

Failure of standard conservation laws at a classical change of signature

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The divergence theorem as usually stated cannot be applied across a change of signature unless it is reexpressed to allow for a finite source term on the signature change surface. Consequently all conservation laws must also be “modified,” and therefore insistence on conservation of matter across such a surface cannot be physically justified. The Darmois junction conditions normally ensure conservation of matter via Israel’s identities for the jump in the energy-momentum density, but not when the signature changes. Modified identities are derived for this jump when a signature change occurs, and the resulting surface effects in the conservation laws are calculated. In general, physical vector fields experience a jump in at least one component, and a source term may therefore appear in the corresponding conservation law. Thus current is also not conserved. These surface effects are a consequence of the change in the character of physical law. The only way to recover standard conservation laws is to impose restrictions that no realistic cosmological model can satisfy.

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INTRODUCTION

Interest in the possibility of a change of the signature of spacetime has been revived recently by Hawking’s [1] “no boundary condition” proposal and by subsequent considerations of quantum cosmology (e.g., [2–6]), and there have been several papers discussing the junction of Lorentzian (L) to Euclidean (E) regions in classical relativity [7–13]. However, none of these has examined the divergence theorem, upon which all conservation equations are based. (Stokes’ theorem is discussed in this context in [14] using differential forms, but conservation is not explicitly discussed.)

To a large extent, the laws of physics in a space of E (Euclidean) signature and at a change of signature are a matter of personal choice—our intuition, which after all is exclusively based on experience of L (Lorentzian) spacetime, cannot be a reliable guide. This paper follows a strictly classical approach, which is not entirely equivalent to the quantum cosmology approach, in which the Euclidean regions are “classically forbidden.”

We here argue that the Darmois (D) junction conditions, which ensure that the geometries on either side of a boundary surface do in fact fit together, are the absolute minimum gravitational requirements for passing through a signature change. While one may wish to impose stronger conditions for reasons of preference, or to achieve some particular physical result, such extra conditions are less fundamental, and may eliminate legitimate and interesting types of transition.

In any case, the primary results obtained here hold for all known junction conditions, and are independent of the choice of coordinates near the signature change. In other words we permit, but do not assume, a lapse function that goes to zero on the signature change surface.

We begin by reviewing the relationship between the Darmois-Israel junction conditions [15,16] and the divergence theorem, for the case when no signature change occurs. We proceed by considering how the theorem and the Israel identities should be adjusted for the case when a change of signature does occur inside the volume of integration. Similar considerations are applied to an electromagnetic field; and finally the significance of the results is discussed.

THE DARMOIS-ISRAEL CONDITIONS

We wish to join two manifolds M^+ and M^- of L (Lorentzian) signature $(-+++)$ with non-null boundary surfaces Σ^\pm , by identifying Σ^+ with Σ^- . Manifolds M^\pm have coordinate systems x_\pm^α and metrics $g_{\alpha\beta}^\pm$, while Σ^\pm have coordinates ξ_\pm^i , which are also identified. Latin indices range 1 to 3, and Greek indices range 0 to 3. The Darmois (D) [15] junction conditions state that the first and second fundamental forms of the surfaces, the intrinsic metric ${}^3g_{ij}$ and the extrinsic curvature K_{ij} , must be continuous across the identified boundary Σ . These conditions have been shown to be the “most convenient and reliable,” whereas those of O’Brien and Synge [17] are too restrictive in general [18].

Using the notation

$$[Z] = Z^+|_\Sigma - Z^-|_\Sigma \quad (1)$$

for the jump in some quantity Z across Σ , where $Z^\pm|_\Sigma$ are the limiting values of Z as Σ is approached from either

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side,

$$e_i^\alpha = \frac{\partial x^\alpha}{\partial \xi^i} \quad (2)$$

for the basis vectors of the surface, and

$$n^\alpha, \quad n^\alpha n_\alpha = \epsilon = \pm 1 \quad (3)$$

for the unit normal to Σ , which may be timelike (i.e., n spacelike, $\epsilon = +1$) or spacelike ($\epsilon = -1$), then the intrinsic metric and extrinsic curvature are

$$\mathring{g}_{ij} = g_{\alpha\beta} e_i^\alpha e_j^\beta, \quad (4)$$

$$\begin{aligned} K_{ij} &= \nabla_\beta n_\alpha e_i^\alpha e_j^\beta \\ &= -n_\gamma \left(\frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right). \end{aligned} \quad (5)$$

The D junction conditions are the minimum requirements for joining M^+ and M^- smoothly:

$$[\mathring{g}_{ij}] = 0, \quad (6)$$

$$[K_{ij}] = 0, \quad (7)$$

and it is important to note that (1) requires the normals on both sides to point from M^- to M^+ for proper evaluation. The great advantage of the D conditions is that these expressions are completely invariant to the coordinates used in M^+ and M^- . They may be implemented without ever finding a common coordinate system on $M^+ \cup M^-$, though one must obviously find a common coordinate system on $\Sigma^+ = \Sigma^-$. Thus they provide an unambiguous algorithm for joining spacetimes. In the above we have assumed an isometry between the points on the surfaces Σ^+ and Σ^- induced in M^+ and M^- . In simple cases this may merely be an identification of induced surface coordinates, $\xi_+^i = \xi_-^i$, but in general one might have to solve the three-dimensional metric equivalence problem before stating whether or not (6) may be satisfied. Any isometry for which (6) and (7) are satisfied results in a valid matching.

It is often convenient to use geodesic normal coordinates (Gaussian coordinates), defined near Σ to consist of the proper time/distance coordinate $\xi^0 = \tau$ along geodesics normal to the surface, increasing from M^- through Σ into M^+ , and the surface coordinates ξ^i which are held constant along each geodesic. Then

$$ds^2 = \epsilon d\tau^2 + \mathring{g}_{ij} d\xi^i d\xi^j = \tilde{g}_{\mu\nu} d\xi^\mu d\xi^\nu, \quad \tilde{g}_{ij}|_\Sigma = \mathring{g}_{ij}, \quad (8)$$

$$\tilde{K}_{ij} = -\tilde{\Gamma}_{0,ij} = \tilde{\Gamma}_{j,0i} = \frac{1}{2} \tilde{g}_{ij,0}, \quad \tilde{K}_{ij}|_\Sigma = K_{ij}, \quad (9)$$

where we use a tilde to indicate four-dimensional quantities expressed in this Gaussian coordinate system. Nevertheless, all of the following may be done without introducing these coordinates.

Israel [16] has shown that these junction conditions lead to the following identities for the Einstein tensor $G_{\alpha\beta}$:

$$[\tilde{G}_{00}] = [G_{\alpha\beta} n^\alpha n^\beta] = 0, \quad (10)$$

$$[\tilde{G}_{\alpha i}] = [G_{\alpha\beta} n^\alpha e_i^\beta] = 0. \quad (11)$$

This means, for a timelike surface, that the flux of energy-momentum through Σ , as measured by an observer moving with the surface, is continuous across Σ . For a spacelike surface, an observer moving orthogonally to it sees no jump in the density of energy-momentum across Σ . However, if only the first fundamental form is continuous, and there is a jump in the second form, then Israel showed that Σ contains a finite amount of matter and a ‘‘surface layer’’ occurs.

The D conditions are more or less equivalent to making the appropriate components of the gravitational field and its first derivatives continuous across Σ , naturally expressed in geometric fashion. Although there are 12 conditions on the 50 independent components of $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma}$, there do exist coordinates in which the 4D metric and its first derivatives are continuous (e.g., [19]), but these are not always trivial to find, the most reliable choice being normal coordinates. (This is the approach of Lichnerowicz [20], which is equivalent to the D conditions [18], but the fact that the Lichnerowicz junction conditions are not invariant makes them less reliable.) Nevertheless, even in normal coordinates, the continuity of all components of the matter tensor $T^{\alpha\beta}$ does not follow from the D conditions. Despite this, conservation of matter right through the boundary is guaranteed.

MATTER CONSERVATION AT A BOUNDARY

Given a volume W enclosed by a surface S with normal m_α , and defining a three-form \mathbf{p} to have components

$$\mathbf{p}_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma\delta} \Psi^\delta = \sqrt{\epsilon g} \epsilon_{\alpha\beta\gamma\delta} \Psi^\delta, \quad (12)$$

where $\eta_{\alpha\beta\gamma\delta}$ is the permutation tensor and $\epsilon_{\alpha\beta\gamma\delta}$ the permutation symbol, then Stokes’ theorem (e.g., [21]) in terms of differential forms

$$\oint_S \mathbf{p} = \int_W d\mathbf{p} \quad (13)$$

applies over any region W bounded by S within which \mathbf{p} is C^1 , and, given a metric, it leads to the divergence theorem (e.g., [21]) in terms of tensor components

$$\oint_S \Psi^\beta m_\beta d^3 S = \int_W \nabla_\beta \Psi^\beta d^4 W, \quad (14)$$

where $d^4 W$ is the metric volume element on W , $d^3 S$ is the induced metric volume element on S , and m^α , the unit vector normal to S , has contravariant components that point outward where it (m^α) is spacelike, and inward where it is timelike—i.e., m_α is always outward. (The character of this normal will be clarified later.) In general both Ψ^α and $g_{\alpha\beta}$ must be C^1 to make \mathbf{p} C^1 . In order to give Stokes’ theorem physical meaning, \mathbf{p} must be related to measurable quantities, which requires a metric. Hence (14) is the physical version of (13) for a three-form. Choosing

$$\Psi^\delta = G^{\sigma\delta} v_\sigma, \quad (15)$$

v_α being some smooth field (e.g., an element of an orthonormal basis), this becomes

$$\begin{aligned} \oint_S G^{\alpha\beta} v_\alpha m_\beta d^3 S &= \int_W \nabla_\beta (G^{\alpha\beta} v_\alpha) d^4 W \\ &= \int_W G^{\alpha\beta} (\nabla_\beta v_\alpha) d^4 W. \end{aligned} \quad (16)$$

Over small enough volumes this, together with the Einstein equations, gives the local conservation of matter. It should be noted that n^α is the normal to the junction surface Σ , and m_α is the normal to S , the closed boundary of W ; the two are quite different in general and, even if the two surfaces coincide partially, they may still differ in sign there.

Consider now a spacelike boundary surface Σ , where no signature change occurs, that divides W and S into two parts W_+ and W_- , S_+ and S_- , $S_0 = \Sigma \cap W$ being the enclosed region of Σ , as in Fig. 1. In general this \mathbf{p} [given by (12) and (15)] is not even C^0 through Σ . However the divergence theorem holds within each part, so adding them gives

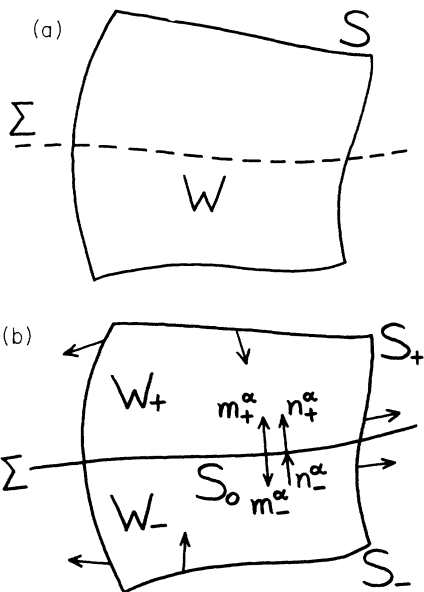


FIG. 1. (a) The volume W with boundary S , which intersects a spacelike surface of discontinuity Σ , where the manifold is only C^2 . Since a manifold must be C^4 (i.e., $g_{\alpha\beta}$ must be C^3) to satisfy the conservation equations $\nabla_\mu G^{\mu\nu} = 0$, we wish to determine whether they hold across Σ . (b) The boundary Σ divides W and S into W_+ , W_- , S_+ , and S_- , and the enclosed region of Σ , S_0 , completes the boundaries. Application of the divergence theorem in each part, supplemented by the Darmois-Israel junction conditions, shows that matter is conserved if the signature stays Lorentzian, even though not all components of $G^{\mu\nu}$ are continuous. Similar results hold for the electromagnetic and other fields. We write m^α for the normal to the volume boundary used in the divergence theorem, and n^α for the normal to the surface of discontinuity Σ used in the junction conditions. The two normals to S_0 are shown on both sides of Σ . Unlabeled vectors are m^α on S_+ and S_- .

$$\begin{aligned} &\int_{S_+} G_+^{\alpha\beta} v_\alpha^+ m_\beta^+ d^3 S + \int_{S_0} G_+^{\alpha\beta} v_\alpha^+ (+n_\beta^+) d^3 S \\ &+ \int_{S_-} G_-^{\alpha\beta} v_\alpha^- m_\beta^- d^3 S + \int_{S_0} G_-^{\alpha\beta} v_\alpha^- (-n_\beta^-) d^3 S \\ &= \int_{W_+} \nabla_\beta^+ (G_+^{\alpha\beta} v_\alpha^+) d^4 W + \int_{W_-} \nabla_\beta^- (G_-^{\alpha\beta} v_\alpha^-) d^4 W, \end{aligned} \quad (17)$$

$$\begin{aligned} &\Rightarrow \oint_S G^{\alpha\beta} v_\alpha m_\beta d^3 S + \int_{S_0} [G_{\alpha\beta} v^\alpha n^\beta] d^3 S \\ &= \int_W \nabla_\beta (G^{\alpha\beta} v_\alpha) d^4 W, \end{aligned} \quad (18)$$

where $d^3 S$ and $d^4 W$ can be made smooth well-defined volume elements, by a suitable choice of coordinates spanning Σ , provided only (6) is satisfied.

We now choose v^γ to be each of the basis vectors of normal coordinates n^γ and e_i^γ in turn, and insert the Israel identities (10) and (11) in (18) to obtain the appearance of the divergence theorem for $G^{\sigma\delta} v_\sigma$ (16) as if no discontinuity were present. (Note that the volume integrands can be evaluated to

$$\nabla_\beta (G_\alpha^\beta e_i^\alpha) = G_\alpha^\beta (\nabla_\beta e_i^\alpha) = \tilde{G}_k^j \tilde{\Gamma}_{ij}^k, \quad (19)$$

$$\nabla_\beta (G_\alpha^\beta n^\alpha) = G_\alpha^\beta (\nabla_\beta n^\alpha) = \tilde{G}_k^j K_j^k, \quad (20)$$

and since $\tilde{G}_{ij} = G_{\alpha\beta} e_i^\alpha e_j^\beta$ contains $\tilde{\partial}_o K_{ij} = \frac{1}{2} \tilde{g}_{ij,o,o}$ [see Eq. (32)] they are not continuous across Σ , and Ψ^δ is not C^1 .) Thus the D conditions provide the necessary link that ensures conservation of matter across boundary surfaces.

For the case when Σ is a surface layer, only the first fundamental form is continuous, so we must impose conservation in some other way, as was done in [22]. (See also [23].) We might then rewrite (18) to include the surface layer in the volume integrals, using a Dirac delta,

$$\begin{aligned} \oint_S G^{\alpha\beta} v_\alpha m_\beta d^3 S &= \int_W (\nabla_\beta (G^{\alpha\beta} v_\alpha) \\ &- \delta(\tau) [G_{\alpha\beta} v^\alpha n^\beta]) d^4 W \end{aligned} \quad (21)$$

or more generally we write

$$\oint_S \Psi^\beta m_\beta d^3 S + \int_{S_0} [\Psi_\beta n^\beta] d^3 S = \int_W \nabla_\beta \Psi^\beta d^4 W, \quad (22)$$

$$\oint_S \Psi^\beta m_\beta d^3 S = \int_W (\nabla_\beta \Psi^\beta - \delta(\tau) [\Psi_\beta n^\beta]) d^4 W. \quad (23)$$

We can think of this double application of (14) in a discontinuous setting as constituting a ‘‘patchwork divergence theorem.’’

MODIFYING ISRAEL'S IDENTITIES

We now turn to the case of a boundary where a change of signature occurs. We continue to use the D conditions for matching the gravitational field across a signature change, since they ensure the geometries of the two manifolds can fit together at Σ , and it turns out they require no modification despite any metric discontinuity in \tilde{g}_{00} . The first condition ensures the induced metric on Σ is the same from either side, and allows the two truncated manifolds to fit over the whole surface. The second ensures continuity of affine structure, as indicated by (9). The metric is clearly less continuous than in the L to L case, and the Lichnerowicz conditions are no longer equivalent, since one can no longer find admissible coordinates in which the full four-metric is continuous and nondegenerate through Σ .

We now follow Israel's procedure very closely. He defines the normal to Σ by

$$n^\alpha n_\alpha = \epsilon = \pm 1 \quad (24)$$

such that $\epsilon = +1$ (or -1) for a spacelike (or timelike) normal (timelike or spacelike Σ), respectively. Of course, ϵ does not change across Σ in his case, as there is no change of signature. For our purposes, we know Σ must be spacelike for a signature change, so instead we set $\epsilon = +1$ on the E side M^+ , and $\epsilon = -1$ on the L side M^- .

At this point the existence of two different normal vectors becomes apparent. Recall that $\tau = \xi^0$ is the proper time/distance coordinate of geodesic normal coordinates, as defined earlier. The gradient of the function $\tau = \tau(x^\gamma)$ is

$$l_\beta = \frac{\partial \xi^0}{\partial x^\beta} = \bar{e}_\beta^0, \quad (25)$$

where \bar{e}_γ^0 is one of the dual basis vectors of geodesic normal coordinates, and the tangent vector to the τ coordinate lines is

$$n^\beta = \frac{\partial x^\beta}{\partial \xi^0} = e_0^\beta. \quad (26)$$

The former gives the sense in which τ increases and the latter points in the positive τ direction; i.e., they both "point" into M^+ , the E region. Thus we have

$$l_\alpha l^\alpha = \epsilon = n^\alpha n_\alpha, \quad l_\alpha n^\alpha = 1, \quad (27)$$

so that $\tilde{l}_\mu = \delta_\mu^0$ and $\tilde{n}^\nu = \delta_0^\nu$ are continuous through Σ , but

$$n_\alpha = \epsilon l_\alpha \quad \text{and} \quad l^\alpha = \epsilon n^\alpha \quad (28)$$

so $\tilde{l}^\mu = \epsilon \delta_\mu^0$ and $\tilde{n}_\nu = \epsilon \delta_\nu^0$ are not. We will call l_α the "gradient normal," and n^α the "tangent normal." Note that $l = l_\alpha \mathbf{d}^\alpha$ is a one-form, and $\mathbf{n} = n^\alpha \boldsymbol{\theta}_\alpha$ is a vector. To establish which of these is the appropriate one to use in the Darmois-Israel matching we specify that we want the three-metric \tilde{g}_{ij} of Eq. (8) to be a C^1 function of the normal coordinate τ , which leads to

$$\begin{aligned} 0 &= [K_{ij}] = [\tfrac{1}{2} \tilde{g}_{ij,0}] \\ &= [\tfrac{1}{2} \{ (e_\sigma^\gamma \partial_\gamma g_{\alpha\beta}) e_i^\alpha e_j^\beta + g_{\alpha\beta} (\tilde{\partial}_\sigma e_i^\alpha) e_j^\beta \\ &\quad + g_{\alpha\beta} e_i^\alpha (\tilde{\partial}_\sigma e_j^\beta) \}] \\ &= [\tfrac{1}{2} e_i^\alpha e_j^\beta (e_\sigma^\gamma g_{\alpha\beta, \gamma} + g_{\beta\gamma} \partial_\alpha e_\sigma^\gamma + g_{\alpha\gamma} \partial_\beta e_\sigma^\gamma)] \\ &= \{ \tfrac{1}{2} e_i^\alpha e_j^\beta [-2e_\sigma^\gamma \Gamma_{\gamma, \alpha\beta} + \partial_\alpha (g_{\beta\gamma} e_\sigma^\gamma) \\ &\quad + \partial_\beta (g_{\alpha\gamma} e_\sigma^\gamma)] \} \\ &= [\tfrac{1}{2} e_i^\alpha e_j^\beta \epsilon (2\partial_\alpha \bar{e}_\beta^0 - 2\bar{e}_\gamma^0 \Gamma_{\alpha\beta}^\gamma)] \\ &= [e_i^\alpha e_j^\beta (\nabla_\alpha n_\beta)]. \end{aligned} \quad (29)$$

At a surface of signature change, then, the extrinsic curvature that must be matched across Σ is defined relative to Σ 's tangent normal—a unit normal vector whose contravariant components point from M^- to M^+ on both sides.

Apart from some sign mistakes, not all corrected in the errata (see reference), Israel's working up to his Eqs. (12)–(15) carries over without change. Using $K = K_m^m = \mathfrak{y}^{ml} K_{ml}$, we have

$$G_{\alpha\beta} n^\alpha n^\beta = \tfrac{1}{2} \{ K^2 - K_{ij} K^{ij} - \epsilon \mathfrak{R} \}, \quad (30)$$

$$G_{\alpha\beta} n^\alpha e_i^\beta = \mathfrak{V}_j K_i^j - \mathfrak{V}_i K, \quad (31)$$

and the remaining components are

$$\begin{aligned} G_{\alpha\beta} e_i^\alpha e_j^\beta &= \mathfrak{G}_{ij} - \epsilon \mathfrak{y}_{ik} \{ \partial_o K_j^k + K K_j^k \\ &\quad - \tfrac{1}{2} \delta_j^k (2\partial_o K + K^2 \\ &\quad + K_{lm} K^{lm}) \}, \end{aligned} \quad (32)$$

where \mathfrak{G}_{ij} , \mathfrak{R} , and \mathfrak{V}_i are the three-dimensional intrinsic Einstein tensor, Ricci scalar, and covariant derivative of Σ . (See also [24] but note that their definition of K_{ij} is the negative of our Eq. (5).) The D conditions keep everything on the right-hand side (RHS) of (30) and (31) unchanged except for ϵ . Thus the modified Israel identities are

$$[G_{\alpha\beta} n^\alpha n^\beta] = [\tilde{G}_{00}] = -\mathfrak{R}, \quad (33)$$

$$[G_{\alpha\beta} n^\alpha e_i^\beta] = [\tilde{G}_{oi}] = 0. \quad (34)$$

Since the operation of raising and lowering indices is not smooth through Σ , this implies

$$[\tilde{G}^{00}] = [G^{\alpha\beta} l_\alpha l_\beta] = -\mathfrak{R}, \quad (35)$$

$$[\tilde{G}_o^0] = [G_\beta^\alpha l_\alpha n^\beta] = K^2 - K_{ij} K^{ij}, \quad (36)$$

$$[\tilde{G}_o^i] = [G_\alpha^\beta n^\alpha \bar{e}_\beta^i] = 0, \quad (37)$$

$$[\tilde{G}^{oi}] = [G^{\alpha\beta} l_\alpha \bar{e}_\beta^i] = 2(\mathfrak{V}_j K^{ij} - \mathfrak{y}^{ij} \mathfrak{V}_j K), \quad (38)$$

$$[\tilde{G}_i^o] = [G_\beta^\alpha l_\alpha e_i^\beta] = 2(\mathfrak{V}_j K_i^j - \mathfrak{V}_i K). \quad (39)$$

GENERALIZING THE PATCHWORK DIVERGENCE THEOREM

A change of signature, being a metric phenomenon, should affect the divergence theorem (14), but not Stokes' theorem (13). In other words, if \mathbf{p} satisfies Stokes' theorem on a particular manifold when the signature does not change, then the same \mathbf{p} must still satisfy it on the

same manifold when the signature does change. Since this reasoning is not valid for Ψ^δ , we must adapt the patchwork approach to the case of signature change at Σ . Consequently we assume that there exists an orientation (a smooth nonzero form) right through Σ , so that Stokes' theorem holds for a sufficiently smooth \mathbf{p} . However we will not actually need to assume that \mathbf{p} is C^1 . Although $\tilde{g}_{00} = \epsilon$ is discontinuous, being double valued on the identified boundary $\Sigma^+ \equiv \Sigma^-$, the volume element in normal coordinates $d^4\tilde{W} = \sqrt{\epsilon\tilde{g}} \epsilon_{\alpha\beta\gamma\delta} d\xi^\alpha d\xi^\beta d\xi^\gamma d\xi^\delta$ is actually smooth through Σ . This does not make (14) valid even if Ψ^δ is smooth through a signature change, since the conversion of Stokes theorem to the divergence theorem involves the metric itself.

In order to preserve some kind of "divergence theorem" in this case, we once again use the usual divergence theorem on either side of Σ and join them by means of junction conditions on Ψ^α appropriate for signature change, such as the modified Israel Identities, which now give a nonzero surface contribution to the volume integral. In other words, we expect a result of the form (22) or (23):

$$\oint_S \Psi^\beta m_\beta d^3S - \int_{S_0} E d^3S = \int_W \nabla_\beta \Psi^\beta d^4W, \quad (40)$$

$$\oint_S \Psi^\beta m_\beta d^3S = \int_W [\nabla_\beta \Psi^\beta + \delta(\tau)E] d^4W, \quad (41)$$

where $\tau = 0$ on Σ , which we now derive, obtaining E for this case. We point out that the normal vector m_α

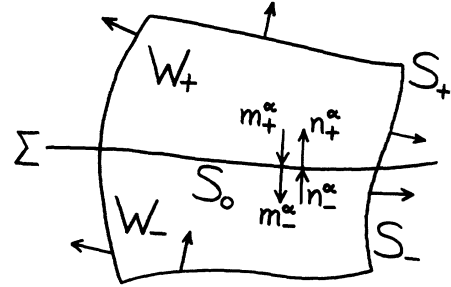


FIG. 2. In the case when the signature changes from $(-+++)$ in W_- to $(++++)$ in W_+ , m_\pm^α changes direction.

is really a gradient normal, that "points" out of W . In other words, if we set up geodesic normal coordinates based on S , with proper time/distance coordinate $\zeta^4 = 0$ on S and increasing outward, and ζ^b , $b = 1, 2, 3$ surface coordinates, then $m_\alpha = \partial_\alpha \zeta^4$. However it is customary to define the direction of m_α via the pointing behavior of its metric dual m^α , given after Eq. (14). For an arbitrary volume W bounded by S and spanning Σ , we again split W as in Fig. 2.

The usual divergence theorem applies to the volume integrals over W_+ and W_- , hence, noting that the gradient normals m_\pm^α , on S_0^+ and S_0^- convert into the gradient normals $-l_\alpha^+$ and $+l_\alpha^-$ on Σ^+ and Σ^- , respectively, and being careful not to change index positions, we find

$$\begin{aligned} \int_{S_+} \Psi_+^\alpha m_\alpha^+ d^3S + \int_{S_0} \Psi_+^\alpha (-l_\alpha^+) d^3S + \int_{S_-} \Psi_-^\alpha m_\alpha^- d^3S + \int_{S_0} \Psi_-^\alpha (+l_\alpha^-) d^3S \\ = \int_{S_+} \Psi_+^\alpha m_\alpha^+ d^3S + \int_{S_-} \Psi_-^\alpha m_\alpha^- d^3S - \int_{S_0} [\Psi^\alpha l_\alpha] d^3S \\ = \int_{W_+} \nabla_\alpha^+ \Psi^\alpha d^4W + \int_{W_-} \nabla_\alpha^- \Psi^\alpha d^4W \end{aligned} \quad (42)$$

which holds for arbitrary W and S_0 , so we conclude that

$$E = (\Psi_+^\alpha l_\alpha^+ - \Psi_-^\alpha l_\alpha^-) = [\Psi^\alpha l_\alpha]. \quad (43)$$

We can represent this in the following two-forms, which allow easier comparison with (14) and (22), and (23):

$$\Rightarrow \oint_S \Psi^\alpha m_\alpha d^3S - \int_{S_0} [\Psi^\alpha l_\alpha] d^3S = \int_W \nabla_\alpha \Psi^\alpha d^4W, \quad (44)$$

$$\oint_S \Psi^\alpha m_\alpha d^3S = \int_W (\nabla_\alpha \Psi^\alpha + \delta(\tau)[\Psi^\alpha l_\alpha]) d^4W. \quad (45)$$

These forms may be justified on the grounds that $d^4\tilde{W}$ and $d^3\tilde{S}$ are smooth through Σ , and no further manipulation with a discontinuous $\tilde{g}_{\mu\nu}$ is required. [However they are not well defined if the coordinates near Σ are defined to be such that $g_{00} \rightarrow 0$ on Σ . Equation (42) is always well defined.] In contrast to the case of no signature change, where the substitution $[\Psi^\alpha l_\alpha] = -[\Psi_\alpha n^\alpha]$ does not affect the validity of Eqs. (22) and (23), it is

important to use only the gradient normal here. If we know how Ψ^α matches across Σ we can use this to determine the surface "singularity" on Σ associated with Ψ^α due to the signature change. The surface term *only* disappears for smooth contravariant Ψ^δ in normal coordinates. (Recall that the Euclidean region is "+," the Lorentzian region is "-", and l_α^\pm "point" into the Euclidean region.) Results (42)–(45) are of course valid whether or not the

signature changes, and for timelike or spacelike Σ , in all viable combinations, whereas (22) and (23) are only valid for constant Lorentzian signature at a spacelike Σ .

NONCONSERVATION OF MATTER?

Returning to the construction of Fig. 2 with $\Psi^\beta = G^{\alpha\beta}v_\alpha$, it is clear that there must be a source term $E(v_\gamma)$ on Σ in the volume integral over $\nabla_\beta(G^{\alpha\beta}v_\alpha)$, which depends on the choice of a *covariant* v_γ . This choice of Ψ^β in (43) gives

$$E(v_\gamma) = G_+^{\alpha\beta}v_\alpha^+l_\beta^+ - G_-^{\alpha\beta}v_\alpha^-l_\beta^- = [G^{\alpha\beta}v_\alpha l_\beta]. \quad (46)$$

For $v_\alpha = l_\alpha$ and \bar{e}_α^i we find, respectively,

$$E(l_\alpha) = [\tilde{G}^{00}] = -{}^3R, \quad (47)$$

$$E(\bar{e}_\alpha^i) = [\tilde{G}^{0i}] = 2({}^3\nabla_j K^{ij} - \mathfrak{y}^{ij}{}^3\nabla_j K). \quad (48)$$

If instead of (15) we choose $\Psi^\beta = G^\beta_\nu v^\alpha$ with smooth *contravariant* v^α , then E depends on v^α and we arrive at

$$E(n^\alpha) = [\tilde{G}^0_0] = K^2 - K_{ij}K^{ij}, \quad (49)$$

$$E(e_i^\alpha) = [\tilde{G}^0_i] = 2({}^3\nabla_j K_i^j - {}^3\nabla_i K). \quad (50)$$

\tilde{G}_-^{00} and \tilde{G}_-^{i0} are the energy density and the three energy fluxes and/or momentum densities on the L side, as measured by an observer moving orthogonally to the transition surface. The meanings of the Euclidean quantities \tilde{G}_+^{00} and \tilde{G}_+^{i0} are open for discussion.

OTHER MATCHING OPTIONS

The results presented above are based on those junction conditions that we regard as the most reasonable. For the sake of completeness we mention two important steps at which a different choice of sign has a large effect on the results.

The first one is in the definition of the extrinsic curvature. If instead of (5) we choose

$$K'_{ij} = (\nabla_\beta l_\alpha) e_i^\alpha e_j^\beta, \quad (51)$$

then we find

$$[K'_{ij}] = 0 \Rightarrow K_{ij}^+ = -K_{ij}^- \quad (52)$$

and the modified Israel identities become

$$[G_{\alpha\beta} n^\alpha n^\beta] = [\tilde{G}_{00}] = -{}^3R, \quad (53)$$

$$[G_{\alpha\beta} n^\alpha e_i^\beta] = [\tilde{G}_{0i}] = 2({}^3\nabla^j K_{ij}^+ - {}^3\nabla_i K^+), \quad (54)$$

so, giving the K_{ij} their values in M^+ , the RH sides of Eqs. (34) and (39) are swapped and the RH sides of Eqs. (37) and (38) are swapped if this sign is changed. This does mean that $E(\bar{e}_\alpha^i) = 0$, but $E(l_\alpha)$ and $E(n^\alpha)$ are unchanged. Also $[K'_{ij}] = 0$ implies $\tilde{g}_{ij,0}^+ = -\tilde{g}_{ij,0}^-$.

The second sign choice relates the orientations of the manifolds M^- and M^+ . If we do not assume that the combined manifold is oriented, then there are two possible relative orientations; one gives Eq. (46) and the other leads to

$$E(v_\gamma) = -(G_+^{\alpha\beta}v_\alpha^+l_\beta^+ + G_-^{\alpha\beta}v_\alpha^-l_\beta^-). \quad (55)$$

Changing this sign factor swaps the RH sides of Eqs. (33)/(35), (34), and (37) with (36), (39), and (38), respectively, and changes all their signs. There are no combinations of choices which will make the surface terms $E(l_\alpha)$ or $E(n^\alpha)$ disappear, because Eq. (30) is second order in the K_{ij} and some but not all of its terms contain ϵ .

ELECTROMAGNETIC FIELDS AND SOURCES

We here take the electromagnetic field as an example of a vector field, and we assume either vacuum or no change in dielectric properties at the transition surface. According to [25] the junction conditions for a macroscopic electromagnetic (EM) field at a dielectric boundary, which for his purposes is actually timelike (i.e., no signature change, L - L), are

$$[\mathbf{D}_\perp] = 0, \quad [\mathbf{E}_\parallel] = 0, \quad [\mathbf{B}_\perp] = 0, \quad [\mathbf{H}_\parallel] = 0, \quad (56)$$

which means that for the microscopic quantities, vectors \mathbf{E} and \mathbf{B} are C^0 . We now try to find junction conditions for the EM potential which are analogous to the D conditions for the gravitational potentials. Working temporarily in normal coordinates at a timelike or spacelike boundary, and assuming that the D conditions are satisfied, this can be ensured by (i) choosing a “normal gauge” (the equivalent of normal coordinates),

$$\tilde{A}_0^\Sigma = 0 \quad (57)$$

on the boundary, which ensures $\tilde{\partial}_i \tilde{A}_0, \tilde{\partial}_j \tilde{\partial}_i \tilde{A}_0$ are all zero, (ii) then requiring

$$[\tilde{A}_i] = 0 \quad \text{and} \quad [\tilde{\partial}_0 \tilde{A}_j] = 0 \quad (58)$$

everywhere on the surface, which of course means $[\tilde{\partial}_i \tilde{\partial}_0 \tilde{A}_j], [\tilde{\partial}_i \tilde{A}_j]$, and $[\tilde{\partial}_k \tilde{\partial}_i \tilde{A}_j]$ are also zero. These compare nicely with the D conditions in normal coordinates: $[\tilde{g}_{ij}] = 0 = [\tilde{\partial}_0 \tilde{g}_{ij}]$. In non-normal coordinates, the gauge and junction conditions are

$$A_\alpha^\Sigma n^\alpha = 0, \quad [A_\alpha e_i^\alpha] = 0, \quad \text{and} \quad [n^\alpha \partial_\alpha (A_\beta e_i^\beta)] = 0 \quad (59)$$

and if $[K_{ij}] = 0$ these can be written as

$$A_\alpha^\Sigma n^\alpha = 0, \quad [A_\alpha e_i^\alpha] = 0, \quad \text{and} \quad [e_i^\beta n^\alpha \nabla_\alpha A_\beta] = 0. \quad (60)$$

In terms of $\tilde{F}_{\mu\nu}$ we get

$$[\tilde{F}_{\alpha i}] = 0, \quad [\tilde{F}_{ij}] = 0, \quad [\tilde{\nabla}_k \tilde{F}_{0i}] = 0, \quad [\tilde{\nabla}_k \tilde{F}_{ij}] = 0, \quad (61)$$

and, for a timelike boundary, Jackson's conditions are recovered. Because these quantities are coordinate invariant and projected onto the boundary surface, they are unaffected by a change in signature. However, non-dummy indices may not be raised and lowered freely, and of course (59) and (60) are not gauge invariant.

Now the current density

$$4\pi J_\beta = \nabla^\alpha F_{\alpha\beta} = \nabla^\alpha \partial_\alpha A_\beta - \nabla^\alpha \partial_\beta A_\alpha \quad (62)$$

and the stress-energy tensor

$$4\pi T_{\alpha\beta} = F_\alpha^\mu F_{\beta\mu} - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu} \quad (63)$$

have normal gauge, normal coordinate components

$$4\pi \tilde{J}_j = \tilde{g}^{im} \tilde{\nabla}_i (\tilde{\partial}_m \tilde{A}_j - \tilde{\partial}_j \tilde{A}_m) + \epsilon \tilde{\nabla}_0 (\tilde{\partial}_0 \tilde{A}_j - \tilde{\partial}_j \tilde{A}_0) \quad (64)$$

$$= \mathfrak{A} \tilde{\nabla}^m \tilde{F}_{mj} + \epsilon (\tilde{\partial}_0 \tilde{F}_{0j} + K \tilde{F}_{0j} - 2K_j^m \tilde{F}_{0m}), \quad (65)$$

$$4\pi \tilde{J}_0 = \tilde{g}^{im} \tilde{\nabla}_i (\tilde{\nabla}_m \tilde{A}_0 - \tilde{\nabla}_0 \tilde{A}_m) = \mathfrak{A} \tilde{\nabla}^m \tilde{F}_{m0} \quad (66)$$

(where the terms containing ϵ in \tilde{J}_0 cancel owing to the antisymmetry of $F_{\alpha\beta}$) and

$$4\pi \tilde{T}_{00} = \frac{1}{2} \tilde{F}_{0k} \tilde{F}_{0l} g^{kl} - \frac{1}{4} \epsilon \tilde{F}^{kl} \tilde{F}_{kl}, \quad (67)$$

$$4\pi \tilde{T}_{0i} = \tilde{F}_{0k} \tilde{F}_{il} \tilde{g}^{kl}, \quad (68)$$

$$4\pi \tilde{T}_{ij} = \epsilon (\tilde{F}_{0i} \tilde{F}_{0j} - \frac{1}{2} \tilde{g}_{ij} \tilde{F}_{0k} \tilde{F}_{0l} \tilde{g}^{kl}) \\ + (\tilde{F}_{ik} \tilde{F}_{jl} \tilde{g}^{kl} - \frac{1}{4} \tilde{g}_{ij} \tilde{F}_{kl} \tilde{F}_{mn} \tilde{g}^{km} \tilde{g}^{ln}). \quad (69)$$

With the foregoing EM junction conditions and the standard D conditions we find

$$4\pi[\tilde{J}_j] = 4\pi[J_\alpha e_j^\alpha] \\ = \epsilon^+ \tilde{\partial}_0^+ \tilde{F}_{0j}^+ - \epsilon^- \tilde{\partial}_0^- \tilde{F}_{0j}^- \\ + (\epsilon^+ - \epsilon^-) (K \tilde{F}_{0j} - 2K_j^m \tilde{F}_{0m}), \quad (70)$$

$$4\pi[\tilde{J}_0] = 4\pi[J_\alpha n^\alpha] = 0, \quad (71)$$

$$4\pi[\tilde{J}^0] = 4\pi[J^\alpha l_\alpha] = (\epsilon^+ - \epsilon^-) \mathfrak{A} \tilde{\nabla}^m \tilde{F}_{m0} \quad (72)$$

(cf. [26]). Across a L - L boundary, $\epsilon^+ = \epsilon^-$, the entire stress energy tensor is continuous, while $[J^\alpha l_\alpha] = 0$ links the divergence theorems on either side of Σ and ensures conservation of four-current. However, a jump in the value of the current parallel to the surface is quite acceptable.

Returning to Eq. (43), for the surface term E in the case of a signature change, we find that

$$E = [J^\alpha l_\alpha] = [\tilde{J}^0] = \mathfrak{A} \tilde{\nabla}^m (\tilde{F}_{m0}^+ + \tilde{F}_{m0}^-) = 2\tilde{\nabla} \cdot \tilde{\mathbf{E}}, \quad (73)$$

where \tilde{J}^0 is the charge density and \tilde{F}_{m0}^\pm is the electric field, both as measured by an orthogonally moving observer on the L side. Furthermore, from

$$4\pi[\tilde{T}_{00}] = 4\pi[\tilde{T}^{00}] \\ = -\frac{1}{4} (\tilde{F}_+^{kl} \tilde{F}_{kl}^+ + \tilde{F}_-^{kl} \tilde{F}_{kl}^-) = -\frac{1}{2} \tilde{\mathbf{B}}^2, \quad (74)$$

$$4\pi[\tilde{T}_i^0] = 4\pi g_{ij} [\tilde{T}^{0j}] \\ = \tilde{g}^{kl} (\tilde{F}_{0k}^+ \tilde{F}_{il}^+ + \tilde{F}_{0k}^- \tilde{F}_{il}^-) = 2(\tilde{\mathbf{B}} \times \tilde{\mathbf{E}})_i, \quad (75)$$

$$4\pi[\tilde{T}_0^0] = \frac{1}{2} \tilde{g}^{kl} (\tilde{F}_{0k}^+ \tilde{F}_{0l}^+ + \tilde{F}_{0k}^- \tilde{F}_{0l}^-) = \tilde{\mathbf{E}}^2, \quad (76)$$

$$4\pi \tilde{g}_{ij} [\tilde{T}_0^j] = 4\pi[\tilde{T}_{0i}] = 0, \quad (77)$$

we see that the EM energy density \tilde{T}^{00} cannot be continuous, unless the magnetic field \tilde{F}_{ij} is zero. This is also true for \tilde{T}_{00} , whereas \tilde{T}_0^0 can only be continuous if the electric field \tilde{F}_{0i} is zero, and the continuity of \tilde{T}_i^0 and \tilde{T}^{0i} requires a zero Poynting vector. If one requires continu-

ity of all components of the EM stress tensor in all index positions, the entire EM field must be zero at a signature change, regardless of sign choices in the matching conditions. In fact, zero field is required just to make \tilde{T}_{00} and \tilde{T}_0^0 continuous, so, although we could recover conservation of four-current by matching $\tilde{F}_\pm^{\mu\nu}$ instead of \tilde{F}_\pm^\pm , there is no way to recover full EM energy conservation without restricting the field configuration at the transition surface. For an electrovac model, the Einstein equations plus (47)–(50) and (74)–(77) lead to

$$\mathfrak{R} = \tilde{\mathbf{B}}^2, \quad (78)$$

$$2(\mathfrak{A} \tilde{\nabla}_j K_i^j - \mathfrak{A} \tilde{\nabla}_i K) = 4(\tilde{\mathbf{B}} \times \tilde{\mathbf{E}})_i, \quad (79)$$

$$K^2 - K_{ij} K^{ij} = 2\tilde{\mathbf{E}}^2, \quad (80)$$

which restrict both the extrinsic curvature of Σ and the EM field configuration.

We managed to sidestep the question of whether to match \tilde{A}_\pm or \tilde{A}_\pm^\pm by choosing $\tilde{A}_0^\pm = 0$, but it definitely seems more natural to match $\tilde{\partial}_0 \tilde{A}_i|_\pm$ and hence \tilde{F}_{0i}^\pm than $\tilde{\partial}^0 \tilde{A}_i|_\pm$ and \tilde{F}_\pm^{0i} .

JUNCTION CONDITIONS AND CONSERVATION LAWS

In this paper we have considered the effect of signature change on conservation laws in classical general relativity. Although we have advocated the use of Darmois junction conditions, we emphasize that the results (42)–(45), (46), (59), (64)–(66) hold for any set of junction conditions that impose at least (6) and any set of coordinates near Σ for which the limiting values $\Psi^\delta l_\delta|_\Sigma$ may be calculated, so the surface effects for any other choice may be calculated this way. Nevertheless, we are not aware of any junction conditions for signature change that are less restrictive than the D conditions.

At a change of the signature of spacetime the Darmois conditions no longer ensure standard matter conservation, as they do through a constant signature boundary, and only if Eqs. (47)–(50) are zero can we have matter conservation in the usual sense, but this means the transition occurs on a surface with zero extrinsic curvature that has ${}^3R = 0$ — a highly restrictive condition, eliminating all realistic cosmological models. [Although the $k = 0$ Friedmann-Lemaître-Robertson-Walker (FLRW) model survives, its perturbed cousin does not.]

Significantly, the condition that the extrinsic curvature be zero at the signature change surface, $K_{ij}^\Sigma = 0$, which is required in the quantum cosmology approach, does not entirely remove the need to modify the matter conservation law, since only three of (47)–(50) become zero. If however, one is satisfied with only $G_\alpha^\beta v^\alpha$ and not $G^{\beta\alpha} v_\alpha$ being conserved, then $K_{ij}^\Sigma = 0$ will suffice.

Since the operation of raising or lowering indices introduces minus signs on one side of a signature change and not on the other, it is clear that a nonempty model cannot have all of $G_{\alpha\beta}$ continuous across a signature change as well as all of $G^{\alpha\beta}$. If G_{oi} is continuous, then G^{oi} has a jump; if G_{00} is continuous then G_o^0 has a jump, and vice versa. The same applies to any nonzero tensor — if $g_{\alpha\beta}$,

$\Gamma_{\alpha,\beta\gamma}$, $R_{\alpha\beta\gamma\delta}$, $F_{\alpha\beta}$, etc., are continuous, then $g^{\alpha\beta}$, $\Gamma_{\beta\gamma}^{\alpha}$, $R^{\alpha\beta\gamma\delta}$, $F^{\alpha\beta}$, etc., are not. Thus we cannot expect both $\nabla_{\alpha}G^{\alpha\beta}$ and $\nabla_{\alpha}G_{\beta}^{\alpha}$ to be free of surface effects.

The jump in the value of the energy density relative to the surface, given by Eq. (33), cannot be viewed as a surface layer of matter on Σ in the usual sense, as Israel's definition for the surface stress tensor is still zero here. Rather it is an effect due to the change in physical law from Lorentzian to Euclidean forms. Momentum changes from mass times velocity into the "momentum" conjugate to the "time" coordinate, which is now a spatial direction, and energy converts to a quantity having the dimensions of "momentum."

For the other matching options too, the situation is not much different. Matching $\tilde{g}_{ij,o}$ to its negative across the signature change does remove two out of four surface effects. But if orientation is not preserved through the signature change, the conservation of matter may become separate from the continuity of the projected Einstein tensor, with conservation breaking down even if all 10 components of $\tilde{G}^{\mu\nu}$ are continuous through Σ .

For any vector field on spacetime, the generalized patchwork divergence theorem, (44) or (45), and the resulting expression for E , Eq. (43), combined with the appropriate matching conditions, then show that the field may also have a surface effect due to the signature change and the modification of its physics. Specifically, the matching conditions for the EM field (60) lead to a surface effect in the associated current density (73) which is zero only if $\tilde{J}_o = J_{\alpha}n^{\alpha} = 0$. This implies the current is conserved only if the net charge density (in geodesic normal coords) is zero everywhere on the signature change surface, the most likely example being that of a source-free field. (It is utterly improbable that a system of charges and currents should have zero charge density everywhere on a spacelike slice of the Universe.) However, the EM energy density is not continuous. This may not be a problem if there are other fields or matter components that can exchange energy and momentum with the EM field, so as to satisfy the overall "modified conservation law." But, for an electrovac solution, (78)-(80) place very strong restrictions on the allowed gravitational and EM fields, which eliminate any radiation at the transition. If there is a nonzero charge density of electrons, say, then the surface effect calculated from the EM junction conditions must be consistent with the results from the Dirac equation, with suitable junction conditions. This would be an interesting avenue of investigation.

When considering matter tensors consisting of several components and/or fields, the above results indicate that

we may specify continuity of momentum density (parallel to Σ) separately for each component, but we should expect the individual energy densities to jump.

Faced with the inapplicability of the standard divergence theorem across a signature change, our intuition that matter must always be conserved in the usual way no longer seems physically justified. In fact, when considering the physics of signature changes, all intuition should be very carefully cross-checked. If we wish to impose extra restrictions in order to describe a particular physical effect or situation, we should review the physical justification in the light of the change in the relevant physical laws.

At a L to L boundary, the D conditions are sufficient to ensure the minimum necessary continuity and the conservation of all fundamental *gravitational* quantities, and any further restrictions then specialize to particular scenarios, and eliminate other possibilities. We have presented a L to E boundary in the same light. The D conditions impose the same number of conditions, and still ensure minimal continuity and modified conservation laws, so further restrictions need only be imposed in order to describe specific physical effects which require them. For example, [9] and [10] proposed criteria for when the signature should change, but these are not specifically required by the Darmois junction conditions. Their condition amounted to requiring continuity of the equation of state (continuous Friedmann equation) across Σ . One might regard this as the equivalent for a fluid of junction conditions for a field. Such extra conditions may often be reasonable and necessary.

On the other hand, a recent investigation [27,28] related to Smolin's [29] idea that Universes evolve in Darwinian fashion, required the Darmois approach. This enabled collapse to a black hole to pass through a double signature change, emerging into a new Universe. Interesting results are not possible if $K_{ij} = 0$.

In the absence of convincing physical arguments or experimental evidence, as mentioned at the beginning, the "correct" way to effect a change of signature remains a matter of conjecture. The relationship between the present results and other approaches found in the literature will be discussed elsewhere.

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