane and possesses an arbitrary mass-quadrupole moment. The multipole structure of the solution obtained, as well as the physical sense of its parameters, are well illustrated by the first four relativistic Simon's multipole moments [10] calculated from (1) with the aid of the Hoenselaers-Perjés procedure [11]

$$\begin{aligned} &M_0=m, \quad M_1=0, \quad M_2=-m(a^2+b), \quad M_3=0 , \\ &J_0=0, \quad J_1=ma, \quad J_2=0, \quad J_3=ma(a^2+2b) , \\ &Q_0=q, \quad Q_1=0, \quad Q_2=-q(a^2+b)-ac, \quad Q_3=0 , \\ &\mu_0=0, \quad \mu_1=aq, \quad \mu_2=0, \\ &\mu_3=-c(a^2+b)-qa(a^2+2b) \end{aligned} \tag{19}$$

(M_i, J_i, Q_i, and μ_i describe, respectively, the distributions of mass, angular momentum, electric charge, and magnetic moment), whence it follows that the solution is asymptotically flat ($J_0=0$), and the parameters m, a, q, c, and b define, respectively, total mass, total angular momentum per unit mass, total charge, magnetic dipole, and mass-quadrupole moments of the source. Because of the symmetry of the solution with respect to the equatorial plane, all its odd mass and electric multipoles M_{2k+1} and Q_{2k+1}, and all even angular momentum and magnetic multipoles J_{2k} and μ_{2k}, $k=0,1,\ldots$, are equal to zero, considerably simplifying the investigation of the physical properties of this spacetime.

We believe that the metric obtained may be potentially important for relativistic astrophysics since it contains all the necessary parameters and admits all the required limits to model correctly the exterior fields of the magnetized rotating objects such as, e.g., neutron stars. It should be mentioned that the potentials $\mathcal{E}$ and Φ defined by (14) and (12) have a remarkably simple form containing only the terms r_ir_j and r_j with some coefficients (in a recent solution [9] the mass-quadrupole parameter causes the respective function $\mathcal{E}$ to contain the terms of the fourth order in r) that is the consequence of a very fortunate form of the functions $e(z)$ and $F(z)$ in (1), which has been discovered only after some other possibilities (more complicated, as can be seen now) were analyzed [1,9]. It should be mentioned also that the existence of the equatorial plane in the solution considered above may be advantageous for the consideration of the relativistic effects in this new spacetime (now, for instance, it is possible to consider the motion of test particles in the equatorial plane, unlike in the case of solutions which have no reflection symmetry).

It is interesting that Eqs. (14) and (16) defining the Ernst potentials of the new solution and the corresponding metric functions admit further simplifications. Up to now we have been able to obtain the simplified formulas for all the particular cases of the potentials $\mathcal{E}$ and Φ, the main three of which are (a) the magnetic generalization of the Kerr-Newman solution ($b=0$), (b) the Kerr-Newman solution endowed with an arbitrary mass-quadrupole moment ($c=0$), and (c) a generalization of the Kerr solution possessing an arbitrary magnetic dipole and mass-quadrupole moments ($q=0$). In what follows we shall give the resulting simplified expressions for the above three cases and discuss some other limits of the solution (14) which are a direct consequence of the formulas obtained.

Case (a) has already been discussed independently in Ref. [12]. However, since it can also be obtained from the general formulas defining the solution (14) below we shall write down the potentials $\mathcal{E}$ and Φ defining the simplest magnetized Kerr-Newman solution bringing the notation of the parameters in accordance with the one used in Eq. (1):

$$\begin{aligned}
&\mathcal{E}=\frac{A-B}{A+B}, \quad \Phi=-\frac{C}{A+B}, \\
&A\equiv\kappa_-^2[(\kappa_+^2-aq-c)(R_-r_-+R_+r_+)+i\kappa_+(a+q)(R_-r_--R_+r_+)] \\
&\qquad+\kappa_+^2[(\kappa_-^2+aq+c)(R_-r_++R_+r_-)+i\kappa_-(a-q)(R_-r_+-R_+r_-)]-4c(aq+c)(R_-R_++r_-r_+), \\
&B\equiv m\kappa_-\kappa_+\{(m^2-a^2-q^2)(r_-+r_+-R_--R_+)+\kappa_-\kappa_+(r_-+r_++R_-+R_+) \\
&\qquad+iq[(\kappa_++\kappa_-)(r_--r_+)+(\kappa_+-\kappa_-)(R_+-R_-)]\}, \\
&C\equiv\kappa_-\kappa_+\{[q(m^2-a^2-q^2)-2ac](R_-+R_+-r_--r_+)-q\kappa_-\kappa_+(R_-+R_++r_-+r_+) \\
&\qquad+i[\kappa_+(q^2+c)(R_--R_++r_+-r_-)+\kappa_-(q^2-c)(R_+-R_-+r_+-r_-)]\}, \\
&R_\pm\equiv\sqrt{\rho^2+[z\pm(\kappa_++\kappa_-)/2]^2}, \quad r_\pm\equiv\sqrt{\rho^2+[z\pm(\kappa_+-\kappa_-)/2]^2}, \\
&\kappa_\pm\equiv\sqrt{m^2-a^2-q^2\pm 2c}\;.
\end{aligned} \tag{20}$$

The Kerr-Newman solution is easily recovered from (20) by simply setting $c=0$. Other nontrivial limits of the solution (20) are the magnetized Kerr solution ($q=0$), the magnetized Reissner-Nordström solution ($a=0$), the magnetized Schwarzschild solution ($q=a=0$), and the solution for a massless magnetic dipole ($m=a=q=0$). In the latter case the potentials $\mathcal{E}$ and Φ assume an extremely simple form: i.e.,

$$\begin{aligned}
&\mathcal{E}=1, \quad \Phi=\frac{i\sqrt{2c}\,[(1-i)(R_+-R_-)+(1+i)(r_+-r_-)]}{(R_++R_-)(r_++r_-)}, \\
&R_\pm\equiv\left[\rho^2+\left[z\pm\frac{\sqrt{2c}}{2}(1+i)\right]^2\right]^{1/2}, \quad r_\pm\equiv\left[\rho^2+\left[z\pm\frac{\sqrt{2c}}{2}(1-i)\right]^2\right]^{1/2}.
\end{aligned} \tag{21}$$

A remarkable feature of the solution (21) is that it has the only nonzero relativistic multipole moment (the magnetic dipole moment $\mu_1=c$), being an exact version of the classical pure magnetic dipole potential in general relativity.

In case (b) the simplified potentials $\mathcal{E}$ and Φ are the following [here and later on we are using the same notations as in the formulas (20) to define $\mathcal{E}$ and Φ giving in each particular case the new expressions of the functions involved]:

$$\begin{aligned}
&\mathcal{E}=\frac{A-B}{A+B}, \quad \Phi=\frac{C}{A+B}, \\
&A\equiv\kappa_-^2[(m^2-2a^2-q^2)(R_-r_-+R_+r_+)+2ia\kappa_+(R_-r_--R_+r_+)] \\
&\qquad+(m^2-q^2)[\kappa_+^2(R_-r_++R_+r_-)-4b(R_-R_++r_-r_+)], \\
&B\equiv m\kappa_-\kappa_+\{(m^2-a^2-q^2)(r_-+r_+-R_--R_+)+\kappa_-\kappa_+(r_-+r_++R_-+R_+) \\
&\qquad+ia[(\kappa_++\kappa_-)(r_--r_+)+(\kappa_+-\kappa_-)(R_+-R_-)]\}, \\
&C\equiv qB/m, \\
&\kappa_+\equiv\sqrt{m^2-a^2-q^2}, \quad \kappa_-\equiv\sqrt{m^2-a^2-q^2-4b},
\end{aligned} \tag{22}$$

where the dependence of $R_\pm$ and $r_\pm$ on $\kappa_\pm$ is the same as Eqs. (20).

The formulas (22) define the Kerr-Newman solution possessing an arbitrary mass-quadrupole moment that turns out to be much simpler than some other known analogous generalizations of the Kerr-Newman solution earlier obtained and containing the exponential functions of the coordinates ρ and z [13,14].

By setting in Eqs. (22) $q=0$, $a=0$, or $a=q=0$ one obtains, respectively, the generalizations of the Kerr [15], Reissner-Nordström [16], or Schwarzschild solutions possessing an arbitrary mass-quadrupole moment.

Case (c) appears to be the most important from the physical point of view since it allows us to describe the exterior field of a rotating magnetized source possessing arbitrary magnetic dipole and mass-quadrupole moments, thus containing four essential parameters defining the field of an axisymmetric neutron star. In contradistinction to the four-parameter solution [9] whose Ernst potentials have a complicated form, the solution given below is determined by remarkably compact expressions involving only the explicit form of the parameters m, a, b, and c: i.e.,

$$
\begin{aligned}
&\mathcal{E}=\frac{A-B}{A+B}, \quad \Phi=\frac{C}{A+B}, \\
&A \equiv \kappa_-^2\{[(m^2-a^2)d-a^2b+c^2](R_-r_-+R_+r_+)+ia\kappa_+(b+d)(R_-r_--R_+r_+)\} \\
&\quad +\kappa_+^2\{[(m^2-a^2)d+a^2b-c^2](R_-r_++R_+r_-)+ia\kappa_-(b-d)(R_+r_--R_-r_+)\} \\
&\quad -4d(c^2+m^2b)(R_-R_++r_-r_+), \\
&B \equiv m\kappa_-\kappa_+\{d[(m^2-a^2)(r_-+r_+-R_--R_+)+\kappa_-\kappa_+(r_-+r_++R_-+R_+)] \\
&\qquad +iab[(\kappa_++\kappa_-)(r_--r_+)+(\kappa_+-\kappa_-)(R_+-R_-)]\}, \qquad (23) \\
&C \equiv ic\kappa_-\kappa_+\{(\kappa_++\kappa_-)[b(r_--r_+)+d(R_+-R_-)] \\
&\qquad +(\kappa_+-\kappa_-)[b(R_+-R_-)+d(r_--r_+)]+2iad(r_-+r_+-R_--R_+)\}, \\
&\kappa_\pm \equiv \sqrt{m^2-a^2+2(\pm d-b)}, \quad d\equiv\sqrt{b^2+c^2},
\end{aligned}
$$

$R_\pm$ and $r_\pm$ similar to the previous case relating to $\kappa_\pm$ as in the formulas (20).

Most limits of the solution (23) are apparently the same as already mentioned above for the cases (a) and (b) since with a particular choice of the parameters the solutions (20), (22), and (23) "intersect." The only limit of solution (23) that cannot be obtained from formulas (20) and (22) corresponds to the case $a=0$ representing a magnetized Schwarzschild mass possessing an arbitrary quadrupole deformation, the resulting solution being magnetostatic.

As a concluding remark we would like to say that it seems likely to obtain the analogous simplified formulas for the general case of the potentials (14) and metric coefficients (16) to make the new space-time suitable for quick estimates. We hope to be able to give the desired simplified form of the solution considered in the present paper in a forthcoming publication.

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