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Classical solutions in three-dimensional cosmological gravity

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Solutions, depending on only one variable, to three-dimensional cosmological gravity are shown to be geodesics of an abstract three-dimensional Minkowski space. These geodesics are timelike, lightlike, or spacelike for positive, zero, or negative values of the cosmological constant. The singularity structure of the solutions depends on the position of the associated geodesics relative to the light cone in solution space. The extension to the case of three-dimensional cosmological gravity with field-theoretical sources is briefly discussed.

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In three space-time dimensions, the Riemann curvature tensor is uniquely determined by the Einstein tensor, so that Einstein's equations with a (positive or negative) cosmological constant Λ are solved by space-times of constant curvature. It is well known [1] that all constant curvature spaces of a given signature and curvature are locally diffeomorphic, the equivalence classes being in the case of Minkowskian signature de Sitter space for $\Lambda>0$, Minkowski space for $\Lambda=0$, and anti–de Sitter space for $\Lambda<0$. However, they can be globally inequivalent owing either to the presence of point sources, or to inequivalent topologies (e.g., wormholes). In a now classic paper [2], Deser and Jackiw constructed static solutions to the cosmological Einstein equations with several point sources. Recently, interest in the subject has been rekindled by the construction of a black-hole stationary solution to cosmological gravity with $\Lambda<0$ [3,4]. The research which shall be reported here was motivated by the desire to gain a better understanding of the relationship between these various solutions and yet other inequivalent solutions.

In the present paper, we assume the existence of two Killing vectors, so that the metric depends on only one variable. According to their signature, such metrics describe stationary translationally symmetric or rotationally symmetric space-times (the more general case of stationary space-times, with only one Killing vector assumed, shall be treated elsewhere [5]), cosmologies (the variable is time), or spaces with Riemannian signature depending on one variable. As we shall show, all such solutions of the cosmological Einstein equations are geodesics of an abstract three-dimensional Minkowski space (the solution space). These geodesics are timelike, lightlike, or spacelike for $\Lambda>0$, $\Lambda=0$, or $\Lambda<0$, respectively. In the case $\Lambda\le 0$ these geodesics are further classified by their geometric relation to the light cone of solution space, i.e., according to whether they do not intersect, are tangent to, or intersect the (future or past) light cone. For $\Lambda<0$ the stationary space-times thus constructed include, in addition to generalizations of previously known solutions, a new class of wormhole solutions.

We first consider the general case of a $(p+1)$-dimensional space-time. Assuming that the metric depends on only one variable r, we choose the canonical parametrization [6]

$$ds^2=\lambda_{ab}(r)dx^a dx^b+\varepsilon dr^2 \tag{1}$$

($\varepsilon=\pm 1$), where λ is a symmetrical $p\times p$ matrix, and define

$$\chi\equiv\lambda^{-1}\frac{d\lambda}{dr},\quad \sigma\equiv|\det\lambda|^{1/2}\,. \tag{2}$$

The action for cosmological gravity (derived from the known expressions of the Ricci tensor [6]),

$$\begin{aligned} I_0&=-\frac{1}{2\kappa}\int d^{p+1}x\sqrt{|g|}(R+2\Lambda)\\ &=-\frac{1}{2\kappa}\int d^p x\int dr\left[-2\varepsilon\frac{d^2\sigma}{dr^2}\right.\\ &\qquad\left.+\frac{\sigma}{4}[-\varepsilon\,\mathrm{Tr}\chi^2+\varepsilon(\mathrm{Tr}\chi)^2+8\Lambda]\right], \end{aligned} \tag{3}$$

is invariant under the action on λ of the group $SL(p,R)$.

Specializing now to $p=2$, the invariance group $SL(2,R)$ is locally isomorphic to the Lorentz group $SO(2,1)$, which suggests the vector parametrization of the matrix λ [7],

$$\lambda=\begin{bmatrix} T+X & Y\\ Y & T-X\end{bmatrix}, \tag{4}$$

such that special linear transformations of λ correspond to Lorentz rotations of the vector $\mathbf{X}=(T,X,Y)$. The signature of the metric (1) depends on the sign of $\det\lambda=T^2-X^2-Y^2\equiv R^2$, stationary space-times corresponding to spacelike $\mathbf{X}$ with $\varepsilon=-1$, cosmologies or spaces with Riemannian signature to past timelike or future timelike $\mathbf{X}$ with $\varepsilon=+1$ (Fig. 1), so that in all cases of

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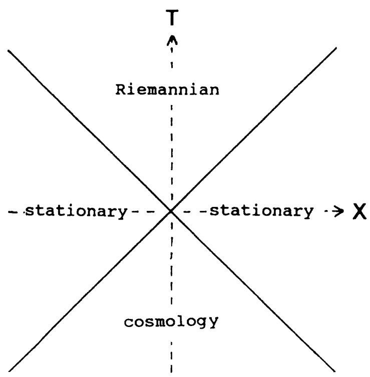

FIG. 1. The three sectors of solution space (the third dimension Y is suppressed).

interest $\varepsilon = \mathrm{sgn}(R^2)$. The parametrization (4) leads to the following form of the action (3):

$$I_0 = -\frac{1}{4\kappa}\int d^2x \int dr \sigma[\sigma^{-2}(T'^2 - X'^2 - Y'^2) + 4\Lambda] , \quad (5)$$

where we have used $R^2 = \varepsilon\sigma^2$, and a prime denotes d/dr. The introduction of a variable ρ such that

$$\sigma dr = \zeta^{-1} d\rho \quad (6)$$

(where ζ is a constant scale) further simplifies the action (5) to

$$I_0 = -\frac{1}{4\kappa}\int d^2x \int d\rho[\zeta(\dot{T}^2 - \dot{X}^2 - \dot{Y}^2) + 4\Lambda\zeta^{-1}] , \quad (7)$$

where an overdot denotes $d/d\rho$.

This form of the action shows that the full invariance group of the solutions $\lambda_{ab}(\rho)$ is actually, for all values of Λ, the Poincaré group ISO(2,1). Varying the action (7) with respect to the "coordinates" X^A ($A = 0,1,2$) leads to the geodesic equations

$$\ddot{X}^A = 0 , \quad (8)$$

while varying with respect to the Lagrange multiplier ζ and fixing afterwards the scale $\zeta = 1$ leads to the first integral

$$\dot{T}^2 - \dot{X}^2 - \dot{Y}^2 = 4\Lambda , \quad (9)$$

showing that the solutions are geodesics of the Minkowski space of metric $d\mathbf{X}^2 = dT^2 - dX^2 - dY^2$, which are timelike for $\Lambda > 0$, lightlike for $\Lambda = 0$, spacelike for $\Lambda < 0$.

Integrating Eq. (8) with the constraint (9), we see that the general solution[1]

$$\mathbf{X}(\rho) = \boldsymbol{\alpha}\rho + \boldsymbol{\beta} \quad (\boldsymbol{\alpha}^2 = 4\Lambda) \quad (10)$$

apparently depends on five arbitrary parameters. The actual number of independent parameters depends on the topology of the sections $\rho = $const. If this topology is $\mathbb{R}\times\mathbb{R}$, then the fivefold arbitrariness corresponds precisely to the residual invariance of the parametrization (1), (6) under the five-parameter group of linear transformations

$$x^a \to \tau^a{}_b x^b, \quad \rho \to |\det\tau|^{-1}\rho , \quad (11a)$$

$$\rho \to \rho + c , \quad (11b)$$

so that the solution is essentially unique. However, if the sections $\rho = $const have the topology $\mathbb{R}\times S^1$ (for instance, in the case of a cylindrical cosmology or of a stationary rotationally symmetric space-time), only the transformations preserving the periodicity condition on the second variable x^2 are allowed, which restricts (11a) by $\tau^a{}_2 = \delta^a_2$, so that we are left with a solution depending generically on two parameters. If this topology is $S^1\times S^1$ (which is the case of the toroidal cosmologies of Ref. [8]), then only the reparametrization (11b) is allowed, and the solution depends on four parameters (more complicated topologies do not allow the existence of two global Killing vector fields).

Recalling the full expression of the physical space-time metric

$$ds^2 = \lambda_{ab}(\rho)dx^a dx^b + R^{-2}(\rho)d\rho^2 , \quad (12)$$

we see that this metric develops a singularity whenever the geodesic $\mathbf{X}(\rho)$ crosses the light cone $\det\lambda \equiv R^2 = 0$. We shall mainly discuss the case of stationary rotationally symmetric space-times corresponding to spacelike $\mathbf{X}$. Then the intersection of a solution $\mathbf{X}(\rho)$ with the future light cone [inside which the signature of the metric (12) would be Riemannian] corresponds either to a naked singularity or to the origin $r = 0$ of the polar coordinates (r,θ). On the other hand, crossing the past light cone does not entail a change of signature but only a simultaneous change of sign of the largest eigenvalue of λ (associated with the time coordinate) and of R^{-2}, corresponding to a horizon.

We now discuss briefly the various cases. For $\Lambda > 0$, all timelike geodesics cross both the future and past light cones of solution space. The general stationary symmetric solution, which is [up to a transformation (11)]

$$ds^2 = \tfrac{1}{2}(1 - 2c\rho)dt^2 + 2\omega dt d\theta - 2\Lambda c^{-2}(1 + 2c\rho)d\theta^2 - (\Lambda c^{-2} + \omega^2 - 4\Lambda\rho^2)^{-1}d\rho^2 \quad (13)$$

($c > 0$) corresponds to a stationary de Sitter metric with two antipodal massive and spinning sources, generalizing the static de Sitter solution with sources of Ref. [2]. The full space-time is obtained by following the solution-space geodesic (Fig. 2) from the "future" singularity $\rho = \rho_0$ with $\rho_0 \equiv -\frac{1}{2}(c^{-2} + \omega^2\Lambda^{-1})^{1/2}$ (a pole of the de Sitter sphere) to the "past" singularity $\rho = -\rho_0$ (the de Sitter equator) and back to $\rho = \rho_0$ (the other de Sitter pole). The sourceless de Sitter metric is obtained from (12) for the choice $\omega = 0$, $c = 2$. Of course, de Sitter space-time is also recovered from solutions with past timelike $\mathbf{X}$, e.g., $T = -2\Lambda^{1/2}\rho$, $X = Y = 0$, with $\rho = \frac{1}{2}\Lambda^{-1/2}\exp(2\Lambda^{1/2}t)$, leading to the well-known metric

$$ds^2 = dt^2 - e^{2\Lambda^{1/2}t}(dx^2 + dy^2) , \quad (14)$$

which covers half of de Sitter space [9].

[1]Essentially the same solution was obtained independently in Sec. II of Ref. [4], using a matrix approach parallel to that followed here.

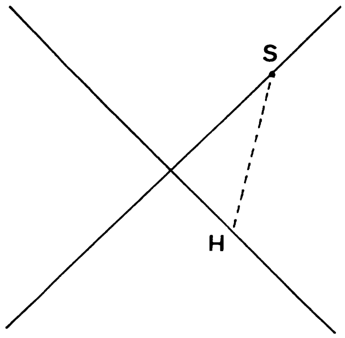

FIG. 2. A stationary rotationally symmetric solution for $\Lambda>0$; S is the singularity, H the horizon (de Sitter equator).

We discuss for completeness the case $\Lambda=0$ (Einstein gravity), for which the solutions are given by (10) with $\boldsymbol{\alpha}^2=0$, leading to

$$R^2=2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\rho+\boldsymbol{\beta}^2 \ . \tag{15}$$

In the generic case $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\neq 0$ (Fig. 3), either the geodesic intersects the future light cone for $\boldsymbol{\alpha}$ past lightlike, an instance being $T+X=1$, $T-X=\omega^2-2\alpha\rho$, $Y=-\omega$, which corresponds to the particlelike metric [10]

$$ds^2=(dt-\omega d\theta)^2-\alpha^2r^2d\theta^2-dr^2 \ , \tag{16}$$

or it intersects the past light cone for $\boldsymbol{\alpha}$ future lightlike, in which case [11,12]

$$ds^2=r^2(dt-\omega d\theta)^2-v^2d\theta^2-dr^2 \tag{17}$$

($r\in]-\infty, +\infty[$). In the special case $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}=0$, the geodesic never intersects the light cone; the corresponding family of regular solutions,

$$ds^2=r(dt-\omega d\theta)^2-2c(dt-\omega d\theta)d\theta-dr^2 \ , \tag{18}$$

was first given in Ref. [11] [the solutions of this family depend on two parameters because in this case the transformation (11b) is equivalent to a combination of transformations (11a)]. The even more special case $\boldsymbol{\alpha}=0$ corresponds not to a geodesic but to an arbitrary point, the associated metric being the Minkowski cylinder $ds^2=dt^2-v^2d\theta^2-dr^2$.

For $\Lambda<0$, we obtain, from (10),

$$R^2=-4l^{-2}p^2+2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\rho+\boldsymbol{\beta}^2 \tag{19}$$

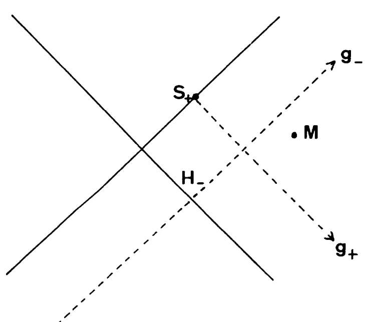

FIG. 3. Three stationary rotationally symmetric solutions for $\Lambda=0$; g_+ is solution (16) with singularity S_+, g_- is solution (17) with horizon H_-, M is the Minkowski cylinder.

($l^{-2}\equiv-\Lambda$). The discussion depends on the sign of the discriminant $\Delta=(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2+4l^{-2}\boldsymbol{\beta}^2$. Let us choose for the generic solution the parametrization

$$ds^2=(2l^{-2}\rho-M/2)dt^2+Jdtd\theta-(2l^{-2}\rho+M/2)l^2d\theta^2 -[4l^{-2}\rho^2+(J^2-M^2l^2)/4]^{-1}d\rho^2 \ , \tag{20}$$

where M and J are (proportional to) the mass and spin associated with the solution [3]. Then $\Delta=M^2-J^2l^{-2}$. For $\Delta>0$ ($J^2<M^2l^2$), the geodesics intersect twice the past (for $M>0$) or future (for $M<0$) light cone (Fig. 4). The corresponding solutions (20) are, as discussed in Ref. [3], black holes for $M>0$, or solutions with naked singularities for $M<0$ (except for anti–de Sitter space with $M=-1$, $J=0$). For $\Delta=0$ ($J^2=M^2l^2$), the geodesics are tangent to the light cone. The corresponding solutions (20) have an apparent singularity ($\rho=0$) at infinite proper distance. In the "extreme black-hole" case $M>0$ this apparent singularity is actually a horizon which is crossed by almost all the physical space geodesics [those of the metric (20)], while for $M<0$ the circle $\rho=0$ is at infinite geodesic distance, so that in all cases the $\Delta=0$ solutions are regular. The horizon size goes to zero in the limiting case $J=Ml=0$ (the "vacuum" solution of Ref. [3]), which corresponds to a geodesic coming from the apex $\mathbf{X}=0$ ($\rho=0$) of the light cone. Finally, for $\Delta<0$ ($J^2>M^2l^2$), the geodesics do not intersect the light cone, so that the space-time metric is regular for all values of $\rho\in\mathbb{R}$. The corresponding solutions are Lorentzian wormholes, with two lines at spatial infinity: $\rho\to\pm\infty$ (a class of null geodesics, along which ρ is the affine parameter, connect these two lines); it is also apparent from (20) that light cones tilt over by $\pi/2$ when ρ varies from $+\infty$ to $-\infty$, with closed timelike curves for $\rho<-Ml^2/4$ (the analogous closed timelike curves of the black hole and extreme black-hole solutions are hidden behind the inner horizon [3]).

The class of solutions (10) with $\alpha_T=\alpha_X$, $\alpha_Y=\pm 2l^{-1}$ cannot be obtained from (20) by a regular transformation (11) (one which preserves the periodicity condition on $x^2=\theta$). For $\beta_T\neq\beta_X(g_{22}\neq 0)$, these solutions can be parametrized by

$$ds^2=\varepsilon dt^2\pm 4l^{-1}\rho dtd\theta-bd\theta^2-[4l^{-2}\rho^2+\varepsilon b]^{-1}d\rho^2 \tag{21}$$

with $\Delta=-4l^{-2}\varepsilon b$, and $b>0$ for θ spacelike, so that we

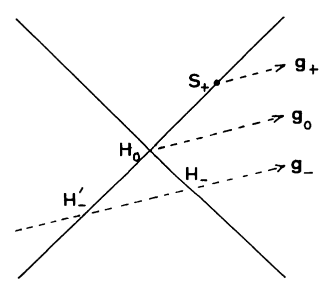

FIG. 4. Three stationary rotationally symmetric solutions for $\Lambda<0$; g_+ has naked singularity S_+, g_0 (a "vacuum" solution) has horizon H_0, while g_- has outer and inner horizons H_- and H'_-. Extreme black-hole and wormhole solution are not shown.

may have either $\varepsilon=-1$ $(\Delta>0)$, $\varepsilon=0$ $(\Delta=0)$, or $\varepsilon=+1$ $(\Delta<0)$. The properties of these three cases are similar to those of the corresponding cases of the generic solution (20). For instance, solution (21) with $\varepsilon=0$, $b>0$ previously reported in Ref. [13] is an extreme black hole. The parametrization (21) does not cover the special case $g_{22}=0$, for which the solution takes the form

$$ds^2=cdt^2\pm 4l^{-1}\rho dt d\theta-[4l^{-2}\rho^2]^{-1}d\rho^2 \tag{22}$$

with $\Delta=0$.

To conclude we discuss briefly the extension of our approach to the case where the gravitational field is coupled to other fields. Consider, for instance, the coupling of a massless scalar field φ to cosmological gravity, described by the action

$$I=-\frac{1}{2\kappa}\int d^3x\sqrt{|g|}[R-\kappa g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}+2\Lambda] . \tag{23}$$

Upon assuming again the metric to be of form (1), (6), and $\varphi=\varphi(\rho)$, this action takes the form

$$I=\int d^2x\int d\rho[\zeta(-\tfrac{1}{2}m\dot{\mathbf{X}}^2+\tfrac{1}{2}R^2\dot{\varphi}^2)-2m\Lambda\zeta^{-1}] \tag{24}$$

[with $m\equiv(2\kappa)^{-1}$], showing that the solutions to the coupled field equations are geodesics of the four-dimensional space of the metric:

$$dS^2=dT^2-dX^2-dY^2-m^{-1}R^2d\varphi^2 \tag{25}$$

$(R^2\equiv\mathbf{X}^2\equiv T^2-X^2-Y^2)$. As in the case of sourceless cosmological gravity, the length of these geodesics is proportional to the cosmological constant Λ:

$$\tfrac{1}{2}m\dot{\mathbf{X}}^2-\tfrac{1}{2}R^2\dot{\varphi}^2=2m\Lambda \tag{26}$$

for $\zeta=1$. Eliminating the cyclic variable φ in terms of its constant conjugate momentum π, we see that the reduced three-dimensional dynamical problem, derived from the Lagrangian

$$L=-\tfrac{1}{2}m\dot{\mathbf{X}}^2+\tfrac{1}{2}\pi^2R^{-2} , \tag{27}$$

is the Minkowskian equivalent of the Euclidean problem of a particle moving in a central R^{-2} potential, and may easily be solved completely by the same methods. The physically more important but more intricate Einstein-Maxwell cosmological problem is treated elsewhere [14].

I am grateful to A. Comtet for a discussion which stimulated my interest in this subject.

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