Scaling, asymptotic scaling, and Symanzik improvement: Deconfinement temperature in SU(2) pure gauge theory

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We report on a high statistics simulation of SU(2) pure gauge field theory at finite temperature, using the Symanzik action. We determine the critical coupling for the deconfinement phase transition on lattices up to 8×24, using finite size scaling techniques. We find that the pattern of asymptotic scaling violation is essentially the same as the one observed with conventional, not improved action. On the other hand, the use of effective couplings defined in terms of plaquette expectation values shows a precocious scaling, with respect to an analogous analysis of data obtained by the use of the Wilson action, which we interpret as an effect of improvement.

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I. INTRODUCTION

In the last few years the increased computer power available for lattice theorists has allowed a number of studies on relatively large lattices and it has been possible to test accurately scaling and asymptotic scaling.

We have in mind in particular a series of works on SU(2) lattice gauge theory at finite temperature ([1,2] and references therein) which have been able to test for asymptotic scaling on lattices up to $N_\tau = 16$, showing still considerable scaling violations.

On the other hand, in the same simulations one observes a rather good scaling of ratios of dimensionful quantities, thus supporting the idea that asymptotic scaling is violated by universal terms, which cancel in the ratio, and that some β function, other than the perturbative one, exists.

Since the renormalization group, parametrized in terms of the "bare" coupling on the lattice, shows large deviations from the asymptotic behavior, we feel that it is very important to test different renormalization schemes, as stressed in a recent work by Lepage *et al.* [3]. For instance, it has become more and more apparent that effective schemes on the one hand improve the asymptotic scaling [4,5], and on the other hand reconcile the perturbative expansion of ultraviolet dominated quantities with the lattice numerical results [3].

In this context, we have decided to test the effect of the use of a different lattice formulation, using an improved Symanzik action in the simulation of four-dimensional lattice gauge theory. We did not expect a dramatic improvement of asymptotic scaling, even if in a pioneering work [6] some indications of such an improvement were found, but we have considered that, by comparison with similar works using Wilson action, something about the origin of scaling violations could be understood.

The study of this particular model, SU(2) gauge theory at finite temperature, is justified by its similarity with QCD, as well as by its greater simplicity. Moreover, the particular observable chosen, the temperature of deconfining phase transition, is well defined and can be exactly determined by use of finite size scaling (FSS) techniques, thus allowing us to pursue our main objective, the study of scaling properties.

We can anticipate our results: the pattern of asymptotic scaling violation observed with the Symanzik action is essentially the same as the one observed with the Wilson formulation; the only apparent effect is a precocious transition from the strong to the weak coupling regime. This result seems to indicate that a universal lattice β function exists.

Analogously the effective coupling approach, formulated in terms of plaquettes expectation values, works well in improving the asymptotic scaling figure and gives results consistent for the two lattice formulations.

In Sec. II we give a short review of the Symanzik approach to the lattice formulation of field theories and we recall the physical characteristics of the model under study.

In Sec. III, we review the FSS techniques used in this work, in order to give a presentation as self-contained as possible.

In Sec. IV we give details about our simulations.

In Sec. V we present the results of our measurements, exploiting the FSS techniques to present various tests of consistency of the results.

We discuss the results in Sec. VI, together with an analysis of the different renormalization schemes.

Appendix A is devoted to technical details on the subtraction of biases and on error estimation.

Analogously in Appendix B we discuss our implementation of the density of states method (DSM).

II. THE SYMANZIK IMPROVEMENT AND SU(2) LATTICE GAUGE THEORY

A. Generalities

The formulation of a field theory on a lattice is a necessary step if we want to study it in a nonperturbative

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way with a computer simulation. The lattice can be seen as an ultraviolet regulator as any other, introducing in a natural way a short distance cutoff.

We can extract from a computer simulation relations between observables (for example, mass ratios) only for this regularized theory, and obviously we cannot recover completely the continuum limit, so our predictions will be affected by systematic errors (the so-called lattice artifacts).

The cutoff dependence of a theory can be reduced by a clever choice of the regularization scheme: this is a well-known technical point which was studied extensively by Symanzik [7,8]. He realized that this fact can be used to minimize the consequences of a nonzero lattice spacing a (for an introductory discussion see the pedagogical work of Parisi [9]).

The key observation is that a field theory can be transcribed on a lattice with considerable arbitrariness: the lattice action must only reduce to the continuum one as $a \rightsquigarrow 0$. So we can redefine the action by adding an arbitrary combination of irrelevant operators, which vanish in this limit, and this is equivalent to changing the regularization scheme. Symanzik has also shown (see for instance [10]) that every lattice regulated theory is perturbatively equivalent to a local effective Lagrangian, in a given renormalization scheme, of the type

$$\mathcal{L}_{\text{latt}} \equiv \mathcal{L}_{\text{eff}} = \sum_i c_i^{(0)} \mathcal{O}_i^{(0)} + a^2 \sum_i c_i^{(2)} \mathcal{O}_i^{(2)} + a^4 \sum_i c_i^{(4)} \mathcal{O}_i^{(4)} + \cdots , \tag{1}$$

where $\mathcal{O}_i^{(n)}$ are local operators of dimension $d+n$, if d is the dimension of space. It follows that it is possible to build a lattice action in such a way that the corresponding effective Lagrangian is

$$\mathcal{L}_{\text{latt,impr}} \equiv \mathcal{L}_{\text{eff,impr}} = \mathcal{L}_{\text{cont}} + a^{2p} \sum_i c_i^{(2p)} \mathcal{O}_i^{(2p)} + \cdots , \tag{2}$$

using an appropriate linear combination of the operators $\mathcal{O}_i^{(q<2p)}$ transcribed in an arbitrary way on the lattice. The determination of the coefficients for this combination is a nontrivial task for an interacting theory: in principle it could be done with an high precision numerical simulation.

We can also compute them perturbatively, if the bare coupling constant is sufficiently small (i.e., near the continuum limit for an asymptotically free theory), using a matching procedure between vertex functions [7], or in the case of gauge theories (for instance) matching the expectation value of gauge invariant quantities, such as the Wilson loops [11] or the interquark potential [12]. An exposition of the different improvement strategies can be found in the work of Lüscher *et al.* [13].

If this program works, we may hope to obtain more accurate predictions for physical quantities with the not extremely big lattices we can use with the current computing resources. The more direct way to verify that lattice artifacts are reduced is to measure an adimensional ratio of physical quantities while making a smaller and smaller. For example using the action in Eq. (2) we expect to find, for a mass ratio (ξ = correlation length),

$$\frac{m_a}{m_b} = K \left(1 + O\left[\left(\frac{a}{\xi} \right)^{2p+2} \ln \left(\frac{a}{\xi} \right) \right] \right) . \tag{3}$$

A more refined test consists in verifying how well the quantities $\mathcal{M}_i$ we can measure on lattice obey the renormalization group equation, which is merely a statement of cutoff independence of physical observables (scaling). The lattice artifacts modify this equation adding non-universal (i.e., $\mathcal{M}$-dependent) scaling-violating terms, so we have

$$\left(-a \frac{\partial}{\partial a} + \left[\bar{\beta}(g) + \bar{\beta}_{\mathcal{M}}(g) \right] \frac{\partial}{\partial g} \right) \mathcal{M}_i = \Lambda_{\mathcal{M}} . \tag{4}$$

Here $\bar{\beta}_{\mathcal{M}}$ is a nonanalytic contribution that cannot be calculated perturbatively (for instance, in the bidimensional σ model it can be evaluated in the large N limit, where one finds $\bar{\beta}_{\mathcal{M}} = O[g e^{-\frac{k}{g}}]$) and $\Lambda_{\mathcal{M}} = O[(a/\xi)^2 \ln(a/\xi)]$. If we use the action (2) with $p=2$, we expect first of all $\Lambda_{\mathcal{M}} = O[(a/\xi)^4 \ln(a/\xi)]$, and then a reduced $\bar{\beta}_{\mathcal{M}}$. In the real case we can calculate the improved action at best to the first perturbative order, and we expect contributions $O[g^4 (a/\xi)^2 \ln(a/\xi)]$ to $\Lambda_{\mathcal{M}}$, which give a little contribution for an asymptotically free theory if we are sufficiently near the continuum.

B. Scaling vs asymptotic scaling

In the general case we know only the first few perturbative terms of the function $\bar{\beta}(g)$, in particular the first two (scheme-independent) terms. These are the relevant terms in the continuum limit if asymptotic freedom holds, while in an intermediate coupling range higher-order contributions can be important. If an observable $\mathcal{M}$ follows the renormalization group (RG) with the perturbative approximation to $\bar{\beta}(g)$ we say we are in an asymptotic scaling region: if this is not true, it does not imply that the data cannot be trusted, but only that we had to use a better approximation to the exact β function. We can try to extract this "improved" $\bar{\beta}(g)$ from the simulation itself and then verify the self-consistency of this procedure. If we can find a function $\bar{\beta}_{\text{eff}}(g)$ so that the RG evolution holds for different quantities we say we are in the scaling regime.

It is important to note the following.

The Symanzik program can improve the scaling, but not necessarily the asymptotic scaling: indeed, the onset of asymptotic scaling is expected in a region where irrelevant operators give negligible contribution.

The scaling test is more conclusive than the stabilization of mass ratios, as we can imagine that there are some $\mathcal{M}$-independent contributions to $\Lambda_{\mathcal{M}}$ which cancel in Eq. (3).

As is well known, asymptotic scaling in the SU(2) gauge theory with the Wilson action has not yet been found (see for instance [1]). So we must extrapolate the

β function in some physically sensible way in the intermediate coupling region.

C. The model under study

The SU(2) gauge theory at finite temperature has been discussed at length for instance in [14]: here we just recall that the model shares with QCD the presence of a deconfining phase transition, characterized by the breaking of the center $Z(2)$ symmetry [15] and by the appearance of a nonzero expectation value for the thermal Wilson loop or Polyakov line P. It is by now well established that the transition is of the second-order and all numerical simulations give results in good agreement with the early ansatz [16] that the model belongs to the same universality class as the three-dimensional (3D) Ising model.

In our simulations we have used a "tree improved" action; i.e., we have only corrected in part lattice artifacts in the classical theory.

This can be done by observing that the usual Wilson action is equivalent to the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{12}a^2\partial_\mu F_{\mu\nu}\partial_\mu F_{\mu\nu} + O(a^4). \quad (5)$$

We can compensate for the $O(a^2)$ term by adding a suitable irrelevant operator. A possible and widely used choice is

$$S_I = S_W + S_{\text{irr}} = \beta\sum U_{1\times1} + \beta\sum\left(\frac{2}{3}U_{1\times1} - \frac{1}{12}U_{1\times2}\right) = \beta\sum\left(\frac{5}{3}U_{1\times1} - \frac{1}{12}U_{1\times2}\right), \quad (6)$$

where $U_{n\times m}$ is the $n \times m$ plaquette.

This choice has already been experimented in other works, both in the study of the finite temperature theory [6] and of the quark-antiquark potential [17], using the ichosaedral approximation to the SU(2) group. To our knowledge the present work is the first where a four-dimensional gauge theory is simulated with the Symanzik action and the full group.

III. FINITE SIZE SCALING ANALYSIS

The finite size scaling (FSS) technique is by now a widely used tool in the investigation of pseudocritical properties of statistical systems in finite volume. Its application to the analysis of second-order phase transitions permitted a detailed numerical test of theoretical predictions on critical exponents. The effectiveness in the study of SU(2) deconfining phase transition has been already demonstrated in a number of papers, where use of the Wilson action was made [1,2,18].

In this section we collect some formulas needed by our investigation, in order to give a presentation as self-contained as possible of our results. A more detailed general introduction can be found for instance in [19], while we refer to the original works [1,2] for the improvements of the method which we have applied to our analysis.

Numerical simulations of statistical systems are limited to finite volumes, characterized by some length scale L. In the vicinity of an (infinite volume) phase transition the system exhibits a pseudocritical behavior, for instance at a second-order phase transition the susceptibility χ shows a peak broadened by the finite volume, while the correlation length ξ, defined in terms of appropriate pair correlation functions, reaches the dimensions of the system. For our purposes, the shift in the critical temperature is the most important effect, which can be easily estimated if one assumes some universality class for the model under consideration and that the correlation length exhibits a power behavior in the vicinity of the pseudotransition:

$$\xi \propto |t|^{-\nu}\ , \quad (7)$$

where $t = \frac{T-T_c}{T_c}$ is the reduced temperature (in units of the infinite volume critical temperature). If the finite volume "critical" temperature $T_c(L)$ is reached when the correlation length is of the order of the physical size of the system, it follows that the expected shift is

$$|T_c(L) - T_c| \propto T_c L^{-\frac{1}{\nu}}\ . \quad (8)$$

Note that in fitting measurements the critical temperature is a very sensitive parameter; hence, the error in its determination should be reduced as much as possible if one is interested in determining universal parameters with a good accuracy.

To extract infinite volume limits an "ansatz" is necessary on the form of the observables.

The main idea is that near a second-order phase transition, as the correlation length ξ approaches the size L of the system, the observables O depend on these two scales only through the ratio $r = L/\xi$, apart from a volume-dependent prefactor

$$O_L = L^\omega \bar{Q}_0(r) \quad (9)$$

or, in terms of the reduced temperature,

$$O_L = L^\omega \bar{Q}_0(Lt^\nu)\ . \quad (10)$$

If for large L this observable exhibits a critical behavior, with exponent $-\rho$, the function $\bar{Q}_O$ should behave like some power of its argument, and the requirement of cancellation of volume dependence requires that $\omega = \rho/\nu$. Hence it results the parametrization

$$O_L = L^{\rho/\nu}\bar{Q}_L(Lt^\nu)\ , \quad (11)$$

or alternatively, and more commonly,

$$O_L = L^{\rho/\nu}Q_L\left(tL^{1/\nu}\right)\ . \quad (12)$$

The important point is that renormalization group arguments show that all the critical behavior can be derived from the singular part of the free energy density, which is then assumed to have the conventional form

$$f_s = L^{-d} Q\left(g_T L^{1/\nu}, g_h L^{(\eta+\gamma)/\nu}, g_i L_i^y\right) , \qquad (13)$$

where g_T, g_h are connected to the reduced temperature and the external magnetic field by linear relations, plus corrections

$$\begin{aligned} g_t &= c_t t + O\left(th, t^2\right) , \\ g_h &= c_h h + O\left(th, h^2\right) . \end{aligned} \qquad (14)$$

The additional dependence on irrelevant scaling fields g_i, with negative exponents y_i, cannot be neglected in many practical simulations, and determines corrections to the scaling behavior. Note also that we name η the critical exponent of the "magnetization," instead of the conventional β, to avoid confusion with the coupling in the latticized theory.

Let us come to our system and recall that we simulate on a lattice of volume L^3, with $L = N_\sigma a$, and temporal extent N_t, determining a temperature $T = 1/(N_t a)$.

The relation between the spacing a and the bare coupling $\beta = \frac{2C_a}{g_0^2}$ is unknown, and one of the motivations of our simulation is the study of the validity of the universal two-loop asymptotic formula

$$a\Lambda = \left(\frac{\beta}{2C_a b_0}\right)^{b_1/2b_0^2} \exp\left(-\frac{\beta}{4C_a b_0}\right) . \qquad (15)$$

As first noted in [1], it is convenient to rewrite the free energy density in terms of the dimensionless combination LT, that is, of the ratio $y = N_\sigma/N_\tau$.

$$f_s\left(t, h; N_\sigma; N_\tau\right) = y^{-d} Q_f\left(g_t y^{\frac{1}{\nu}}, g_h y^{\frac{\eta+\gamma}{\nu}}, g_i y^{y_i}\right) . \qquad (16)$$

Define P the spatial average of the Polyakov line,

$$\begin{aligned} L\left(\mathbf{x}\right) &= \prod_{t=1}^{N_t} U\left(\mathbf{x}, t\right) , \\ P &= N_\sigma^{-d} \sum_{\mathbf{x}} \mathrm{Tr}\left[L\left(\mathbf{x}\right)\right] , \end{aligned} \qquad (17)$$

and introduce a source h for this quantity in the partition function, through the operator $hZ\left(a, N_\tau\right) N_\sigma^d P$, where the renormalization factor Z cancels the divergent self-energy contributions to the Polyakov loop.

Derivatives with respect to the source h allows us to define the physical order parameter[1] and the susceptibilities

$$\begin{aligned} \langle P_p \rangle &= -\left.\frac{\partial f_s}{\partial h}\right|_{h=0} = y^{-\eta/\nu} Q_P\left(g_t y^{\frac{1}{\nu}}, g_i y^{y_i}\right) , \\ \chi &= \left.\frac{\partial^2 f_s}{\partial h^2}\right. = y^{\gamma/\nu} Q_\chi\left(g_t y^{\frac{1}{\nu}}, g_i y^{y_i}\right) , \end{aligned} \qquad (18)$$

where use has been made of the first hyperscaling relation

$$\gamma/\nu + 2\eta/\nu = d \qquad (19)$$

implied by Eq. (18).[2]

A useful quantity in the determination of the critical temperature is then the Binder cumulant g_4, defined as the fourth derivative of the free energy with respect to "magnetic field," normalized to the susceptibility:

$$g_4 = \left.\frac{\partial^4 f_s}{\partial h^4}\right|_{h=0} \Big/ \left(\chi^2 \left(N_\sigma/N_\tau\right)^d\right) . \qquad (20)$$

In fact, its expression is a directly scaling function, where explicit dependence on N_σ is canceled by the use of the hyperscaling relation: moreover, the multiplicative renormalization Z of the Polyakov loop cancels, and therefore the expression for g_4 can be safely given in terms of expectation values of moments of Polyakov loop:

$$g_4 = \frac{\langle P^4 \rangle}{\langle P^2 \rangle^2} - 3. \qquad (21)$$

Its directly scaling form can be expressed as

$$g_4\left(t; N_\sigma; N_\tau\right) = Q_g(g_t\left(t, N_\tau\right) y^{1/\nu},\ g_1 y^{y_1}) , \qquad (22)$$

where we have conserved the first irrelevant scaling field. The localization of the critical point is then possible by finding the intersections of g_4 curves, as functions of the temperature t, at various values of the spatial size N_σ.

Taking into account the first irrelevant field one expands the expression given in Eq. (22) in the vicinity of

[1] In a finite volume system, as tunneling between inequivalent "vacua" tends to restore the $Z(2)$ symmetry, the experimental order parameter is defined as $|P|$.

[2] In fact, assuming that fluctuations contribute to the singular part of the free energy density as the ratio of the unit volume to the volume $\xi(t)^d$,

$$f_s\left(t\right) \sim \xi\left(t\right)^{-d} \sim t^{\nu d} ,$$

which is incorporated in Eq. (16), if, for small values of x,

$$Q_f\left(x, 0, \ldots\right) \sim \left(x\right)^{\nu d} .$$

Now derivatives in Eq. (18) can be written as

$$\begin{aligned} \langle P \rangle &= -\left(N_\sigma/N_\tau\right)^{-d+(\eta+\gamma)/\nu} Q_1(g_t \left(N_\sigma/N_\tau\right)^{1/\nu}) , \\ \chi &= \left(N_\sigma/N_\tau\right)^{-d+2(\eta+\gamma)/\nu} Q_2(g_t \left(N_\sigma/N_\tau\right)^{1/\nu}) , \end{aligned}$$

and so, combining the dependence of $Q_{1,2}$ on g_t (that is, the same as Q_f) with the requirement that in the infinite volume $N_\sigma \to \infty$ limit the critical behavior is specified by the exponents η, γ,

$$P \sim |t|^\eta , \quad \chi \sim |t|^{-\gamma} ,$$

Eq. (19) results.

the transition

$$g_4(t; N_\sigma; N_\tau) \simeq g_{4,0} + g_{4,1} t y^{\frac{1}{\nu}} + g_{4,2} y^{y_1} \ . \qquad (23)$$

Hence the intersection of two g_4 curves at values of the spatial size N_σ, N'_σ corresponds to a shifted temperature

$$t = \frac{g_{4,2}}{g_{4,1}} \frac{\left[(y')^{y_1} - y^{y_1}\right]}{\left[y^{\frac{1}{\nu}} - (y')^{\frac{1}{\nu}}\right]} \propto \left(\frac{N_\sigma}{N_\tau}\right)^{y_1 - 1/\nu} \frac{1 - b^{y_1}}{b^{1/\nu} - 1} \ , \qquad (24)$$

where $b = N'_\sigma / N_\sigma$.

As $y_1 < 0$, this shift in the crossing goes to zero as N_σ grows, while also at fixed N_σ as b grows the shift goes down, governed by the exponent $-1/\nu$.

Actually, in the simulation of SU(N) gauge theories, the connection between the bare coupling β and the lattice spacing, which allows one to determine the temperature in units of the reference scale Λ, is known only in the asymptotic regime $g_0 \to 0$. As noted in [1], this means that expressing the reduced temperature t in terms of β through an approximate formula such as

$$t = (\beta - \beta_{c,\infty}) \frac{1}{4 C_a b_0} \left[1 - \frac{2 C_a b_1}{b_0} \beta_{c,\infty}^{-1}\right] \qquad (25)$$

introduces an error $O\left(\beta_{c,\infty}^{-1}\right)$ in the determination of t. This shift is of the order of 8% in the relevant coupling regime explored, but one should be aware that this estimate would be *a posteriori* justified by the observation of an asymptotic scaling behavior, which is absent.

In our search for the determination of the critical temperature this shift is immaterial, as long as a relation of the form

$$t \propto (\beta - \beta_{c,\infty}) \qquad (26)$$

is valid.

Therefore Eq. (24) translates in the form

$$\beta_c(N_\sigma, N'_\sigma) = \beta_{c,\infty}(1 - c\varepsilon) \ , \qquad (27)$$

where

$$\varepsilon = N_\sigma^{y_1 - 1/\nu} \frac{1 - b^{y_1}}{b^{1/\nu} - 1}, \quad b = \frac{N'_\sigma}{N_\sigma} \qquad (28)$$

and the constant c maintains a dependence on N_τ and $\beta_{c,\infty}$ which is irrelevant in the $\varepsilon \to 0$ extrapolation.

IV. DETAILS OF THE SIMULATION

Most of the work has been done on the APE supercomputer [20]. The model operating at Pisa is the so-called "tube" machine, a 128 processor parallel computer with a peak performance of 6 GFlops.

All of the code has been written in the high-level language [21] proper of this machine (APESE), and the program runs at about 35% of the peak speed: this result is quite good if one takes into account the complications both in addressing and in memory access due to the use of the Symanzik action. In other words, in our code the floating point performance is slowed by the large number of integer and addressing operations required to load the elements of group in the register file: this means that an analogous implementation for the SU(3) gauge group should result in a better performance, as the ratio of floating point to integer operations would be increased both in computation and in I/O. A detailed account of the implementation will be given elsewhere.

In the update of lattices with $N_\tau = 3, 4, \ldots, 8$ we have used an overrelaxed heat bath update. An exact heat bath algorithm has been implemented, in the modified Kennedy-Pendleton form [22] which results in higher acceptance and is well suited to a parallel machine such as APE; a number of complete overrelaxation sweeps ranging from 10 to 16 has been used to decorrelate between subsequent heat bath sweeps.

As the Symanzik action couples next-to-nearest-neighbors sites, on a lattice $2 \times N_\sigma$ a self-coupling of spatial links results, and therefore the action is quadratic and prevents the use of an heat bath algorithm: hence on these smaller lattices we have used a Metropolis algorithm running on RISC workstations.

In Tables I–VI we present the simulation parameters, together with the estimated autocorrelation time of the Polyakov line. The density of states method has been used to interpolate between simulated data points. See Appendix B for a discussion of the meaning of the last column, where the "reweighting range" used in the interpolation is listed.

We have also performed a series of runs on large symmetric lattices to measure the expectation values of plaquettes, $U_{1\times1}$, $U_{1\times2}$, to be used in determining effective couplings "in the manner of Parisi." We list in Table VII the corresponding run parameters.

V. RESULTS

A. Determination of critical point by FSS techniques

Runs on lattices with $N_\tau = 2, 3, \ldots, 8$ and up to $N_\sigma = 24$ have been devoted to explore the transition region and measure accurately the Polyakov line.

TABLE I. Run parameters.

N_σ	N_τ	β	$N_{\rm meas}$	$\tau_{\rm int}$	Rew. range
4	2	1.340000	40960	32.8	1.28–1.40
4	2	1.360000	40960	51.3	1.30–1.42
4	2	1.380000	40960	105.7	1.32–1.44
4	2	1.400000	16384	79.4	1.34–1.46
8	2	1.340000	16384	55.0	1.31–1.37
8	2	1.360000	38912	154.4	1.33–1.39
8	2	1.380000	38912	314.9	1.35–1.41
8	2	1.400000	38912	1264.8	1.37–1.43
12	2	1.360000	36864	142.9	1.34–1.37
12	2	1.380000	45056	407.1	1.36–1.39
12	2	1.400000	49152	3349.7	1.39–1.41

TABLE II. Run parameters.

N_σ	N_τ	β	N_{meas}	τ_{int}	Rew. range
6	3	1.5600	32768	3.0	1.52–1.60
6	3	1.5700	32768	3.4	1.53–1.61
6	3	1.5800	32768	4.2	1.54–1.62
6	3	1.5900	32768	4.9	1.55–1.63
6	3	1.6000	32768	6.7	1.56–1.64
6	3	1.6100	32768	7.9	1.57–1.65
6	3	1.6200	32768	10.6	1.58–1.66
6	3	1.6300	32768	15.3	1.59–1.67
10	3	1.5600	30720	5.9	1.54–1.58
10	3	1.5700	30720	8.9	1.55–1.59
10	3	1.5800	30720	15.9	1.56–1.60
10	3	1.5900	30720	22.3	1.57–1.61
10	3	1.6000	30720	53.5	1.58–1.62
10	3	1.6100	30720	103.6	1.59–1.63
10	3	1.6200	30720	138.7	1.60–1.64
10	3	1.6300	30720	327.6	1.61–1.65
14	3	1.5600	30336	6.5	1.551–1.569
14	3	1.5700	65536	10.3	1.560–1.578
14	3	1.5800	65536	17.6	1.572–1.590
14	3	1.5900	65536	33.6	1.579–1.599
14	3	1.6000	65536	84.8	1.590–1.612
14	3	1.6100	30336	206.9	1.602–1.621
14	3	1.6200	30336	902.2	1.612–1.629
14	3	1.6300	30336	3.9	1.620–1.639
18	3	1.5800	22464	12.3	1.572–1.587
18	3	1.5900	22464	22.1	1.584–1.597
18	3	1.6000	22464	81.1	1.592–1.606
18	3	1.6100	22464	563.7	1.602–1.617

1. Binder method

In Figs. 1–7 we show the plots of the Binder cumulant, Eq. (21), for different values of N_τ and N_σ.

TABLE III. Run parameters.

N_σ	N_τ	β	N_{meas}	τ_{int}	Rew. range
8	4	1.6500	65536	3.2	1.625–1.675
8	4	1.6700	65536	4.8	1.646–1.694
8	4	1.6900	65536	7.6	1.665–1.716
8	4	1.6950	131072	8.7	1.670–1.721
8	4	1.7050	131072	11.0	1.680–1.731
8	4	1.7100	196608	12.9	1.686–1.736
8	4	1.7200	262144	18.5	1.694–1.746
8	4	1.7300	131072	23.0	1.704–1.757
12	4	1.6500	4096	3.5	1.637–1.665
12	4	1.6700	4096	6.6	1.657–1.684
12	4	1.6900	69632	12.5	1.676–1.704
12	4	1.6950	131072	15.5	1.81–1.709
12	4	1.7000	65536	22.7	1.686–1.714
12	4	1.7050	131072	29.1	1.691–1.719
12	4	1.7100	135168	50.5	1.704–1.724
12	4	1.7200	131072	84.4	1.714–1.734
12	4	1.7300	131072	200.7	1.717–1.744
12	4	1.7400	131072	351.8	1.726–1.755
16	4	1.6900	28672	16.3	1.681–1.699
16	4	1.6950	57344	25.4	1.686–1.704
16	4	1.7000	57344	52.1	1.691–1.707
16	4	1.7050	40960	55.4	1.696–1.714
16	4	1.7100	12288	71.2	1.701–1.719

TABLE IV. Run parameters.

N_σ	N_τ	β	N_{meas}	τ_{int}	Rew. range
8	5	1.7500	16384	6.7	1.727–1.772
8	5	1.7600	40960	7.4	1.738–1.785
8	5	1.7650	40960	8.4	1.741–1.791
8	5	1.7700	40960	9.0	1.745–1.796
8	5	1.7750	40960	10.0	1.750–1.801
8	5	1.7800	57344	11.3	1.755–1.806
8	5	1.7850	40960	12.0	1.759–1.811
8	5	1.7900	40960	12.4	1.762–1.815
8	5	1.7950	40960	13.1	1.770–1.822
12	5	1.7600	39936	14.0	1.746–1.774
12	5	1.7650	39936	18.2	1.751–1.777
12	5	1.7700	39936	22.9	1.756–1.783
12	5	1.7750	39936	28.5	1.761–1.789
12	5	1.7800	23552	27.5	1.766–1.792
12	5	1.7850	23552	43.1	1.771–1.798
12	5	1.7900	23552	55.8	1.776–1.804
12	5	1.7950	23552	51.6	1.780–1.808
16	5	1.7600	28672	24.0	1.751–1.767
16	5	1.7650	28672	24.2	1.756–1.774
16	5	1.7700	28672	35.6	1.761–1.779
16	5	1.7750	28672	46.4	1.768–1.784
20	5	1.7670	18432	52.4	1.761–1.773
20	5	1.7690	18432	58.5	1.763–1.774

As the Binder cumulant is not a self-averaging quantity, the resulting bias has been computed, and found to be relevant for the larger lattices, where statistics is relatively poor: details on the evaluation and subtraction of the bias are reported in Appendix A.

The solid lines, as well as the dashed lines which are an estimate of the error, are obtained with our implementation of the density of states method [23,24], described in

TABLE V. Run parameters.

N_σ	N_τ	β	N_{meas}	τ_{int}	Rew. range
8	6	1.7750	32768	3.9	1.754–1.799
8	6	1.8000	32768	5.4	1.7775–1.824
8	6	1.8250	32768	7.5	1.800–1.851
8	6	1.8750	32768	11.2	1.850–1.902
12	6	1.7750	49152	5.6	1.762–1.7875
12	6	1.8000	49152	10.1	1.787–1.812
12	6	1.8125	16384	14.7	1.799–1.826
12	6	1.8250	49152	19.5	1.811–1.837
12	6	1.8375	16384	30.6	1.824–1.850
12	6	1.8750	49152	87.9	1.861–1.889
16	6	1.7750	28672	7.5	1.767–1.783
16	6	1.8000	28672	12.0	1.792–1.809
16	6	1.8125	16384	21.9	1.805–1.820
16	6	1.8250	28672	30.1	1.817–1.832
16	6	1.8375	16384	51.2	1.829–1.846
16	6	1.8750	28672	466.3	1.886–1.884
20	6	1.8200	16384	30.3	1.814–1.825
20	6	1.8250	18920	55.0	1.819–1.831
20	6	1.8300	16384	57.6	1.825–1.836
24	6	1.8220	12288	28.5	1.818–1.826
24	6	1.8250	27008	62.9	1.821–1.829
24	6	1.8280	27008	80.2	1.823–1.832

TABLE VI. Run parameters.

N_σ	N_τ	β	$N_{\rm meas}$	$\tau_{\rm int}$	Rew. range
14	7	1.8700	9216	31.3	1.861–1.880
14	7	1.8750	11600	27.2	1.867–1.884
14	7	1.8800	12288	23.5	1.872–1.889
14	7	1.8850	9216	32.0	1.878–1.892
20	7	1.8725	15904	40.7	1.867–1.877
20	7	1.8750	17216	47.8	1.870–1.880
20	7	1.8775	18400	58.3	1.871–1.884
20	7	1.8800	14720	77.4	1.874–1.885
16	8	1.9000	13312	19.0	1.891–1.909
16	8	1.9150	11536	34.0	1.907–1.922
16	8	1.9200	13312	46.2	1.911–1.929
16	8	1.9250	16384	39.2	1.918–1.932
24	8	1.9000	18432	33.1	1.896–1.904
24	8	1.9150	29184	54.5	1.911–1.919
24	8	1.9180	10144	27.4	1.914–1.922
24	8	1.9200	20736	54.9	1.916–1.924
24	8	1.9220	10144	41.4	1.918–1.926
24	8	1.9250	6144	54.9	1.921–1.929

Appendix B.

The determination of the crossing points as well is based on the DSM and we report in Table VIII the values of intersections $\beta_c\,(N_\sigma, N_{\sigma'})$ for the different lattices.

We make use of the extrapolation in Eq. (27) based on the introduction of a single irrelevant field, with exponent $y = -1$, to obtain for lattices $3-6$ the infinite volume limit of the critical coupling:

$$\begin{array}{llll} N_\tau = 3, & \beta_{c,\infty} = 1.596\,24(13), & a = 0.23(5)\ , \\ N_\tau = 4, & \beta_{c,\infty} = 1.699(1), & a = 0.20(6)\ , \\ N_\tau = 5, & \beta_{c,\infty} = 1.769\,48(3), & a = 0.85(10)\ , \\ N_\tau = 6, & \beta_{c,\infty} = 1.8287(11), & a = 3(1)\ . \end{array} \tag{29}$$

For what concerns the lattice with $N_\tau = 2$, we have not enough statistics to distinguish the different intersection points. As this point is in the strong coupling region, a high precision determination is not necessary.

For the $N_\tau = 7, 8$ lattices we have only a single crossing, so we can estimate the critical coupling by assuming some value for the a coefficient. By choosing $a \simeq 3-5$ for the $N_\tau = 7$ lattice we obtain, having $b = 20/14 = 1.43$, a value for $\varepsilon \simeq 4.2 \times 10^{-4}$; hence, $a\varepsilon \simeq 1 - 2 \times 10^{-3}$. Analogously one obtains $a\varepsilon \simeq 3 \times 10^{-3}$ for the $N_\tau = 8$ lattice, so we estimate a systematic error of order 10^{-3}, giving

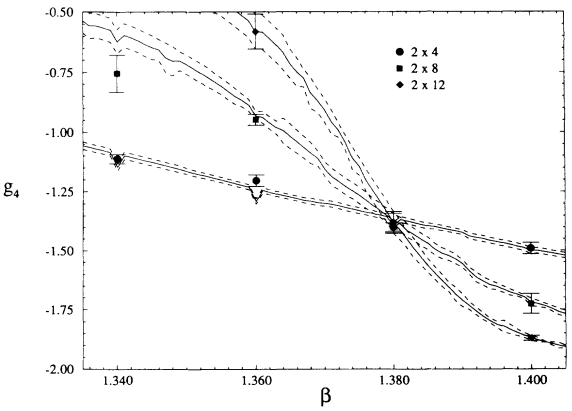

FIG. 1. Binder cumulant: lattices with $N_\tau = 2$.

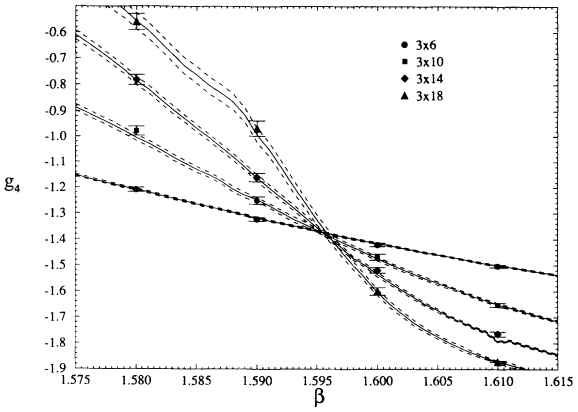

FIG. 2. Binder cumulant: lattices with $N_\tau = 3$.

$$\begin{array}{l} N_\tau = 7, \quad \beta_{c,\infty} = 1.8747 \pm 0.002 \pm 0.002\ , \\ N_\tau = 8, \quad \beta_{c,\infty} = 1.920 \pm 0.004 \pm 0.003\ . \end{array} \tag{30}$$

Using the two-loop asymptotic formula, Eq. (15), these values are reexpressed in terms of the $T_c/\Lambda_{\rm I}$ ratio in Table IX .

2. χ method

A different way to determine the critical point has been recently proposed by Engels *et al.* in [18]. The idea is to exploit the specific form of the scaling law for the susceptibility, which can be written as

$$\chi_v = N_\sigma^3 \left\langle P^2 \right\rangle\ , \tag{31}$$

taking into account that in finite volume the expectation value of Polyakov loops is set to zero by spin flips between degenerate vacua.

Expanding in the vicinity of the transition the expression in Eq. (18) one obtains

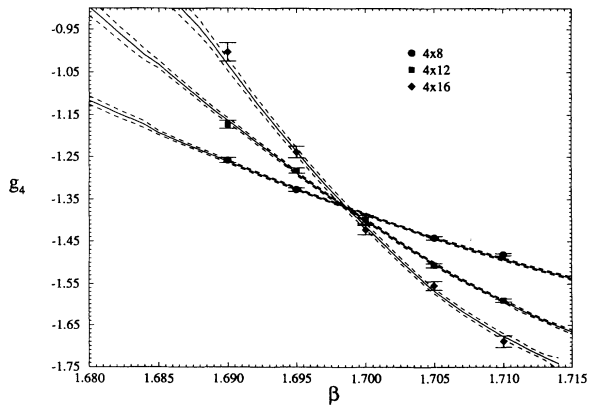

FIG. 3. Binder cumulant: lattices with $N_\tau = 4$.

TABLE VII. Run parameters on symmetric lattices, and expectation values of plaquettes.

N_σ	N_τ	β	$N_{\rm meas}$	$\tau_{\rm int}$	$U_{1\times1}$	$U_{1\times2}$
12	12	1.375000	384	1.0	0.51106(11)	0.25133(14)
12	12	1.385000	384	1.0	0.51488(13)	0.25561(20)
14	14	1.590000	384	1.0	0.596484(95)	0.35759(14)
14	14	1.595000	384	1.0	0.598358(84)	0.36011(12)
14	14	1.600000	384	1.0	0.600174(94)	0.36261(15)
16	16	1.690000	384	1.0	0.631531(63)	0.40691(11)
16	16	1.695000	384	1.1	0.633037(61)	0.409088(95)
16	16	1.700000	384	1.0	0.634699(72)	0.41150(12)
16	16	1.705000	384	1.0	0.636203(78)	0.41371(10)
20	20	1.769000	2304	1.0	0.654686(17)	0.440752(27)
20	20	1.770000	2304	1.0	0.654941(13)	0.441124(22)
20	20	1.827000	2048	1.0	0.669217(16)	0.462206(26)
20	20	1.828500	3072	1.0	0.669557(10)	0.462711(17)
20	20	1.830000	2048	1.0	0.669923(13)	0.463250(22)
20	20	1.872500	1536	1.2	0.679420(18)	0.477325(29)
20	20	1.877500	1536	1.0	0.680491(14)	0.478897(24)
20	20	1.915000	1536	1.0	0.688180(16)	0.490325(27)
20	20	1.925000	1536	1.0	0.690137(15)	0.493229(24)

$$\chi_v = \left(\frac{N_\sigma}{N_\tau}\right)^{\gamma/\nu}\left\{c_0 + \left[c_1 + c_2\left(\frac{N_\sigma}{N_\tau}\right)^{y_1}\right] t\frac{N_\sigma}{N_\tau}^{1/\nu} + c_3\left(\frac{N_\sigma}{N_\tau}\right)^{y_1}\right\},$$

$$\ln(\chi_v)|_{t=0} = \ln(c_0) + \frac{\gamma}{\nu}\ln\left(\frac{N_\sigma}{N_\tau}\right) + \frac{c_3}{c_0}\left(\frac{N_\sigma}{N_\tau}\right)^{y_1}. \tag{32}$$

Hence, assuming that the effect of the irrelevant is small ($y_1 \simeq -1$), at the phase transition the true susceptibility χ_v, plotted as a function of the spatial size on a doubly logarithmic scale, should exhibit a linear behavior.

Let us apply this method to the $N_\tau = 3$ lattice. We use the density of states method to determine χ_v at various values of β, and fit the data to a doubly logarithmic law. The resulting value for the minimum of χ^2 is at $\beta_c = 1.5956(4)$ at 95% C.L., which is in good agreement with the g_4 determination. One obtains also the slope $\gamma/\nu = 1.91(3)$, which exhibits a 3% difference from the known value in the 3D Ising model $\gamma/\nu = 1.965(11)$.

The same procedure has been applied to the other lattices: we find that, when applicable to our data, this method gives essentially the same result as the method of the Binder cumulant, so we have used it as a cross-check of the validity of the analysis.

3. *Universal scaling behavior*

We may test the universal scaling laws, expressed by Eqs. (22) and (18), by plotting $g_4(t,y)$ ($y = N_\sigma/N_\tau$), as a function of the combination $ty^{1/\nu}$, and the analogous expression for the Polyakov loop, $N_\tau y^{\eta/\nu}\langle|P|\rangle$: we choose $\nu = 0.628$, $\eta/\nu = 0.516$, $\gamma/\nu = 1.965$ as given by the 3D Ising model. In Fig. 8 one may see that the g_4

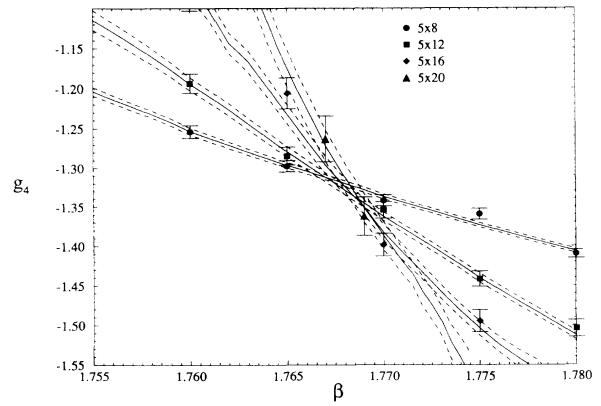

FIG. 4. Binder cumulant: lattices with $N_\tau = 5$.

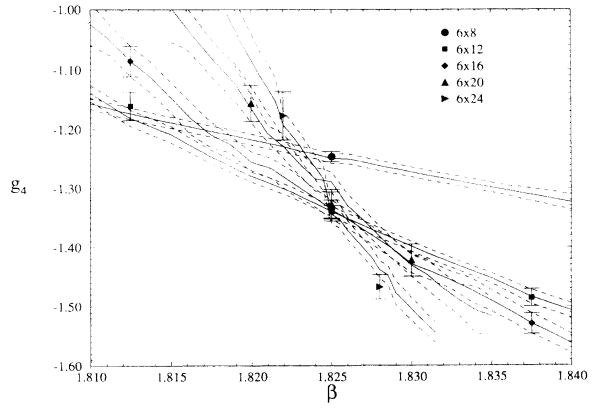

FIG. 5. Binder cumulant: lattices with $N_\tau = 6$.

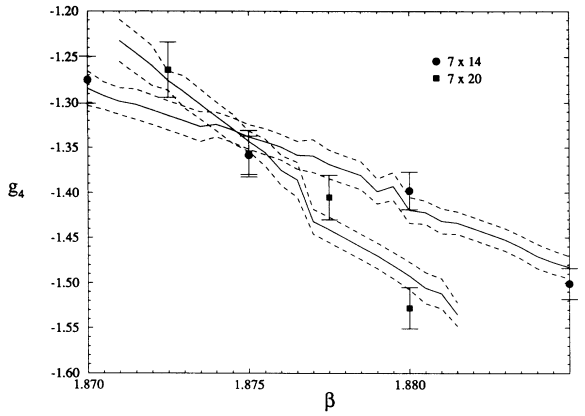

FIG. 6. Binder cumulant: lattices with $N_\tau = 7$.

values cluster around a straight line, thus supporting the theoretical conclusions. The agreement is not so good for the Polyakov line, Fig. 9, which is sensitive to the self-energy contribution.

B. Critical temperature in the bare scheme

Let us give in Fig. 10 a cumulative plot of the various determinations of the critical temperature. The data on the Wilson action are taken from [1] and normalized to the point at $N_\tau = 8$ (instead of using, for instance, the perturbative Λ ratio).

A few comments are in order, the weak coupling regime appears to be reached earlier in the simulation with the Symanzik action; in the weak coupling regime the pattern of asymptotic scaling violation appears to be essentially the same, by using Wilson and Symanzik actions.

This result is in clear disagreement with the earlier work by Curci *et al.* [6], where asymptotic scaling was found with the Symanzik action already at $N_\tau = 5$. We have then analyzed our data with the same method, based on a fit to the critical behavior of the Polyakov line. For $\beta > \beta_c$, the form

$$\langle |L| \rangle = A \ (\beta - \beta_c)^\eta \tag{33}$$

is assumed, while for $\beta < \beta_c$ the Polyakov line is set to zero: this introduces some arbitrariness in the fitting procedure. In fact, using Eq. (33) to determine β_c one is forced to discard data points assumed to be on the "left" of the transition. Moreover, on the "right" side of the transition (smaller lattice spacing) there are stronger renormalization effects, and these introduce a systematic which could be reduced by subtracting perturbative tails.

In Fig. 11 we show the data points, together with the results of a three-parameter fit to the critical behavior: the study is done for lattices $3 \times 10, 4 \times 12, 5 \times 16$.

TABLE VIII. Crossings for g_4 curves.

N_τ	N_σ	$N_{\sigma'}$	β_c
2	4	8	1.380(4)
2	4	12	1.380(2)
2	8	12	1.380(4)
3	6	10	1.595(1)
3	6	14	1.5955(8)
3	6	18	1.5958(5)
3	10	14	1.5956(12)
3	10	18	1.5960(7)
3	14	18	1.5963(15)
4	8	12	1.6984(7)
4	8	16	1.6987(5)
4	12	16	1.699(1)
5	8	12	1.7669(12)
5	8	16	1.7678(8)
5	8	20	1.7682(7)
5	12	16	1.7684(15)
5	12	20	1.769(12)
5	16	20	1.769(3)
6	8	12	1.812(4)
6	8	16	1.817(2)
6	8	20	1.822(2)
6	8	24	1.824(1)
6	12	16	1.823(5)
6	12	20	1.828(2)
6	12	24	1.825(1)
6	16	20	1.830(4)
6	16	24	1.825(2)
6	20	24	1.824(3)
7	14	20	1.8747(20)
8	16	24	1.920(4)

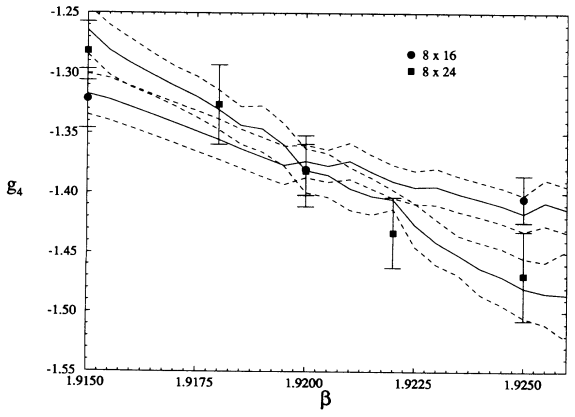

FIG. 7. Binder cumulant: lattices with $N_\tau = 8$.

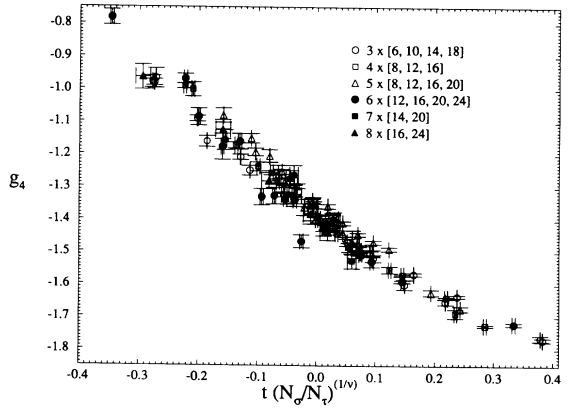

FIG. 8. Universal scaling behavior of Binder cumulant.

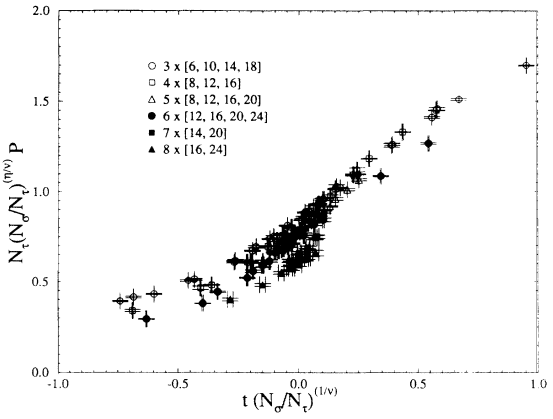

FIG. 9. Universal scaling behavior of Polyakov loop.

Results of the fit are shown in Table X, together with the value of the best reduced χ^2 found. Errors are estimated with the method of χ^2 equisurfaces.

The resulting critical temperatures are also reported in Fig. 10. The high value of the reduced χ^2, especially on the 4×12 lattice, shows the bad quality of the fit, but for our purposes it is important to point out that the finite volume "critical" temperature determined by this method shows no sign of asymptotic scaling. On the contrary, it is present only a constant shift from the infinite volume limit, and we conclude that the results of [6] were hampered by a insufficient statistic.

1. *Perturbative Λ ratio*

We can exploit the high precision data obtained with the Wilson action by Fingberg *et al.* [1] to give in Table XI the resulting determination of Λ ratios, by using the asymptotic formula

$$\frac{\Lambda_{\rm I}}{\Lambda_{\rm L}} = \left(\frac{\beta_{\rm I}}{\beta_{\rm L}}\right)^{b_1/2b_0^2} \exp\left(\frac{\beta_{\rm L}-\beta_{\rm I}}{4C_a b_0}\right) . \qquad (34)$$

These values are to be confronted with the perturbative results from Weisz and Wohlert [25],

$$\frac{\Lambda_{\rm I}}{\Lambda_{\rm L}} = 4.130\,89(1) , \qquad (35)$$

showing a discrepancy of the order of 10%.

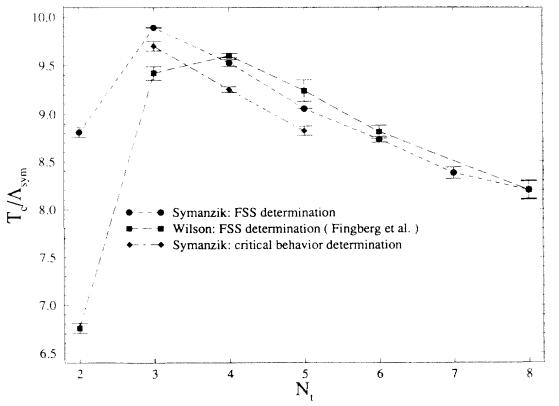

FIG. 10. Plot of the T_c/Λ ratio by use of the "bare" scheme.

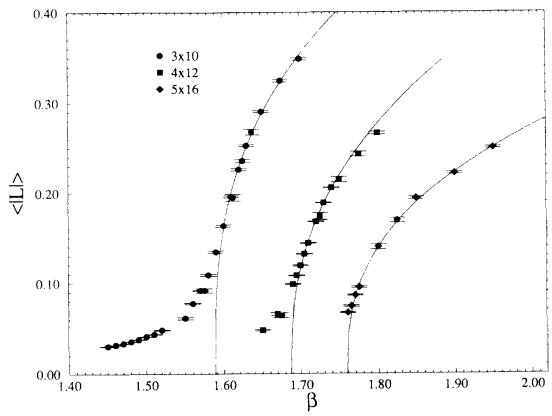

FIG. 11. Critical behavior of Polyakov line.

VI. DISCUSSION OF RESULTS

We can resume the outcome of our simulations as follows: Symanzik and Wilson actions give consistent results in the so-called "bare" scheme; that is, the renormalization group evolution driven by a scheme-independent part of the perturbative β function does not seem to show the scaling of the temperature. The pattern of scaling violation is similar, at least for $N_\tau > 4$.

On the other hand it is well known that adimensional ratios of physical quantities show a good scaling in the coupling range considered in this work, at least in the case of Wilson action, where the ratio $T_c/\sigma^{1/2}$ has been showed to scale over the entire $\beta = 2.30 - 2.74$ range [1].

In our opinion these two facts, that is, the good scaling and the similar pattern of asymptotic scaling violation in the two actions, are consistent: as anticipated, in the scaling region the improvement has no effect on the asymptotic scaling.

Let us first elaborate on the statement that the two scaling figures are compatible with each other: to do this, we set up a perturbative framework and discuss the effect of the coupling redefinition implied by the use of the Symanzik action.

We split $\beta(g)$ in the universal $\beta_u = -b_0 g^3 - b_1 g^5$ and in the scheme-dependent (unknown) part $\beta_{\rm s.d.} = O(g^7)$, where

$$b_0 = \frac{1}{(4\pi)^2}\frac{11C_a}{3}, \quad b_1 = \frac{1}{(4\pi)^4}\frac{34C_a^2}{3} . \qquad (36)$$

The β function determines the dependence of lattice spacing a on the g value:

TABLE IX. Critical temperature by two loop asymptotic scaling formula.

N_τ	β_c	$T_c/\Lambda_{\rm I}$
2	1.380(4)	8.81(8)
3	1.59624(13)	9.889(3)
4	1.699(1)	9.526(30)
5	1.76948(3)	9.0562(7)
6	1.8287(11)	8.73(3)
7	1.8747(30)	8.38(6)
8	1.920(5)	8.2(1)

$$\ln(a) = -\int \frac{dg}{\beta(g)} + C = \int \left(\frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} + \sum_{n=0}^{\infty} c_n g^{2n+1} \right) dg + C = -\frac{1}{2b_0 g^2} - \frac{b_1}{b_0^2} \ln(g) + \sum_{n=0}^{\infty} c_n \frac{g^{2n+2}}{2n+2},$$

$$a(g) = \frac{1}{\Lambda_L} \left(b_0 g^2\right)^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 g^2}\right) \left(1 + g^2 \sum_{n=0}^{\infty} d_n g^{2n}\right) . \tag{37}$$

The coefficients d_n in Eq. (37) are generated by $\beta_{\text{s.d.}}$; they play the role of asymptotic scaling-violating terms. For instance, in terms of the first subasymptotic term in $\beta_{\text{s.d.}}$, the b_2 coefficient, one easily finds

$$d_0 = \frac{1}{2b_0} \left(\frac{b_2}{b_0} - \left(\frac{b_1}{b_0}\right)^2 \right) . \tag{38}$$

If we redefine the coupling constant

$$g = g(\bar{g}) = \bar{g} + \alpha_1 \bar{g}^3 + \alpha_2 \bar{g}^5 + \cdots + \alpha_n \bar{g}^{2n+1} + \cdots \tag{39}$$

we obtain

$$a(\bar{g}) = \frac{e^{\alpha_1/b_0}}{\Lambda_L} \left(b_0 \bar{g}^2\right)^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 \bar{g}^2}\right) \left(1 + \bar{g}^2 \sum_{n=0}^{\infty} d_n^{\star} \bar{g}^{2n}\right) . \tag{40}$$

This is the evolution dictated by a new β function $\beta^{\star}(g)$,

$$\beta^{\star}(\bar{g}) = \beta[g(\bar{g})] \left(\frac{\partial g(\bar{g})}{\partial \bar{g}}\right)^{-1} = \beta_u(\bar{g}) + \beta^{\star}_{\text{s.d.}}(\bar{g}) , \tag{41}$$

and the nonuniversal terms are changed, for instance, one has

$$b_2^{\star} = b_2 - 2\alpha_1 b_1 - 3\alpha_1^2 b_0 + 2b_0 \alpha_2 . \tag{42}$$

The prefactor $\exp(\alpha_1/b_0)$ redefines the scale parameter; consequently,

$$\alpha_1 = -b_0 \ln\left(\frac{\Lambda^{\star}}{\Lambda_L}\right) . \tag{43}$$

On the other hand the asymptotic scaling-violating contribution is changed,

$$d_0^{\star} = d_0 - \alpha_1 \left(\frac{b_1}{b_0^2} + \frac{3\alpha_1}{2b_0} \right) + \frac{\alpha_2}{b_0} , \tag{44}$$

and one may hope to reduce its effect by a clever (and physically motivated) choice of $\bar{g}$.

To study if the asymptotic scaling violation can be ascribed to the d_2 terms, we have performed an exploration of values of b_2 for both actions: in Fig. 12 we show how setting values of b_2 for Wilson and Symanzik actions to $0.0018, 0.0011$, respectively, one is able to obtain a good scaling: the $N_\tau = 2$ term has been dropped for clarity.

On the other hand, we are well aware of the arbitrariness of the procedure, as we have checked that this fit procedure is unstable if also the coefficient b_3 is introduced. This is easily understood as values of b_2 so large are hard to justify on a perturbative ground: too see this consider the running coupling equation in terms of the characteristic QCD coupling, $\alpha_s/(4\pi)$,

$$\begin{aligned} a\frac{d}{da}\left(\frac{\alpha_s}{4\pi}\right) &= \frac{22C_a}{3}\left(\frac{\alpha_s}{4\pi}\right)^2 + \frac{68C_a^2}{3}\left(\frac{\alpha_s}{4\pi}\right)^3 + (4\pi)^6 b_2 \left(\frac{\alpha_s}{4\pi}\right)^4 + \cdots \\ &\simeq \frac{44}{3}\left(\frac{\alpha_s}{4\pi}\right)^2 + \frac{272}{3}\left(\frac{\alpha_s}{4\pi}\right)^3 + 4330\left(\frac{\alpha_s}{4\pi}\right)^4 + \cdots , \end{aligned} \tag{45}$$

for instance at $\beta \simeq 1.9$ one has $\alpha_s/(4\pi) \simeq 0.013$, which means that the last term in Eq. (45) is of the same order of magnitude as the preceding one. So we can merely regard this procedure as the definition of an effective b_2, much in the spirit of the effective Λ used in [26].

TABLE X. Results of the fit.

N_τ	β_c	T_c/Λ	A	η	χ^2
3	1.588(2)	9.70(5)	0.76(4)	0.35(2)	5.4
4	1.687(2)	9.25(3)	0.66(4)	0.40(2)	15.3
5	1.759(3)	8.82(5)	0.47(2)	0.38(2)	4.0

A. Perturbative approach: α_v coupling

Let us consider more closely the poor convergence of the speculative expansion given in Eq. (45): in the preceding subsection we looked for an appropriate b_2 coefficient without any physical motivation. On the other hand, Lepage and Mackenzie [3] point out how a bad choice of the expansion parameter may give rise to a poorer convergence of the perturbation series, and they suggest strategies to design improved couplings, defined in terms of physical quantities.

This is much in the spirit of the old suggestions of

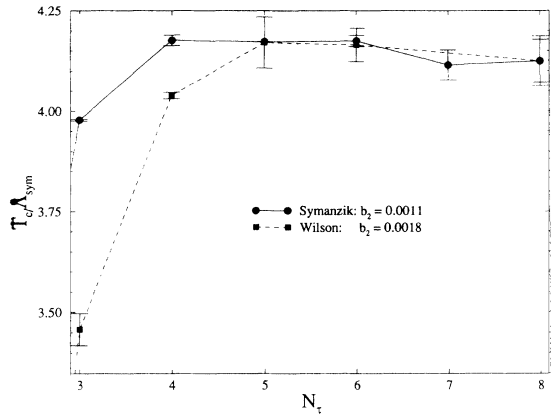

FIG. 12. "Bare" scheme corrected with the first subasymptotic coefficient of the β function.

Parisi, to define an effective coupling by inverting the perturbative expression for some ultraviolet-dominated quantity, such as the single plaquette.

In [3] it is suggested to define an effective coupling α_v in terms of the quark-quark potential by the relation

$$V(q) = -\frac{C_f\,(4\pi)\,\alpha_v\,(q^2)}{q^2} \ . \tag{46}$$

This coupling can be perturbatively related to α_0, the bare lattice coupling, by the formula

$$\alpha_v(q^2) = \alpha_0 \left[1 + \alpha_0 \left((4\pi)\, b_0 \ln\left(\frac{\pi}{aq}\right)^2 + K\right)\right] \ , \tag{47}$$

where the constant K can be extracted by Kovacs' work [27]: she reports

$$V(q) \propto -\frac{1}{q^2}\left(b_0 \ln\left(\frac{q^2}{\pi^2 \Lambda_L^2}\right) + \frac{C_a}{(4\pi)^2} J_L\right)^{-1} \ . \tag{48}$$

Numerically one finds

$$J_L = \begin{cases} -16.954 & \text{for } N = 2 \ , \\ -19.695 & \text{for } N = 3 \ . \end{cases} \tag{49}$$

To translate the result for the Symanzik action it is sufficient to rescale Λ factors and impose equality in the physical result. The defining relation is

$$b_0 \ln\left(\frac{1}{\Lambda_L^2}\right) + \frac{C_a}{(4\pi)^2} J_L \equiv b_0 \ln\left(\frac{1}{\Lambda_I^2}\right) + \frac{C_a}{(4\pi)^2} J_I \ . \tag{50}$$

One obtains easily, for SU(2),

$$J_I = J_L + \frac{11}{3} \ln\left(\frac{\Lambda_I}{\Lambda_L}\right)^2 \ . \tag{51}$$

It is then possible to use the result of Weisz and Wohlert [25] on the Λ ratio:

$$\begin{aligned} \frac{\Lambda_I}{\Lambda_L} &= \exp\left[\frac{1}{(4\pi)\, b_0}\left(C_a\,[0.043\,290\,17(1)] - \frac{1}{C_a}\,[0.041\,414\,17(1)]\right)\right] \\ &= \begin{cases} 4.130\,89(1), & \text{for } N = 2 \ , \\ 5.292\,10(1), & \text{for } N = 3 \ , \end{cases} \end{aligned} \tag{52}$$

and it follows that

$$J_I = \begin{cases} -6.548\,59 & \text{for } N = 2 \ , \\ -7.476\,09 & \text{for } N = 3 \ . \end{cases} \tag{53}$$

Translating into the expression for K, one easily obtains

$$K_L = \begin{cases} 2.698\,31 & \text{for } N = 2 \ , \\ 4.701\,8 & \text{for } N = 3 \ , \end{cases}$$

$$K_I = \begin{cases} 1.042\,24 & \text{for } N = 2 \ , \\ 1.784\,79 & \text{for } N = 3 \ . \end{cases} \tag{54}$$

We can now use the value of α_0 to set the value $\alpha_v\,(\pi/a)$: one has obviously

$$\alpha_v\,(\pi/a) = \alpha_0\left(1 + \alpha_0 K_{L(I)}\right) \ . \tag{55}$$

For a given value $\beta_c = C_a/(2\pi\alpha_c)$ of the measured critical coupling, the corresponding critical temperature in the α_v scheme may be obtained by using the relation

$$\frac{T}{\Lambda_v} = \frac{1}{\pi N_\tau}\frac{\pi}{a\Lambda_v} \tag{56}$$

and the two-loop scaling formula

$$\frac{q}{\Lambda_v} = \exp\left[\frac{1}{(8\pi)\, b_0}\frac{1}{\alpha_v\,(q)} - \frac{b_1}{2b_0^2}\ln\left(\frac{1}{4\pi b_0 \alpha_v\,(q)}\right)\right] \ . \tag{57}$$

Let us report in Tables XII and XIII the results of the analysis both for the Wilson and the Symanzik case.

It appears that the use of a perturbative expression for the effective coupling α_v does not give a better asymptotic scaling. It should be stressed however that this was merely an exercise, as the Lepage scheme is intended to work in conjunction with a nonperturbative determina-

TABLE XI. Λ ratios from asymptotic formula.

N_τ	β_I	β_L	Λ_I/Λ_L
2	1.380(4)	1.8800(30)	3.599(46)
3	1.59624(13)	2.1768(30)	4.470(35)
4	1.699(1)	2.2986(6)	4.713(14)
5	1.76948(3)	2.3726(45)	4.766(56)
6	1.8287(11)	2.4265(30)	4.709(39)
8	1.920(5)	2.5115(40)	4.644(77)

TABLE XII. Perturbative α_v: Wilson case.

N_τ	α_0	α_v	T/Λ_L	T/Λ_v
2	0.1693(3)	0.2466(6)	29.7(2)	2.27(2)
3	0.1462(2)	0.2039(3)	41.4(3)	2.89(2)
4	0.13848(4)	0.19022(7)	42.1(1)	2.848(4)
5	0.1342(2)	0.1828(3)	40.6(5)	2.69(2)
6	0.1312(2)	0.1776(3)	38.7(3)	2.54(2)
8	0.1267(2)	0.1700(3)	36.0(4)	2.32(2)
16	0.1162(4)	0.1526(6)	32.0(8)	1.97(4)

tion of the coupling α_v, based on the measurement of the heavy-quark potential.

B. Nonperturbative approach

The bare coupling itself has no meaning in itself, so it must be eliminated in favor of some physical quantity, connected to it. In the perturbative approach this is a way to hide infinities and obtain a renormalized field theory.

A simple way to implement this program on the lattice is the elimination of the bare coupling in favor of some UV dominated quantity, such as the expectation value of plaquettes. To this end, let us recall the results of the work of Weisz and Wohlert [25] on their perturbative expansion: by defining, for simplicity in the adjoint representation,

$$\ln\left[\frac{1}{C_A}\langle \mathrm{Tr}\, U(L,T)\rangle\right] = -\sum_{n=1}^{\infty}\frac{g^{2n}}{(2n)!}w_n(L,T)\ , \quad (58)$$

the first coefficient is given by

$$w_1(L,T) = C_f I(L,T) \quad (59)$$

and the following values are given:

$$I(1,1) = \begin{cases} \frac{1}{2} & \text{Wilson}\ , \\ 0.366\,262 & \text{Symanzik}\ , \end{cases}$$

$$I(1,2) = \begin{cases} 0.862\,251 & \text{Wilson}\ , \\ 0.662\,624 & \text{Symanzik}\ . \end{cases} \quad (60)$$

Coefficients for the Wilson case are known up to fourth w_4, while for Symanzik we shall limit ourselves to lowest order.

This allows one to define two effective couplings in the Symanzik case, depending on the two expectation values

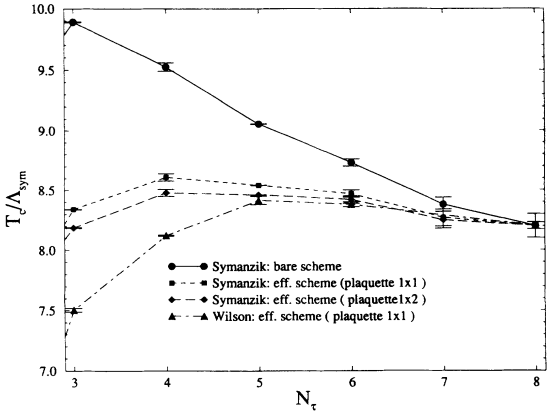

FIG. 13. Comparison of "bare" and effective schemes.

measured. That is, knowing that

$$\ln\left[\frac{1}{C_A}\langle \mathrm{Tr}\, U(L,T)\rangle\right] = -\frac{g^2}{2}C_f I(L,T) \quad (61)$$

and given $\beta = \frac{2C_a}{g^2}$, we define

$$\beta_{1(2)} = C_a C_f \frac{I(1,1\,(2))}{1 - U_{1\times 1(1\times 2)}}\ . \quad (62)$$

Numerically, for the SU(2) case, this corresponds to

$$\beta_{1(2)} = \frac{0.549\,393\,(0.993\,936)}{1 - U_{1\times 1(1\times 2)}}\ . \quad (63)$$

By using the results of measurements on large symmetric lattices we can obtain via interpolation the values of plaquettes needed to compute the effective coupling: we report the results in Table XIV.

We plot in Fig. 13 the comparison of the different schemes, together with the Wilson data, taken from [1], in the effective scheme deduced from the single plaquette. As usual we normalize to the last point: we drop the point at $N_\tau = 2$ to show in a larger scale the other points. It is worth noting that the two effective schemes used for the Symanzik action agree with each other, they both show a good scaling starting at $N_\tau = 4$, and the same scheme used with Wilson action seems to show, starting at $N_\tau = 5$ the same behavior.

We think that these results can be interpreted as a confirmation of the value of effective schemes, and as an

TABLE XIII. Perturbative α_v: Symanzik case.

N_τ	α_0	α_v	T/Λ_I	T/Λ_v
2	0.2306(7)	0.286(1)	8.81(8)	1.49(1)
3	0.19941(2)	0.24085(3)	9.889(3)	1.627(6)
4	0.1873(1)	0.2239(1)	9.526(30)	1.549(2)
5	0.179889(3)	0.213616(4)	9.0562(7)	1.461(1)
6	0.1741(1)	0.2057(1)	8.73(3)	1.398(2)
7	0.1698(3)	0.1998(4)	8.38(6)	1.34(1)
8	0.1658(4)	0.1944(5)	8.2(1)	1.30(1)

TABLE XIV. Data for the effective schemes.

β	β_1	T_1/Λ_1	β_2	T_2/Λ_2
1.380(4)	1.1280(35)	1.832(15)	1.331(30)	2.953(21)
1.59624(13)	1.36942(16)	2.15542(83)	1.55480(16)	3.3651(13)
1.699(1)	1.5026(13)	2.2247(73)	1.6876(13)	3.485(12)
1.76948(3)	1.59156(3)	2.2072(2)	1.77784(3)	3.4781(3)
1.8287(11)	1.6628(13)	2.1876(70)	1.850(1)	3.462(11)
1.8747(30)	1.7163(34)	2.136(18)	1.9041(34)	3.391(29)
1.920(5)	1.7674(56)	2.119(29)	1.9557(56)	3.371(47)

indication that Symanzik action gives an enlargement of the scaling window.

VII. CONCLUSIONS

In this work we have performed a high statistics simulation of the SU(2) pure gauge theory at finite temperature, using the tree-improved Symanzik action and exploring a range of lattices up to $N_\tau = 8$; we have determined, with the help of finite size scaling methods, the critical couplings for the deconfinement transition, with the purpose of studying the scaling properties of the critical temperature and the effect of improvement.

We can resume as follows our findings.

Asymptotic scaling violations are present, and are of the same size as in the case of Wilson action. The pattern of violation is similar and can be interpreted as driven by the same lattice β function, modulo scheme redefinitions. This means either that lattice artifacts are small, or they are the same for the two actions, in accordance with previous determinations of the scaling window.

The use of a nonperturbative coupling derived from the plaquette expectation value gives a better scaling figure: this effective coupling was already used in the same context but with the Wilson action, and comparison of the two analyses shows that Symanzik action apparently gives a slightly precocious scaling.

In our study we are unable to directly test the scaling, so it could be a desirable zero temperature study of some other dimensionful quantity, like masses or the string tension.

From a practical point of view, if we are willing to trust the precocious onset of scaling in the effective scheme, we can say that the gain in volume obtained by use of the improved action compensates the increased complexity in the update routine, thus slightly favoring the use of this technique in simulations. This should be even more true in the case of the SU(3) group.

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APPENDIX A: STATISTICAL EVALUATION OF BINDER'S CUMULANT

In order to extract in a reliable way the quantities we are interested in from a set of N measures we have to correct for the distortions induced by the finite sampling. It is well-known that often the "naive" estimator for a statistical quantity is not the best one. The much simpler and well-known example is the estimator of variance σ^2 for a set of N uncorrelated data x_i. One may try

$$E_b(\sigma^2) = E(\langle x^2\rangle) - E(\langle x\rangle)^2 = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)\left(\frac{1}{N}\sum_{j=1}^{N} x_j\right) \tag{A1}$$

but, owing to the fact that $E(\langle x\rangle)$ is a random variable with a finite variance the estimator's expectation value is

$$\langle E_b(\sigma^2)\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle x^2\rangle - \frac{1}{N^2}\sum_{i=1,j=1}^{N}\left[\delta_{ij}\langle x^2\rangle + (1-\delta_{ij})\langle x\rangle^2\right] = \left(1-\frac{1}{N}\right)\sigma^2, \tag{A2}$$

which is "biased" by a finite sample effect $O(\frac{1}{N})$. To evaluate in the best way σ^2 we must correct for this bias: this can be done using a new redefined estimator

$$E(\sigma^2) = N/(N-1)\; E_b(\sigma^2) \tag{A3}$$

which gives the correct result $\langle E(\sigma^2)\rangle = \sigma^2$.

Now we want to study Binder's cumulant g_4 of a random variable L, where the general cumulant g_{2n} (in the $\langle L^{2n+1}\rangle = 0$ case which is of interest to us) is defined as

$$g_{2n} = \frac{\langle L^{2n}\rangle_c}{\langle L^2\rangle_c^n}, \tag{A4}$$

$$g_4 = \frac{\langle L^4\rangle_c}{\langle L^2\rangle_c^2} = \frac{\langle L^4\rangle}{\langle L^2\rangle^2} - 3, \tag{A5}$$

$$g_6 = \frac{\langle L^6\rangle - 15\langle L^2\rangle\langle L^4\rangle + 30\langle L^2\rangle^3}{\langle L^2\rangle^3}. \tag{A6}$$

The "natural" estimator for g_4 over a set of N mea-

surements is given by

$$\bar{g}_4 = \left(\frac{1}{N}\sum_i L_i^4\right)\left(\frac{1}{N}\sum_i L_i^2\right)^{-2} - 3, \qquad \text{(A7)}$$

but as we will see this is not a self-averaging quantity, in the sense that $\langle \bar{g}_4 \rangle \neq g_4$, and we have to redefine it. To calculate the bias of g_4 we introduce the $O(\frac{1}{\sqrt{N}})$ fluctuations η_2 and η_4 defined by

$$\eta_2 = \frac{1}{N}\sum_i L_i^2 - \langle L^2 \rangle,$$
$$\eta_4 = \frac{1}{N}\sum_i L_i^4 - \langle L^4 \rangle, \qquad \text{(A8)}$$

and we expand Eq. (A7) around $\eta = 0$, to obtain

$$\bar{g}_4 = g_4\left[1 - 2\frac{\eta_2}{\langle L^2\rangle} + \frac{\eta_4}{\langle L^4\rangle} + 3\frac{\eta_2^2}{\langle L^2\rangle^2} - 2\frac{\eta_2\eta_4}{\langle L^2\rangle\langle L^4\rangle} + o\left(\frac{1}{N}\right)\right]. \qquad \text{(A9)}$$

Now to evaluate the corrections we take the expectation value, and we get

$$\langle \bar{g}_4 \rangle = g_4\left[1 + \frac{3}{\langle L^2\rangle^2}\langle \eta_2^2 \rangle - \frac{2}{\langle L^2\rangle\langle L^4\rangle}\langle \eta_2\eta_4 \rangle + o\left(\frac{1}{N}\right)\right], \qquad \text{(A10)}$$

where

$$\langle \eta_2\eta_4 \rangle = \frac{1}{N}\left(\langle L^6\rangle - \langle L^4\rangle\langle L^2\rangle\right)\left(\sum_{\delta=0}^{N-1}\frac{N-\delta}{N}\left[C(L^2,L^4,\delta) + C(L^4,L^2,\delta)\right] - 1\right),$$
$$\langle \eta_2^2 \rangle = \frac{1}{N}\left(\langle L^4\rangle - \langle L^2\rangle^2\right)\left(\sum_{\delta=0}^{N-1}\frac{N-\delta}{N}\left[2C(L^2,L^2,\delta) - 1\right]\right), \qquad \text{(A11)}$$

and

$$C(X,Y,\delta) = \frac{\langle (X_i - \langle X\rangle)(Y_{i+\delta} - \langle Y\rangle)\rangle}{\langle (X_i - \langle X\rangle)(Y_i - \langle Y\rangle)\rangle}. \qquad \text{(A12)}$$

If N is much bigger than the correlation times of interest then $C(X,Y,\delta) \simeq 0$ for $\delta > \bar{\delta}$, with $\bar{\delta} < n$, so we can write approximately

$$\langle \eta_2\eta_4 \rangle \simeq \frac{1}{N}\left(\langle L^6\rangle - \langle L^4\rangle\langle L^2\rangle\right)\left[\tau_{\rm int}(L^2,L^4) + \tau_{\rm int}(L^4,L^2) - 1\right], \qquad \text{(A13)}$$
$$\langle \eta_2^2 \rangle \simeq \frac{1}{N}\left(\langle L^4\rangle - \langle L^2\rangle^2\right)\left[2\tau_{\rm int}(L^2,L^2) - 1\right], \qquad \text{(A14)}$$

where we have defined the integrated autocorrelation time

$$\tau_{\rm int}(X,Y) = \sum_{\delta=0}^{\infty} C(X,Y,\delta). \qquad \text{(A15)}$$

In order to correct for the $O(\frac{1}{N})$ bias we note that in Eq. (A9) the terms linear in the fluctuations do not contribute to the expectation value. So it is sufficient to define an "improved" estimator $g_4^\star$ as

$$g_4^\star = \bar{g}_4\left[1 - 3\frac{\eta_2^2}{\langle L^2\rangle^2} + 2\frac{\eta_2\eta_4}{\langle L^2\rangle\langle L^4\rangle}\right] = g_4\left[1 - 2\frac{\eta_2}{\langle L^2\rangle} + \frac{\eta_4}{\langle L^4\rangle} + o\left(\frac{1}{N}\right)\right]. \qquad \text{(A16)}$$

To the order we are working we can substitute the quadratic fluctuations terms with the "naive" estimator for their expectation value, and we obtain

$$g_4^\star = \frac{\frac{1}{N}\sum_i L_i^4}{\frac{1}{N^2}\sum_{i,j} L_i^2 L_j^2}\left[1 - \frac{3}{N}\left(\frac{\frac{1}{N}\sum_i L_i^4}{\frac{1}{N^2}\sum_{i,j} L_i^2 L_j^2} - 1\right) E\left[2\tau_{\rm int}(L^2,L^2) - 1\right]\right.$$
$$\left. + \frac{2}{N}\left(\frac{\frac{1}{N}\sum_i L_i^6}{\frac{1}{N^2}\sum_{i,j} L_i^2 L_j^4} - 1\right) E\left[\tau_{\rm int}(L^2,L^2) + \tau_{\rm int}(L^2,L^4) - 1\right]\right]. \qquad \text{(A17)}$$

To evaluate the variance of $g_4^\star$ we can write, using Eq. (A16),

$$\sigma^2(g_4^\star) = \langle (g_4^\star)^2 \rangle - g_4^2 = g_4^2 \left[4\frac{\langle \eta_2^2 \rangle}{\langle L^2 \rangle^2} + \frac{\langle \eta_4^2 \rangle}{\langle L^4 \rangle^2} - 4\frac{\langle \eta_2 \eta_4 \rangle}{\langle L^2 \rangle \langle L^4 \rangle} \right]. \tag{A18}$$

We can estimate this quantity to the lowest order with

$$\begin{aligned} E\left[\sigma^2(g_4^\star)\right] = & \left(\frac{\frac{1}{N}\sum_i L_i^4}{\frac{1}{N^2}\sum_{i,j} L_i^2 L_j^2} \right)^2 \left[\frac{4}{N} \left(\frac{\frac{1}{N}\sum_i L_i^4}{\frac{1}{N^2}\sum_{i,j} L_i^2 L_j^2} - 1 \right) E\left[2\tau_{\rm int}(L^2, L^2) - 1\right] \right. \\ & + \frac{1}{N} \left(\frac{\frac{1}{N}\sum_i L_i^8}{\frac{1}{N^2}\sum_{i,j} L_i^4 L_j^4} - 1 \right) E\left[2\tau_{\rm int}(L^4, L^4) - 1\right] \\ & \left. - \frac{4}{N} \left(\frac{\frac{1}{N}\sum_i L_i^6}{\frac{1}{N^2}\sum_{i,j} L_i^2 L_j^4} - 1 \right) E\left[\tau_{\rm int}(L^2, L^2) + \tau_{\rm int}(L^2, L^4) - 1\right] \right]. \end{aligned} \tag{A19}$$

APPENDIX B: DENSITY OF STATES METHOD

To extract the value of an observable $\mathcal{O}$ from a Monte Carlo simulation we average over the sequence of values O_i generated by the algorithm. This is nothing else than an approximation of the path integral formula

$$\langle \mathcal{O} \rangle(\beta) = \frac{\int \mathcal{D}\psi \mathcal{O}[\psi] \exp(-\beta S[\psi])}{\int \mathcal{D}\psi \exp(-\beta S[\psi])} \simeq \frac{1}{N} \sum_i O_i. \tag{B1}$$

Using the fact that we know the probability distribution for the fundamental fields we can extrapolate the observable's value around the β value we have used in the simulation. To this effect we note that

$$\langle \mathcal{O} \rangle(\beta + \Delta\beta) = \frac{\int \mathcal{D}\psi \mathcal{O}[\psi] \exp(-\Delta\beta S[\psi]) \exp(-\beta S[\psi])}{\int \mathcal{D}\psi \exp(-\Delta\beta S[\psi]) \exp(-\beta S[\psi])} = \frac{\langle \mathcal{O} \exp(-\Delta\beta S) \rangle(\beta)}{\langle \exp(-\Delta\beta S) \rangle(\beta)}, \tag{B2}$$

$$\langle \mathcal{O} \rangle(\beta + \Delta\beta) \simeq \frac{\sum_i O_i \exp(-\Delta\beta S_i)}{\sum_i \exp(-\Delta\beta S_i)}, \tag{B3}$$

so it is sufficient to measure the action S_i correlated with each O_i value, and resum each measure according to Eq. (B3). The "resummed" value will be reliable only for small $\Delta\beta$,because we have a good statistical sampling of action distribution only in a small range around the mean value, so we must do a preliminary determination of the allowed resummation range. During our simulation we have used essentially the same method as Alves *et al.* in [28]: we plot histograms of the energy distribution, we set a minimum and maximum energy by imposing that at least 2.5% of the energy distribution is present below the minimum and above the maximum, and we translate this energy range in a β range by finding the β shifts that via resummation give the found energy limits.

If the resumming ranges of different simulations overlap we can "patch" the values coming from different extrapolations to obtain a more accurate estimate.

The determination of an unbiased estimator for the resummed Binder cumulant and its variance can be obtained with the same method as in Appendix A. We only note that there are some complications when we combine together different extrapolations, because in this case to cancel the bias we must use an improved estimator for the variances, depending on many-point correlations which can be estimated reliably only with very high statistics.

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