Singularities formed by the focusing of cylindrical null fluids

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Motivated by the recent studies of the late stages of gravitational collapse, the collision and interaction of cylindrically symmetric null fluids are studied. By investigating analytic models, we find that a naked singularity can be developed. This singularity is different from the ones found previously in the sense that it is solely formed by the mutual focus of the null fluids.

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I. INTRODUCTION

One of the most thorny and important problems in general relativity (GR) is gravitational collapse. It is well known that GR admits solutions with singularities, and that such singularities can be formed by the gravitational collapse of matter satisfying nonsingular, physically reasonable initial conditions. If clothed by event horizons these singularities would not cause any problem [1]. However, if the other alternative exists, that is, if these singularities are not clothed and instead are naked, it would be a disaster, since this will mean that no prediction can be made about the further evolution of a region containing such a singularity and that new information could emerge in a completely arbitrary way. To prevent such events from happening, several conjectures have been proposed [1–3]. So far, these conjectures can be considered only as hypotheses; even more, they have not yet been properly formulated.

To study the above issue in its most general terms is extremely difficult; instead, the attention has been focused on finding counterexamples [4–7]. In this way, we can hone the conditions to prevent the formation of naked singularities. In the past, most of the studies were restricted to spherical collapse [4]. Recently, the numerical stimulations of Shapiro and Teukolsky [6] and the analytic investigations of Barrabès, Israel, and Letelier [7] indicate that naked singularities may also arise in nonspherical collapse. In particular, Shapiro and Teukolsky found that a spheroid of collisionless gas always collapses into a spindle singularity. If the spheroid is sufficiently compact, the singularity is hidden inside a horizon. If the spheroid is elongated enough, no apparent horizon is developed, and most likely naked singularities are formed at the sharp ends of the imploding spheroid. These results are consistent with the earlier conclusions of Thorne [5], obtained by studying the collapse of a cylinder, which can be considered as an infinite version of a finite spheroid.

In this paper, we shall study the collision and interaction of two null fluids in the course of the gravitational collapse. The motivation is twofold. First, let us consider the collapse of a spheroid. According to the results presented in Ref. [8], the spheroid will emit gravitational radiation. This radiation then interacts with the space-time curvature, and will be partially backscattered and move inwards. Before the backscattered radiation is absorbed by the collapsing spheroid, it will collide and interact with the outgoing radiation. Because of the nonlinearity of the Einstein field equations, such collisions and interaction could play an important role in the gravitational collapse. If the collision and interaction happen when the radius of the curvature of the background is large, the graviton geometric optics approximation should be an accurate approximation, and the ingoing and outgoing fluxes can be approximately considered as null fluids [9,10]. Alternatively, we can consider the collapsing spheroid as consisting of some kind of fluid. During its collapse, some of the fluid will be reflected by the symmetry axis and move outwards. Therefore, in the latter case, the space-time will be filled with both ingoing and outgoing fluids. Since a realistic spheroid would tend to fall at nearly the speed of light during the late stages of the collapse, it is reasonable to consider the ingoing and outgoing fluids as being null, too.

It is clear that the analytic study of this problem in the space-time of spheroids is a very difficult task. In this paper, instead, we shall borrow Thorne's arguments [2,5], and study it in a space-time with cylindrical symmetry. This simplification will allow us to consider analytic models and to explore the global properties of the space-time, including the formation of event horizons.

The rest of this paper is organized as follows. In Sec. II the space-time with cylindrical symmetry is briefly reviewed, while in Sec. III the gravitational collapse of a single null fluid is studied. In particular, it is found that the collapse always forms an intermediate singularity at the axis, although it does not form a scalar one. Thus, Morgan's earlier conclusions [9] regarding to the singularity behavior of null fluids need to be revised. In Sec. IV we study some specific examples that represent the collision and interaction of two null fluids on certain space-time backgrounds. It is found that, due to the nonlinear interaction of the two null fluids, a space-time

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singularity can be formed in the interaction region. Section V contains our concluding remarks.

II. THE SPACE-TIME WITH CYLINDRICAL SYMMETRY AND THE NULL FLUID SOLUTIONS

To facilitate our discussions, in this section we shall review some general properties of the space-time with cylindrical symmetry, and the corresponding null fluid solutions of the Einstein field equations. Cylindrical space-times with two orthogonal Killing vectors are characterized by the metric [11]

$$ds^2=e^{-\Omega}(dt^2-dr^2)-e^{-h}\{e^{\Phi}dz^2+e^{-\Phi}d\varphi^2\} , \tag{2.1}$$

where Ω, h, and Φ are functions of t and r only. t is the timelike coordinate, r the spacelike radial coordinate, and z and φ are the axial and azimuthal coordinates, respectively, with $-\infty<t,\ z<+\infty$, $0\leq r<+\infty$, and $0\leq\varphi\leq 2\pi$.

Since we are concerned with null fluids, it is convenient to work with a double null coordinate system (u,v), which is defined as

$$u=\frac{t+r}{\sqrt{2}} , \quad v=\frac{t-r}{\sqrt{2}} . \tag{2.2}$$

In terms of u and v, Eq. (2.1) takes the form

$$ds^2=2e^{-\Omega}du\,dv-e^{-h}\{e^{\Phi}dz^2+e^{-\Phi}d\varphi^2\} , \tag{2.3}$$

but Ω, h, and Φ are now functions of u and v through the relations (2.2).

To study the null fluids, it is also convenient to introduce the null vectors l_μ and n_μ:

$$l_\mu=\delta^0_\mu , \quad n_\mu=\delta^1_\mu , \tag{2.4}$$

where the coordinates are numbered as $\{x^\mu\}=\{u,v,z,\varphi\},\mu=0,1,2,3$. From Eqs. (2.3) and (2.4) we find

$$l_{\mu;\nu}l^\nu=n_{\mu;\nu}n^\nu=0 , \tag{2.5}$$

where a semicolon denotes covariant differentiation, while a comma denotes partial differentiation. Equation (2.5) shows that each of the null vectors l_μ and n_μ defines an affinely parametrized null geodesic congruence. Thus, the quantity

$$Q_l\equiv -l^\lambda{}_{;\lambda}=e^{\Omega}h_{,v} \tag{2.6}$$

represents the rate of contraction of the null geodesic congruence defined by l_μ, and the quantity

$$Q_n\equiv -n^\lambda{}_{;\lambda}=e^{\Omega}h_{,u} \tag{2.7}$$

represents the rate of contraction of the null geodesics defined by n_μ.

Assuming that $\{\Omega^{(0)},h^{(0)},\Phi^{(0)}\}$ represents a solution of the Einstein field equations

$$R^{(0)}_{\mu\nu}-\tfrac{1}{2}g^{(0)}_{\mu\nu}R^{(0)}=-T^{(0)}_{\mu\nu} , \tag{2.8}$$

where $R^{(0)}_{\mu\nu}$ denotes the Ricci tensor built by the metric $g^{(0)}_{\mu\nu}$, and $T^{(0)}_{\mu\nu}$ the corresponding energy-stress tensor, then [11]

$$\{\Omega,h,\Phi\}=\{\Omega^{(0)}+a(u)+b(v),h^{(0)},\Phi^{(0)}\} , \tag{2.9}$$

represents a new solution of the Einstein field equations with the energy-stress tensor given by

$$T^\mu_\nu=\rho_1 l^\mu l_\nu+\rho_2 n^\mu n_\nu+e^{a+b}T^{(0)\mu}_\nu , \tag{2.10}$$

where $T^{(0)\mu}_\nu$ is given by Eq. (2.8); ρ_1 and ρ_2 are defined as

$$\rho_1=a'(u)h_{,u} , \quad \rho_2=b'(v)h_{,v} , \tag{2.11}$$

and $a(u)$ and $b(v)$ are functions of their indicated arguments. A prime denotes ordinary differentiation. The first term in the right-hand side of Eq. (2.10) represents a null fluid moving along the null geodesics defined by l_μ, while the second represents a null fluid moving in the opposite direction, namely, along the null geodesics defined by n_μ.

Although the quantities ρ_1 and ρ_2 have no definite physical meaning, the combination of the two null fluids indeed has [12]. As a matter of fact, the first two terms in Eq. (2.10) can be cast into the form [12]

$$\rho_1 l_\mu l_\nu+\rho_2 n_\mu n_\nu=\rho\left\{U_\mu U_\nu+\chi_\mu\chi_\nu\right\} , \tag{2.12}$$

where

$$U_\mu=e^{-\Omega/2}\left[\frac{\rho_2}{4\rho_1}\right]^{1/4}\left\{n_\mu+\left[\frac{\rho_1}{\rho_2}\right]^{1/2}l_\mu\right\} ,$$

$$\chi_\mu=e^{-\Omega/2}\left[\frac{\rho_1}{4\rho_2}\right]^{1/4}\left\{l_\mu-\left[\frac{\rho_2}{\rho_1}\right]^{1/2}n_\mu\right\} , \tag{2.13}$$

$$\rho=e^{\Omega}(\rho_1\rho_2)^{1/2} ,$$

with

$$U_\alpha U^\alpha=-\chi_\alpha\chi^\alpha=1 , \quad U_\alpha\chi^\alpha=0 . \tag{2.14}$$

From Eqs. (2.12)–(2.14) we can see that the sum of the two null fluids behaves like an anisotropic fluid, the pressure of which has only one component along the χ_μ direction, and the amplitude of the pressure is equal to the energy density of the anisotropic fluid. Moreover, this fluid satisfies all the (weak, dominant, and strong) energy conditions [13].

It is interesting to note that the above physical interpretation holds even for the case where $\rho_1,\rho_2<0$. The interaction of two null fluids was recently considered by Taub in the context of colliding plane waves [14].

III. THE GRAVITATIONAL COLLAPSE OF A SINGLE NULL FLUID

To have a better understanding about the collision and interaction of two null fluids, it is found useful first to consider the gravitational collapse of a single null fluid. This problem was first studied by Morgan [9]. By considering the continuity of the energy density of the null fluid at the axis, it was found that no space-time singularities were developed. However, recent studies of the interaction of waves with cosmic strings [15] show that it might not be the case. We can have that although the Kretschmann scalar vanishes, the tidal forces felt by freely falling test particles may become unbounded, which indicates that a nonscalar or intermediate singularity is formed there [15,16].

Motivated by the above considerations and the observations that the quantities Q_l and Q_n defined by Eqs. (2.6) and (2.7) become unbounded at the axis $r=0$, we reexamined the above problem, and found that Morgan's earlier conclusions [9] regarding singularity behavior need to be revised. When a cylindrical null fluid collapses, an intermediate singularity is always formed. To show this explicitly, let us consider the timelike geodesics in such a space-time. The metric that represents the gravitational collapse of a null fluid in an otherwise flat background takes the form

$$ds^2=2e^{-a(u)}du\,dv-dz^2-r^2d\varphi^2\ , \tag{3.1}$$

with the energy-stress tensor being given by

$$T_{\mu\nu}=-\frac{a'(u)}{\sqrt{2}r}l_\mu l_\nu\ . \tag{3.2}$$

In the following we shall assume $a'(u)<0$. Then, the timelike radial geodesics are given by

$$\ddot{u}-a'(u)\dot{u}^2=0\ ,\quad \ddot{v}=0\ , \tag{3.3}$$

where an overdot denotes the differentiation with respect to the proper time τ. The first integration of Eq. (3.3) yields

$$\dot{u}=e^{a_0+a(u)}\ ,\quad \dot{v}=e^{b_0}\ , \tag{3.4}$$

where a_0 and b_0 are integration constants. The condition $\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}=1$ requires $\exp[a_0+b_0]=1/2$. Setting $t^\mu\equiv\dot{x}^\mu$, we find that the unity vectors

$$\begin{aligned} t^\mu&=e^{a_0+a(u)}\delta^\mu_u+e^{b_0}\delta^\mu_v\ ,\\ \lambda^\mu_{(1)}&=e^{a_0+a(u)}\delta^\mu_u-e^{b_0}\delta^\mu_v\ ,\\ \lambda^\mu_{(2)}&=\delta^\mu_z\ ,\quad \lambda^\mu_{(3)}=r^{-1}\delta^\mu_\varphi\ , \end{aligned} \tag{3.5}$$

form an orthogonal tetrad

$$t^\mu\lambda_{(a)\mu}=0\ ,\quad \lambda^\mu_{(a)}\lambda_{(b)\mu}=-\delta_{ab}\quad (a,b=1,2,3)\ , \tag{3.6}$$

and have the properties

$$\lambda_{(a)\mu;\nu}t^\nu=t_{\mu;\nu}t^\nu=0\ . \tag{3.7}$$

The last equations show that the tetrad is parallel transported along the timelike geodesics defined by t^μ. Therefore, they define a freely falling frame.

From Eqs. (3.1) and (3.5) we find that in this frame there is only one independent component of the Riemann tensor:

$$R_{\mu\nu\sigma\delta}t^\mu\lambda^\nu_{(3)}t^\sigma\lambda^\delta_{(3)}=e^{2a_0+2a(u)}\frac{a'(u)}{\sqrt{2}r}\ . \tag{3.8}$$

Obviously, when it is approaching the hypersurface $r=0$, this component becomes unbounded. On the other hand, one can also show that in the present case all the physical scalar-invariant quantities are finite. Therefore, we conclude that the collapse of a cylindrical null fluid always forms an intermediate singularity. Note that in Ref. [15] it was shown that an intermediate singularity was also formed due to the interaction of a cosmic string with an electromagnetic wave.

By the study of scalar-wave propagation, King [17] suggested that these intermediate or nonscalar singularities might not be stable, and give origin to strong curvature singularities. The solutions to be presented in the next section can be considered as an example in favor to King's conjecture, since we shall show that the intermediate singularity appearing in the collapse of a single cylindrical null fluid is indeed turned into a scalar one due to the mutual focus of the ingoing and outgoing null fluids.

IV. COLLIDING NULL FLUIDS

The collision of two spherical null fluids was recently considered by Poisson and Israel [10]. They found that such solutions can give rise of the phenomenon of mass inflation. Following a similar line, in this section we consider the collision of two cylindrical null fluids. Let us consider a situation where a cylinder is collapsing. Assume that at a moment, say, $t=t_1$, the cylinder emits some particles with zero mass. These particles consist of a null dust cloud moving outwards. Meanwhile, we also assume that there is another null dust cloud which moves inwards. The latter could be created by the backscattering of an outgoing null dust cloud that was emitted by the collapsing cylinder at a moment $t=t_2$ $(t_2<t_1)$. Alternatively, this situation can also be justified by considering a collapsing cylinder consisting of a null fluid, as described in the Introduction. Before the two fluxes meet each other, a region of the space-time remains unperturbed (cf. Fig. 1). At the moment $t=t_0>t_1$, the two null dust clouds collide on the surface $r=r_0$, and afterwards they will mix each other and behave as an anisotropic fluid as described by Eqs. (2.12)–(2.14).

In the (u,v) plane, the above various regions are numbered as follows (cf. Fig. 2). Region IV $(u<u_0, v<v_0)$: This is the region where the two null dust clouds have not met each other yet. So the space-time in this region remains unperturbed. Region III $(u<u_0, v>v_0)$: In this region, the outgoing null dust cloud is propagating outwards along the null geodesics defined by n_μ with the hypersurface $v=v_0$ as its leading wave front. Region II $(u>u_0, v<v_0)$: In this region, the ingoing cylindrical

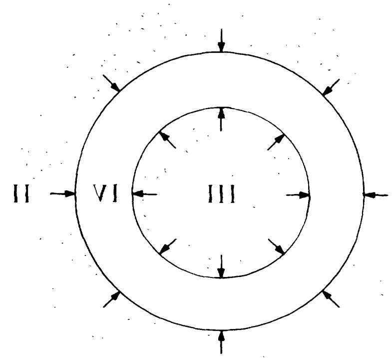

FIG. 1. The space-time at a particular moment of time before the two null fluids collide. The z coordinate is suppressed.

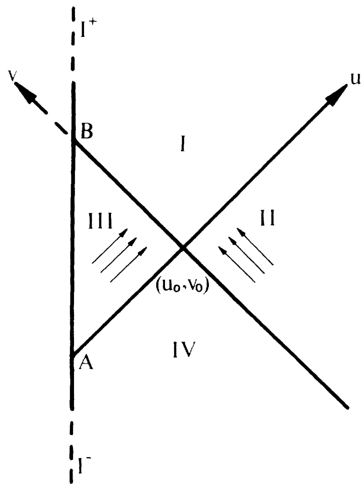

FIG. 2. The space-time projected onto the (u,v) plane. The line I^-I^+ represents the axis of symmetry $r=0$. Region IV is the region in which the exploding and imploding null fluids have not met each other yet. The space-time in this region remains unperturbed. In region III (II) the exploding (imploding) null fluid is incident. Region I is the region where the two null fluids interact.

null dust cloud contracts, and its leading wavefront is the hypersurface $u=u_0$. Region I ($u>u_0, v>v_0$): This is the region where the two null dust clouds collide, interact, and behave as an anisotropic fluid. The solution in this region is uniquely determined once the solution in regions II–IV is given. As a matter of fact, it exactly forms the initial value problem with the initial data imposed on the two characteristic hypersurfaces $u=u_0, v>v_0$ and $u>u_0, v=v_0$.

By using the results presented in Sec. II, in the following we shall study two classes of analytic solutions: one representing the collision and interaction of two null fluids in a flat space-time background, in which the nonlinear interaction between the two null fluids is manifested, the other representing the collision and interaction in a curved space-time background, in which the interaction of the null fluids with the background is shown clearly.

The first class of solutions is given by Eqs. (2.3) and (2.9) with

$$\begin{aligned} &h^{(0)}=-\Phi^{(0)}=-\ln(r)\ , \quad \Omega^{(0)}=0\ , \\ &a(u)=-[A(u)-A(u_0)]H(u-u_0)\ , \\ &b(v)=[B(v)-B(v_0)]H(v-v_0)\ , \end{aligned} \tag{4.1}$$

where $H(x)$ denotes the Heaviside function, which is one for $x\geq 0$, and zero for $x<0$, and $A(u)$ and $B(v)$ are smooth functions. The corresponding energy-stress tensor is given by

$$T_{\mu\nu}=\frac{1}{\sqrt{2}r}\{A'(u)H(u-u_0)l_\mu l_\nu+B'(v)H(v-v_0)n_\mu n_\nu\}\ . \tag{4.2}$$

From the above equation we can see that, in region IV where $u<u_0$ and $v<v_0$, the energy-stress tensor vanishes identically. As a matter of fact, one can show that the space-time in this region is flat and free of any (conical, intermediate, and scalar) singularities on the half infinite line AI^- (where $r=0$) [cf. Fig. 2]. Therefore, the above solutions represent the collision and interaction of two null fluids in a flat background. In region III ($u<u_0, v>v_0$), the first term in the right-hand side of Eq. (4.3) vanishes. Then, the corresponding energy-stress tensor represents a null fluid moving outwards along the null geodesics defined by n_μ from the axis $r=0$. In this region we have [cf. Eq. (2.7)]

$$Q_n=-e^{b(v)}\frac{1}{\sqrt{2}r}\ . \tag{4.3}$$

Clearly, as $r\to 0^+$, we have $Q_n\to-\infty$. Note that because of the presence of the outgoing null fluid at the axis, as shown in the last section, the space-time now has an intermediate singularity on the segment AB [cf. Fig. 2].

In region II ($u>u_0, v<v_0$), the last term of Eq. (4.3) vanishes, and only the first term remains. So the corresponding energy-stress tensor represents a null fluid moving inwards along the null geodesics defined by l_μ. The rate of contraction of these null geodesics is given by

$$Q_l=e^{a(u)}\frac{1}{\sqrt{2}r}\ . \tag{4.4}$$

Since in this region we always have $r\geq r_0$ $[=(u_0-v_0)/\sqrt{2}>0]$, we can see that Q_l is always finite, which indicates that the space-time in this region is regular. By studying all the physical quantities, we find that this is true.

In region I ($u>u_0, v>v_0$), the two terms in Eq. (4.3) are different from zero, and become unbounded as $r\to 0^+$. Therefore, a space-time singularity appears on the half infinite line BI^+. By studying the Kretschmann scalar

$$\mathcal{R}=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=-2e^{2(a+b)}\frac{A'B'}{r^2}H(u-u_0)H(v-v_0)\ , \tag{4.5}$$

we find that this singularity is actually a strong one. Since all the physical quantities, including the metric coefficients, are finite except on BI^+, one can see that this singularity is naked. Therefore, it is concluded that the solutions given by Eqs. (4.1) and (4.2) represent the collision and interaction of two null fluids in a flat background. Because of the mutual focus, the nonscalar singularity appearing on the segment AB is turned into a scalar one. This supports the conjecture given by King in Ref. [17].

The other class of solutions can be obtained from Senovilla's solution [18]

$$\begin{aligned} &\Omega^{(0)}=-4\ln[\cosh(\alpha t)]-2\ln[\cosh(3\alpha r)]\ , \\ &h^{(0)}=-\ln[\cosh(\alpha t)]-\ln[\sinh(3\alpha r)] \\ &\qquad +\frac{2}{3}\ln[\cosh(3\alpha r)]+\ln(3\alpha)\ , \end{aligned} \tag{4.6}$$

$$\Phi^{(0)}=-3\ln[\cosh(\alpha t)]-\ln[\sinh(3\alpha r)]+\ln(3\alpha)\ ,$$

where α is a positive constant. Corresponding to this solution, the energy-stress tensor is given by

$$T^{(0)}_{\mu\nu}=(\mu+p)u_\mu u_\nu-pg^{(0)}_{\mu\nu}\ , \tag{4.7}$$

where

$$\begin{aligned} u_\mu&=\{2\cosh^4(\alpha t)\cosh^2(3\alpha r)\}^{-1/2}\{1,1,0,0\}\ ,\\ \mu&=3p=15\alpha^2\{\cosh(\alpha t)\cosh(3\alpha r)\}^{-4}\ . \end{aligned} \tag{4.8}$$

As shown in Ref. [18], this solution is geodesically complete, singularity-free, and satisfies all the energy conditions.

Using the above solution as the seed, according to Eq. (2.19) we can construct the solutions

$$\begin{aligned} \Omega&=\Omega^{(0)}+a(u)+b(v)\ ,\\ h&=h^{(0)}\ ,\quad \Phi=\Phi^{(0)}\ , \end{aligned} \tag{4.9}$$

where $a(u)$ and $b(v)$ are given by Eq. (4.1). Then, one can show that the corresponding energy-stress tensor takes the form of Eq. (2.10) with

$$\begin{aligned} \rho_1&=-A'(u)h_{,u}H(u-u_0)\\ &=\frac{\alpha}{\sqrt{2}}\{3\coth(3\alpha r)-2\tanh(3\alpha r)+\tanh(\alpha t)\}\\ &\quad\times A'(u)H(u-u_0)\ ,\\ \rho_2&=B'(v)h_{,v}H(v-v_0)\\ &=\frac{\alpha}{\sqrt{2}}\{3\coth(3\alpha r)-2\tanh(3\alpha r)-\tanh(\alpha t)\}\\ &\quad\times B'(v)H(v-v_0)\ , \end{aligned} \tag{4.10}$$

and $T^{(0)}_{\mu\nu}$ being given by Eqs. (4.7) and (4.8). With the same arguments as those for the solutions of Eqs. (4.1) and (4.2), one can show that this class of solutions represents the collision and interaction of two null fluids on the background described by Eq. (4.6). The Kretschmann scalar now is given by

$$\begin{aligned} \mathcal{R}&=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\\ &=e^{2(a+b)}\mathcal{R}_0+4e^{2\Omega}\{\Phi^{(0)}_{00}\rho_1+\Phi^{(0)}_{22}\rho_2\}+2e^{2\Omega}\rho_1\rho_2\ , \end{aligned} \tag{4.11}$$

where $\rho_{1,2}$ are given by Eq. (4.10), $\mathcal{R}_0$ is the Kretschmann scalar corresponding to the background, and $\Phi^{(0)}_{00}$ and $\Phi^{(0)}_{22}$ are the Ricci "scale-invariant" scalars defined in Ref. [19], and in the present case they are

$$\Phi^{(0)}_{00}=\Phi^{(0)}_{22}=\frac{5\alpha^2}{\cosh(3\alpha r)}\ . \tag{4.12}$$

The expression of the Kretschmann scalar in Eq. (4.11) contains three terms, each of them has the following physical interpretation: The last term represents the interaction between the two null fluids. This term is singular as $r\to0^+$, as one can see from Eq. (4.10), and behaves like r^{-2}. Thus, because of the mutual focus of the two null fluids, a naked singularity is again formed on the half infinite line BI^+. The second term represents the interaction between the null fluids and the perfect fluid of the background. Because of this interaction, we can see that a scalar singularity appears on the segment AB in Fig. 2. This is different from the case described by Eqs. (4.1) and (4.2) where such interaction does not exist and on the segment AB there is only an intermediate singularity. The singularity behavior on this segment now is like r^{-1}, which is weaker than the one on BI^+ formed by the mutual focus of the two null fluids. The first term represents the contribution of the background, which is finite in all the space. It should be noted that in this case the half infinite line AI^- in Fig. 2 is still free of all kinds of singularities.

V. CONCLUSIONS AND REMARKS

In the late stages of collapse of an object, such as a spheroid or a star, it is expected that the object will lose most of its mass through gravitational radiation before it settles down to its final form (a black hole or a naked singularity). Because of the backscattering of the space-time curvature, it is also expected that ingoing radiation exists. In this paper, we studied the collision and interaction of two cylindrical null fluids at the attempt of modeling the above mentioned process. It was found that, because of their nonlinear interaction, a naked singularity was finally developed. The formation of this naked singularity is different from the ones found in Refs. [4–7] in the sense that it is due to the mutual focus of the two null fluids. As long as the amplitude [characterized by the functions $A'(u)$ and $B'(v)$] of the null fluids is different from zero (no matter how weak it is), the naked singularity is inevitably developed. This remarkable feature is not only shared by the solution considered in Sec. IV. In fact, for any given solution $\{\Omega^{(0)},\Phi^{(0)},h^{(0)}\}$ of the Einstein field equations (2.8), *which is free of any kind of singularities at the axis* $r=0$, then the solutions

$$\begin{aligned} \Omega&=\Omega^{(0)}-[A(u)-A(u_0)]H(u-u_0)\\ &\quad+[B(v)-B(v_0)]H(v-v_0)\ ,\\ h&=h^{(0)}\ ,\quad \Phi=\Phi^{(0)}\ , \end{aligned} \tag{5.1}$$

represent the collision and interaction of two null fluids on the space-time background $\{\Omega^{(0)},\Phi^{(0)},h^{(0)}\}$, with the energy-stress tensor given by Eqs. (2.10) and (2.11) and the Kretschmann scalar given by

$$\begin{aligned} \mathcal{R}&=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\\ &=e^{2(a+b)}\mathcal{R}_0+4e^{2\Omega}\{\Phi^{(0)}_{00}\rho_1+\Phi^{(0)}_{22}\rho_2\}+2e^{2\Omega}\rho_1\rho_2\ , \end{aligned} \tag{5.2}$$

where $\rho_{1,2}$ are defined by Eq. (2.11), $\mathcal{R}_0$ is the Kretschmann scalar corresponding to the background, and $\Phi^{(0)}_{00}$ and $\Phi^{(0)}_{22}$ are

$$\begin{aligned} \Phi^{(0)}_{00}&=\tfrac14\{2h^{(0)}_{,vv}-h^{(0)2}_{,v}+2h^{(0)}_{,v}\Omega^{(0)}_{,v}-\Phi^{(0)2}_{,v}\}\ ,\\ \Phi^{(0)}_{22}&=\tfrac14\{2h^{(0)}_{,uu}-h^{(0)2}_{,u}+2h^{(0)}_{,u}\Omega^{(0)}_{,u}-\Phi^{(0)2}_{,u}\}\ . \end{aligned} \tag{5.3}$$

The regular condition at the axis for the background solution requires $h^{(0)}$ behaves like $-\ln(r)$ as $r\to0^+$. Then, from Eq. (2.11) we find that

$$\rho_1 \to \frac{A'(u)}{r} H(u-u_0) ,$$
$$\rho_2 \to \frac{B'(v)}{r} H(v-v_0) . \tag{5.4}$$

Combining Eqs. (5.2) and (5.4) we find that, *because of the mutual focus of the two null fluids, a space-time singularity is always developed on the half infinite line BI^+* in Fig. 2.

Although in our studies Thorne's approximation was adapted, i.e., taking the spheroid as a finite version of an infinitely long cylinder, we believe that our main results are equally well applicable to the collapse of a more realistic spheroid, since if the spheroid is long enough (which is essentially the condition for a spheroid to form a naked singularity [6]), the gravitational field at the ends of the spheroid will not influence very much on the one in the middle of it. Therefore, the gravitational field in the middle of the spheroid can be considered approximately as produced by a long cylinder.

An alternative to the above studies is to consider the collision and interaction of two null fluids in the space-time with spherical symmetry. The case that the interaction happens inside an event horizon was studied recently by Poisson and Israel [10] in the effort of understanding the mass inflation phenomenon. Clearly, the interaction happening outside the horizon is also important, since it might turn out that, due to the nonlinear interaction of the fluids, a naked singularity is developed outside of the collapsing star, similar to what happened in the case of a collapsing spheroid [6].

Finally we would like to mention that the theorem given by Eqs. (2.8)–(2.11) is not restricted to the metric (2.1). It also holds for the metrics which describe cylindrical gravitational waves with two degrees of freedom [11], and one or several perfect fluids with a $p=\rho$ equation of state [20], or alternatively a multiplet of noninteracting scalar fields [20]. So, the method presented in Sec. IV is also valid for these cases.

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