Analytic calculation of the vacuum wave function for (2+1)-dimensional SU(2) lattice gauge theory

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The long wavelength vacuum wave function of (2+1)-dimensional SU(2) lattice gauge theory is calculated by the method of truncated eigenvalue equation. Third order results are consistent with Monte Carlo measurement and display a good scaling behavior extending to the deep weak coupling region.

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I. INTRODUCTION

Non-Abelian gauge theory possesses a nontrivial vacuum state which leads to confinement. There have been many discussions on the properties of the vacuum state. Confinement properties had been attributed to condensation of monopoles and instantons in the gauge field [1]. In order to study the low energy physics of hadrons, one needs more detailed information on the structure of the vacuum wave function. Lattice gauge theory (LGT) provides a framework for studying nonperturbative aspects of non-Abelian gauge fields. The vacuum state of non-Abelian gauge theory has been studied in the continuum and on the lattice.

It is generally accepted that the vacuum state of a gauge field is highly disordered for physical scales larger than the confinement scale [1,2]. Accordingly, the long wavelength behavior of the vacuum state can be approximately represented by the wave function [2,3]

$$\Psi(A)=\exp\left[-\mu\int d^4x\,\mathrm{tr}(F_{ij})^2\right] . \quad (1.1)$$

On the lattice, the corresponding vacuum wave function is approximately represented by [3]

$$\Psi(U)=\exp\left[\mu\sum_p \mathrm{tr}(U_p+U_p^\dagger)\right] . \quad (1.2)$$

The vacuum wave function in LGT was also studied by Monte Carlo simulations [4] using slow varying gauge field configurations. The result is consistent with (1.2), with μ satisfying respectively the scaling behaviors in 3 and 4 dimensions.

Recently, Arisue [5] gave a more detailed Monte Carlo measurement of the vacuum wave function for SU(2) gauge theory in 3 dimensions. In the continuum limit, his result is

$$\Psi_0(U)=\exp\left[-\mu\int d^2x\,\mathrm{tr}F(x)^2-\mu_2\int d^2x\,\mathrm{Tr}[D_iF(x)]^2 + \text{higher order terms}\right] , \quad (1.3)$$

where $F=F_{12}$, and

$$\mu_0=(0.91\pm0.02)/e^2 , \quad (1.4a)$$

$$\mu_2=-(0.19\pm0.05)/e^6 . \quad (1.4b)$$

e is the invariant charge which is related to the dimensionless coupling constant g by $g^2=e^2a$.

It is interesting to study the vacuum wave function by analytic methods. It was noted in Ref. [6] that the independent plaquette vacuum state (1.2) is the exact vacuum state of a modified lattice Hamiltonian. Later investigations [7] showed that the modified Hamiltonian differs from the Kogut-Susskind (KS) Hamiltonian by a relevant operator whose continuum limit is proportional to

$$\int d^2x\,\mathrm{Tr}(D_iF)^2 .$$

Hence it is reasonable that the vacuum wave function of the KS Hamiltonian has the form (1.3) with coefficients μ_0 and μ_2 to be determined.

Variational methods are usually used in analytical calculations of mass spectrum in LGT [8]. An approximate vacuum state was obtained by minimizing the expectation value $\langle H\rangle_0$ of the lattice Hamiltonian in a variational vacuum state. Integrations over all gauge field configurations must be performed in evaluating $\langle H\rangle_0$ and other matrix elements. Short wavelength configurations $[\lambda=O(a)]$ dominate the gauge field integrations in the continuum limit. Unless we have a sophisticated vacuum wave function that takes due account of the short wavelength configurations, it is improbable to obtain a good long wavelength vacuum wave function with the correct scaling in the weak coupling region by simple variational methods.

The vacuum state of LGT can also be studied by directly solving the eigenvalue problem in the Hamiltonian formulation. Approximation schemes, including strong coupling expansion and truncated eigenvalue equations, were described in Ref. [3]. As far as we know, solutions with correct scaling behavior have not been obtained by this method. Recently, Llewellyn Smith and Watson [9] proposed the shifted coupled cluster method to solve the eigenvalue equation. It seems to work well for the vacuum energy. However, results for the mass gap are far from scaling in the weak coupling region. In this paper, we present a different approximation scheme to the eigenvalue problem. A long wavelength vacuum state consistent with (1.3) with satisfactory scaling behavior extending to the deep weak coupling region is obtained in a third order calculation.

In Sec. II we present the formulation and approximation scheme. In Sec. III we calculate the vacuum wave function up to third order graphs. Section IV is devoted to conclusions and discussions.

II. FORMULATION AND METHOD OF APPROXIMATION

For simplicity, we study the (2+1)-dimensional SU(2) LGT in the Hamiltonian formulation. The Kogut-Susskind Hamiltonian is

$$H=\frac{g^2}{2a}\left[\sum_l E_l^2-\frac{4}{g^4}\sum_p \mathrm{tr}U_p\right] . \quad (2.1)$$

We write the vacuum wave function in the exponential form

$$|\Psi_0\rangle=e^{R(U)}|0\rangle , \quad (2.2)$$

where $R(U)$ contains closed loops and the state $|0\rangle$ is defined as

$$E_l^a|0\rangle=0 .$$

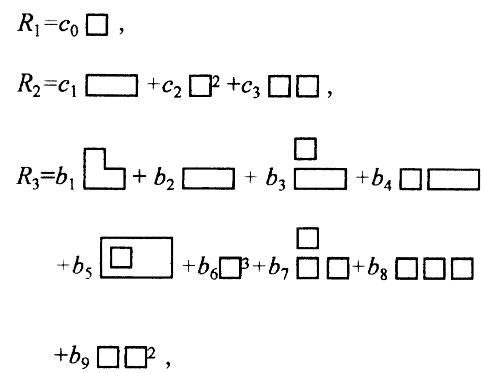
The eigenvalue equation for H is

$$\sum_l ([E_l^a,[E_l^a,R]]+[E_l^a,R][E_l^a,R]) -\frac{4}{g^4}\sum_p \mathrm{Tr}(U_p)=\frac{2a}{g^2}\varepsilon_0 . \quad (2.3)$$

Defining the order of a graph as the number of plaquettes involved (overlapping plaquettes are also counted), we expand R in order of graphs:

$$R=R_1+R_2+\cdots . \quad (2.4)$$

In SU(2) theory we have $\mathrm{tr}U_p=\mathrm{tr}U_p^\dagger$; all loops with crossing can be transformed into loops without crossing. The complete set of graphs in lower orders are



(2.5)

where

$$\square=\mathrm{Tr}U_p=\mathrm{Tr}U_1U_2U_3U_4 ,$$

$$\sqsubset\!\sqsupset=\mathrm{Tr}U_1U_2\cdots U_6, \ldots .$$

In general

$$[E_l^a,[E_l^a,R_n]]\in R_n+\text{lower order terms} ,$$
$$[E_l^a,R_n][E_l^a,R_{n'}]\in R_{n+n'}+\text{lower order terms} . \quad (2.6)$$

We now derive a general formula for $[E_l,\mathrm{Tr}U_\Gamma][E_l,\mathrm{Tr}U_{\Gamma'}]$ in the long wavelength limit, where Γ and Γ' are two rectangular loops with sides N_1,N_2 and N_1',N_2', respectively. Consider a definite link l. For SU(2) we have

$$[E_l^a,\mathrm{Tr}U_\Gamma][E_l^a,\mathrm{Tr}U_{\Gamma'}]=[E_l^a,\tfrac{1}{2}\mathrm{Tr}(U_\Gamma+U_\Gamma^\dagger)][E_l^a,\tfrac{1}{2}\mathrm{Tr}(U_{\Gamma'}+U_{\Gamma'}^\dagger)]$$
$$=\frac{1}{4}\sum_{\Gamma,\Gamma'\supset l}\mathrm{Tr}[\Lambda^a(U_\Gamma-U_\Gamma^\dagger)]\,\mathrm{Tr}[\Lambda^a(U_{\Gamma'}-U_{\Gamma'}^\dagger)] . \quad (2.7)$$

To lowest order, we have

$$U_\Gamma-U_\Gamma^\dagger\cong-2iea^2N_pF , \quad (2.8)$$

where $N_p=N_1N_2$ is the number of plaquettes in Γ.

For a link l in the two-direction, the right-hand side (RHS) of (2.7) is

$$-\frac{e^2a^4}{4}N_pN_p'N_2N_2'(N_1D_1F^a)(N_1'D_1F^a) .$$

Adding the contribution from links in direction 1, we finally obtain

$$\sum_l [E_l^a,\mathrm{tr}U_\Lambda][E_l^a,\mathrm{tr}U_\Lambda']=-\tfrac{1}{2}e^2a^4N_p^2N_p^{2\prime}\sum_p \mathrm{Tr}(D_iF)^2 . \quad (2.9)$$

We now give a recipe for truncating the eigenvalue equation. Let R contain up to Mth order graphs:

$$R=R_1+R_2+\cdots+R_M . \quad (2.10)$$

The term $[E_l,[E_l,R]]$ contains no new graphs, but the term $[E_l,R][E_l,R]$ creates new graphs. Hence we must truncate the latter term. The simplest way is just preserving the terms

$$\sum_{n+n'\le M}[E_l^a,R_n][E_l^a,R_{n'}] .$$

Our truncated eigenvalue equation is

$$\sum_l [E_l^a,[E_l^a,R]]+\sum_{l,n+n'\le M}[E_l^a,R_n][E_l^a,R_{n'}] -\frac{4}{g^4}\mathrm{tr}U_p=\text{const} . \quad (2.11)$$

This equation differs from the truncated eigenvalue equation in Ref. [3] in the treatment of the new graphs form $[E_l,R][E_l,R]$.

For example, in $[E_l,R_1][E_l,R_2]$, there is a term

$$[E^\alpha_\ell,\square][E^\alpha_\ell,▭] = -2\,⌐▭ - ▭ + \overset{\square}{▭}$$

$$+\frac{1}{2}\,\square▭ + \frac{3}{2}\,[\square] - 3\,\square\,. \quad (2.12)$$

All graphs are in R_3 with the exception of the last graph, which is in R_1. Greensite [3] argued that if we truncate the equation at second order then the term -3 is preserved while all others are discarded. However, by (2.9), all terms in (2.12) combine to give the continuum limit

$$-2e^2a^6\,\mathrm{Tr}(D_iF)^2\,.$$

If we preserve the term $-3\square$ and discard all other terms, then the long wavelength limit will be seriously altered. Hence it is more consistent to discard all terms in (2.12) for calculations up to second order. To third order, all terms in (2.12) are included.

III. SECOND ORDER AND THIRD ORDER CALCULATIONS

Calculations of the truncated eigenvalue equation (2.11) are straightforward. At second order, the equation is

$$[E^a_l,[E^a_l,R_1+R_2]]+[E^a_l,R_1][E^a_l,R_1] - \frac{4}{g^4}\mathrm{Tr}U_p = \mathrm{const}\,. \quad (3.1)$$

Substituting R_1 and R_2 from (2.5), we obtain

$$(3c_0-\frac{4}{g^4})\,\square + (\frac{9}{2}c_1 - c_3 - 2c_0^2)\,▭ + (8c_2+c_0^2)\,\square^2$$

$$+(\frac{13}{2}c_3+c_0^2)\,\square\square = const\,. \quad (3.2)$$

Equating the coefficients of each graph to zero, the result is

$$c_0=\tfrac{1}{4}(\tfrac{1}{3}\beta^2),\quad c_1=\tfrac{1}{39}(\tfrac{1}{3}\beta^2)^2\,,$$
$$c_2=-\tfrac{1}{128}(\tfrac{1}{3}\beta^2)^2,\quad c_3=-\tfrac{1}{104}(\tfrac{1}{3}\beta^2)^2 \quad (3.3)$$

where $\beta=4/g^2$. This is just the strong coupling expansion given in Ref. [3].

The long wavelength limit of a graph can be obtained by direct evaluation of the graph to lowest order in a or by using (2.9). For example,

$$[E^\alpha_\ell,\square][E^\alpha_\ell,\square] = -4 - 2\,▭ + \square^2 + \square\square$$

$$= -\frac{1}{2}e^2a^6\mathrm{Tr}(D_\ell F)^2\,,$$

$$\square^2 \cong \square\square \cong [\mathrm{Tr}(1-\tfrac{1}{2}e^2a^4F^2)]^2 = 4-2e^2a^4\,\mathrm{Tr}F^2\,. \quad (3.4)$$

Hence

$$▭ = 2-2e^2a^4\,\mathrm{Tr}F^2+\tfrac{1}{4}\mathrm{Tr}(D_iF^2)^2\,. \quad (3.5)$$

The long wavelength vacuum wave function is

$$\Psi(U)=N\exp\left[-\frac{\mu_0}{e^2}\int d^2x\,\mathrm{Tr}F^2-\frac{\mu_2}{e^6}\int d^2x\,\mathrm{Tr}(D_iF)^2\right]\,, \quad (3.6)$$

where

$$\mu_0=[\tfrac{1}{2}c_0+2(c_1+c_2+c_3)]g^4\,, \quad (3.7a)$$

$$\mu_2=-\frac{c_1}{4}g^8=-0.1823\,. \quad (3.7b)$$

The curves for μ_0 and μ_2 are plotted in Fig. 1. While μ_0 blows up at weak coupling, μ_2 has the correct scaling with its value close to that given in Ref. [5]. However, the excellent agreement of (3.7b) with (1.4b) is somewhat illusive, because at this order μ_2 happens to contain only the term with correct scaling.

We now turn to third order calculation. The truncated eigenvalue equation is

$$\sum_l\{[E^a_l,[E^a_l,R_1+R_2+R_3]]+[E^a_l,R_1][E^a_l,R_1] + 2[E^a_l,R_1][E^a_l,R_2]\}-\frac{4}{g^4}\square=\mathrm{const}\,. \quad (3.8)$$

Substituting (2.5) in (3.8), we obtain

$$[3c_0-\frac{4}{g^4}-3b_5-24b_6-8b_9-6c_0c_1-16c_0(c_2+c_3)]\,\square$$

$$+(\frac{9}{2}c_1-c_3-2c_0^2)\,▭ + (8c_2+c_0^2)\,\square^2 + (\frac{13}{2}c_3+c_0^2)\,\square\square$$

$$+(6b_1-b_3-4c_0c_1)\,⌐▭ + (6b_2-b_4-2c_0c_1)\,▭▭$$

$$+(8b_3-2b_7+2c_0c_1-4c_0c_3)\,\overset{\square}{▭}$$

$$+(8b_4-2b_8+c_0c_1-2c_0c_3)\,\square▭$$

$$+(9b_5-2b_9+3c_0c_1-8c_0c_2-2c_0c_3)\,[\square] + (15b_6+4c_0c_2)\,\square^3$$

$$+(10b_7+2c_0c_3)\,\overset{\square}{\square}\square + (10b_8+c_0c_3)\,\square\square\square$$

$$+(12b_9+5c_0c_3+4c_0c_2)\,\square\square^2 = const\,. \quad (3.9)$$

From the algebraic equations for c_i and b_i, we obtain

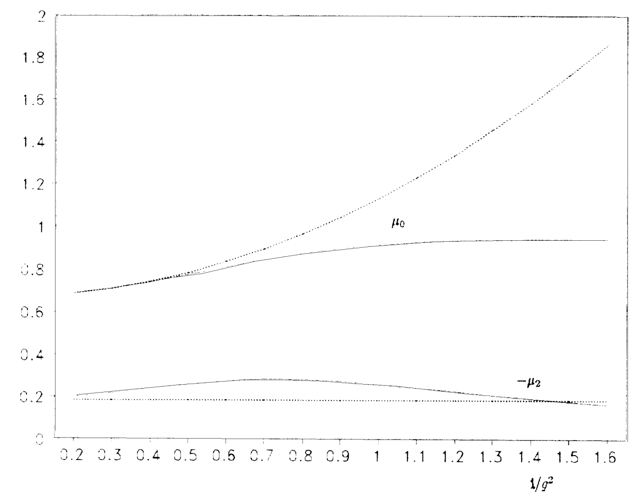


FIG. 1. μ_0 and $-\mu_2$ versus $1/g^2$. Dashed line: second order result. Solid line: third order result.

$$\begin{aligned}
&c_1=\tfrac{16}{39}c_0^2,\quad c_2=-\tfrac{1}{8}c_0^2,\quad c_3=-\tfrac{2}{16}c_0^2\ ,\\
&b_1=\tfrac{121}{780}c_0^3,\quad b_2=\tfrac{191}{1560}c_0^3,\quad b_3=-\tfrac{67}{390}c_0^3\ ,\\
&b_4=-\tfrac{67}{780}c_0^3,\quad b_5=-\tfrac{121}{468}c_0^3,\quad b_6=\tfrac{1}{30}c_0^3\ ,\\
&b_7=\tfrac{2}{65}c_0^3,\quad b_8=\tfrac{1}{65}c_0^3,\quad b_9=\tfrac{11}{104}c_0^3\ ,\\
&3c_0-\frac{4}{g^4}+\tfrac{881}{780}c_0^3=0\ .
\end{aligned} \tag{3.10}$$

Evaluating the long wavelength limit of all graphs up to third order, we obtain

$$\begin{aligned}
\mu_0=\{&\tfrac{1}{2}c_0+2(c_1+c_2+c_3)+\tfrac{9}{2}(b_1+b_2)\\
&+6(b_6+b_7+b_8+b_9)+5(b_3+b_4+b_5)\}g^4\ ,
\end{aligned} \tag{3.11a}$$

$$\mu_2=-\{\tfrac{1}{4}c_1+b_1+\tfrac{3}{2}b_2+\tfrac{1}{2}(b_3+b_4+b_5)\}g^8\ . \tag{3.11b}$$

Results for μ_0 and μ_2 in second and third order approximations are plotted in Fig. 1. Nonperturbative corrections to the strong coupling expansion appear at third order. We observed that the third order result for μ_0 is greatly improved in the weak coupling region ($\beta=4.0-8.0$) with a value close to the Monte Carlo (MC) result (1.4a). The third order result for μ_2 is also in reasonable agreement with (1.4b). Higher order approximations are needed to obtain better scaling behavior for μ_2.

IV. CONCLUSION AND DISCUSSIONS

The result of third order calculations of the vacuum wave function is very encouraging. It is seen that μ_0 has a nearly constant value extending to the deep weak coupling region. The results of this paper support the suggestion that, for long wavelength configurations, the vacuum state of non-Abelian gauge theory can be effectively represented by a few low order graphs in the exponential. This form of vacuum wave function is also supported by calculations of the low energy spectrum [8]. Further works will include (1) extensions to fourth order and higher order calculations, (2) extensions to (3+1)-dimensional LGT and other gauge groups, (3) extensions to calculations of excited states and LGT with fermions.

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