

String-dominated cosmology

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If string theory controls physics at the string scale, the dynamics of the early Universe before the GUT era will be governed by the low-energy string equations of motion. Studying these equations, we find that depending on the initial conditions when the stringy era starts, and on the time when this era ends, there are a variety of qualitatively distinct types of evolution. Among these is the possibility that the Universe underwent a period of inflation. A by-product of this analysis is the observation that it is often possible to erase any evidence of a dilaton at late times.

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In this paper we discuss the solutions of the lowest-order string β -function equations that represent homogeneous isotropic four-dimensional spacetimes. The basic ingredient of this calculation is therefore the massless states of the string, and for simplicity, we consider only the closed bosonic string. It is quite straightforward to extend our technique to other types of string theory. This type of solution is of importance for a number of reasons. The first is that it illustrates some of the possible solutions to the string equations of motion as a problem in its own right. One might think that the only solution to the bosonic string equations that is consistent with having spatial sections of spacetime being surfaces of constant curvature would be the well-known example of 26-dimensional Minkowski spacetime. However, it is possible to trade off some of these extra dimensions in return for spacetime curvature, and this is precisely what we do here.

Another reason for studying the problem is physically motivated. If string theory really controls physics at the string scale (presumed to be roughly Planckian although it could be at a significantly lower scale), then from this era down to whenever the stringy symmetries are broken to yield the physics of (presumably) the grand unified theory (GUT) era universe, it seems reasonable to suppose that the dynamics of the Universe will be governed by the low-energy string equations of motion. These are the equations to be studied in the remainder of this paper.

Similar, but much more restricted results have previously been obtained by Antoniadis, Bachas, Ellis, and Nanopoulos [1], Love and Bailin [2], Campbell, Kaloper, and Olive [3], Tseytlin and Vafa [4], Tseytlin [5], and Kaloper and Olive [6]. A number of other related references can be found in [5,6]. Their solutions are all special cases of our results which in some sense are a complete set of solutions that are consistent with symmetries of the spacetime that we are imposing.

In the closed bosonic string theory, the long-range massless fields are the dilaton Φ , the axion field strength $H_{abc} = H_{[abc]} = 6\partial_{[a}B_{bc]}$, which is derived from the two-form potential B_{ab} , and the graviton, or equivalently, the spacetime metric tensor g_{ab} . Condensates of these fields can be treated in a way consistent with the symmetries of the string and then obey the β -function equations [7]. If we work in the string frame, these equations can all be derived from the action

$$I = \int_M d^4x g^{1/2} e^{\phi} [c - R - (\nabla\phi)^2 + \frac{1}{12} H_{abc} H^{abc}], \quad (1)$$

where R is the Ricci scalar of the metric g_{ab} , c is a constant, related to the central charge of the string theory, and the integral is taken over all of spacetime M . The scale of string physics is determined by α' , the inverse string tension. Provided that the spacetime curvature is small on the string scale, then Eq. (1) is a complete description of the massless modes of the string. However, for curvatures large on the scale of string physics, this action needs to be modified by terms in α' ; however, in the era we are considering, they are negligible.

Variation of this action with respect to the metric yields the string analogue of Einstein's equation

$$R_{ab} = \nabla_a \nabla_b \phi + \frac{1}{4} H_{acd} H_b^{cd}. \quad (2)$$

Variation with respect to B_{ab} gives the axionic analogue of Maxwell's equation

$$\nabla_a H^{abc} + \nabla_a \phi H^{abc} = 0. \quad (3)$$

Finally, variation of the dilaton gives rise to

$$c = R - (\nabla\phi)^2 - 2\Box\phi + \frac{1}{12} H_{abc} H^{abc}. \quad (4)$$

The constant c is given by $(D-26)/3\alpha'$ for the bosonic string, and $(\frac{3}{2}D-15)/3\alpha'$ for the heterotic or superstring, where D is the dimensionality of spacetime and α' is the inverse string tension. However, c can be changed from these values by coupling some conformal field theory to the string, and so c can have any value in practice. One often studies critical string theory where $c=0$. The reason for such a choice is that it enables Minkowski

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spacetime to be a stable ground state for the string. However, we are interested in cosmological solutions in string theory, and so this is no longer a meaningful restriction. We regard c as an arbitrary constant in what follows.

The β -function equations (1)–(4) describe physics as seen from the viewpoint of the string. However, they are not convenient for understanding gravitational phenomena, because the coefficient of the Ricci scalar depends on the dilaton field. An easy way to look at spacetime physics is to perform a conformal transformation on the metric so as to eliminate the dilaton-dependent term. This conformal frame is usually referred to as the Einstein frame, whereas the original one is termed the string frame. Such a conformal transformation yields a new metric \bar{g}_{ab} given by

$$\bar{g}_{ab} = e^{-\phi} g_{ab} \quad (5)$$

so that the action becomes

$$I = \int d^4x \bar{g}^{1/2} [c e^{-\phi} - \bar{R} - \frac{1}{2}(\nabla\phi)^2 + \frac{1}{12} e^{2\phi} H_{abc} H^{abc}]. \quad (6)$$

We see from the new action that in the Einstein frame, gravitation is described by the minimal gravitational action but matter fields couple to gravity via the dilaton with various conformal weights. This is reminiscent of the situation in Brans-Dicke-type of theories. Variation of (6) leads to the field equations in the Einstein frame. The ϕ equation of motion is

$$\square\phi = c e^{-\phi} - \frac{1}{6} e^{2\phi} H_{abc} H^{abc}. \quad (7)$$

The H equations of motion are

$$\nabla_{[a} H_{bcd]} = 0, \quad (8a)$$

$$\nabla_a H^{abc} - 2\nabla_a \phi H^{abc} = 0, \quad (8b)$$

and the Einstein equation is

$$\begin{aligned} R_{ab} - \frac{1}{2} R g_{ab} = & \frac{1}{2} [\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla\phi)^2] \\ & + \frac{1}{4} e^{2\phi} (H_{acd} H_b^{cd} - \frac{1}{6} g_{ab} H_{cde} H^{cde}) \\ & + \frac{1}{2} c g_{ab} e^{-\phi}. \end{aligned} \quad (9)$$

Thus, the Einstein tensor couples to the minimal energy-momentum tensor of the dilaton, and the minimal energy-momentum tensor of the H -field weighted by $e^{2\phi}$. The central charge term c appears in a way reminiscent of the cosmological constant, but again with a weight factor now of $e^{-\phi}$. One might worry about the consistency of this set of equations. On general grounds, one knows that (4) follows from (2) and (3) as a consequence of the Bianchi identities. In a similar way, one can derive (7) as a first integral from (8) and (9).

As is apparent from (9), the right-hand side looks like a conventional gravitational theory, and therefore one should not be surprised to discover singularities in the solutions of (9) where the curvature blows up. Such singularities are, however, unlike general relativity, not necessarily unphysical. As far as string physics is con-

cerned, one only needs to ask if quantities in the string frame blow up, since only then will the string be badly behaved. In other words, the types of singularity predicted by the singularity theories in general relativity do not necessarily cause breakdown of physics in string theory.

The Universe on very large scales looks like a four-dimensional Friedmann-Robertson-Walker spacetime with a value of k which cannot be observationally determined. Therefore, on large scales the spacetime metric is

$$ds^2 = -dt^2 + a^2(t) d\sigma_k^2, \quad (10)$$

where $d\sigma_k^2$ is the line element on a unit three-sphere, Euclidean three-space, or the unit hyperboloid depending on whether $k=1, 0$, or -1 , respectively, and $a(t)$ is the cosmological scale factor. Consistent with the assumption of homogeneity and isotropy implicit in this metric, we choose the dilaton field to be independent of the special coordinates, and the axion field $H_{abc} = f(t)\epsilon_{abc}$, where ϵ_{abc} is the volume form on the surfaces of $t = \text{const}$. Then the requirement that H_{abc} be closed immediately implies that $f(t)$ is a constant, f . We can then reduce the β -function equations down to a set of three coupled ordinary differential equations:

$$c = 6 \frac{\dot{a}}{a} \dot{\phi} + 6 \left[\frac{\dot{a}}{a} \right]^2 + \dot{\phi}^2 - \frac{2f^2}{a^6} + \frac{6k}{a^2}, \quad (11)$$

$$\ddot{\phi} = -3 \frac{\ddot{a}}{a} = 6 \left[\frac{\dot{a}}{a} \right]^2 + \frac{3\dot{a}\dot{\phi}}{a} - \frac{6f^2}{a^6} + \frac{6k}{a^2} \quad (12)$$

where an overdot denotes d/dt . We now aim to characterize all the solutions of (11) and (12).

Although (11) and (12) are the equations in the string frame, it is straightforward to translate the solutions into the Einstein frame explicitly. In the Einstein frame, the metric is

$$d\bar{s}^2 = e^{-\phi} ds^2 = -d\tau^2 + b^2(\tau) d\sigma^2, \quad (13)$$

where

$$\tau = \int dt e^{-\phi/2} \quad (14a)$$

and

$$b(\tau) = e^{-\phi(\tau)/2} a(\tau). \quad (14b)$$

To simplify the discussion of these equations, first we consider the special case of the critical string ($c=0$) and a flat universe ($k=0$). We can now eliminate f and find that

$$\ddot{\phi} = -\frac{3\ddot{a}}{a} = -12 \left[\frac{\dot{a}}{a} \right]^2 - 15 \frac{\dot{a}}{a} \dot{\phi} - 3\dot{\phi}^2. \quad (15)$$

Perhaps the easiest way to find all the solutions to these equations is to change variables into

$$p = \dot{\phi}, \quad (16)$$

$$\chi = \dot{a}/a,$$

yielding

$$\begin{aligned} \dot{p} &= -12\chi^2 - 15p\chi - 3p^2, \\ \dot{\chi} &= 3\chi^2 + 5p\chi + p^2. \end{aligned} \quad (17)$$

There are several simple analytic solutions that can easily [1,5,6] be found from (17). They are

$$p = \frac{A}{t-t_0}, \quad \chi = \frac{B}{t-t_0} \quad (18)$$

with any of

- (i) $A=0, B=0$,
- (ii) $A=-\frac{2}{3}, B=\frac{1}{3}$,
- (iii) $A=1-\sqrt{3}, B=1/\sqrt{3}$,
- (iv) $A=1+\sqrt{3}, B=-1/\sqrt{3}$.

Case (i) is flat Minkowski spacetime (in either the Einstein frame or the string frame) and was first discussed by Antoniadis, Bachas, Ellis, and Nanopoulos [1]. Case (ii) is $a(t)=a_0 t^{1/3}$ and $\phi(t)=\phi_0 - \frac{2}{3} \ln t$ (setting $t_0=0$). If we transform this into the Einstein frame ($\phi_0=0$), then

$$\tau = \frac{2}{3} t^{4/3} \quad \text{and} \quad b = b_0 \tau^{1/2}. \quad (20)$$

In other words, case (ii) is identical to a radiation-dominated Universe. It is singular at $t=0$, even in the string frame, and unphysical since one finds that $f^2 < 0$. Cases (iii) and (iv) were originally found by Tseytlin [5], have $f^2=0$, and similar power-law expansions and contractions. Case (iii) is $a(r)=a_0 t^{1/\sqrt{3}}$ while case (iv) is $a(r)=a_0 t^{-1/\sqrt{3}}$.

The remaining solutions can be explored by examining the phase-plane portrait using the fact that Eq. (17) results in

$$\frac{dp}{d\chi} = -\frac{12\chi^2 + 15p\chi + 3p^2}{3\chi^2 + 5p\chi + p^2}. \quad (21)$$

It is shown in Fig. 1, the region of physical solutions is bounded by the exact solutions of cases (iii) and (iv)

which are given by

$$p = (-3 \mp \sqrt{3})\chi. \quad (22)$$

The evolution is in the direction of the arrows indicated. With the exception of the exact solutions (iii) and (iv), the spacetimes are nonsingular. There are two types of evolution depending on which side of the phase plane the trajectories begin. In both cases tracing the solutions backwards in time reveals that at earlier times they were repelled from the contracting version of either the exact solution (iii) or (iv). For solutions that expand rapidly at late times, the exact solution (iv) is an attractor. These type of solutions occupy the left-hand half of Fig. 1. The solutions that occupy the right-hand half of Fig. 1 are attracted to solution (iii) and on it to the (0,0) point; i.e., the Universe comes to a halt asymptotically. In Fig. 2 we plot $a(t)$ and $\chi(t)$ for two typical cases.

We turn now to a more complicated situation in which the spatial sectors of the Universe are not flat. In order to get a phase-plane picture similar to the one we got in the flat case, we change variables from t to θ such that

$$\frac{d\theta}{dt} = \frac{1}{a}. \quad (23)$$

Using θ as a dynamical variable we now define χ and P to be

$$\chi = \frac{a'}{a}, \quad P = \phi', \quad (24)$$

where a prime denotes $d/d\theta$. The analogue of Eq. (17) can then be rewritten as

$$\begin{aligned} P' &= -14\chi P - 12\chi^2 - 12k - 3P^2, \\ \chi' &= 5\chi P + 4\chi^2 + 4k + P^2. \end{aligned} \quad (25)$$

To check consistency, the solutions of Eqs. (25) must satisfy Eq. (11), which in terms of the new variables takes the form

$$6\chi P + 6\chi^2 + P^2 + 6k = 2f^2/a^4. \quad (26)$$

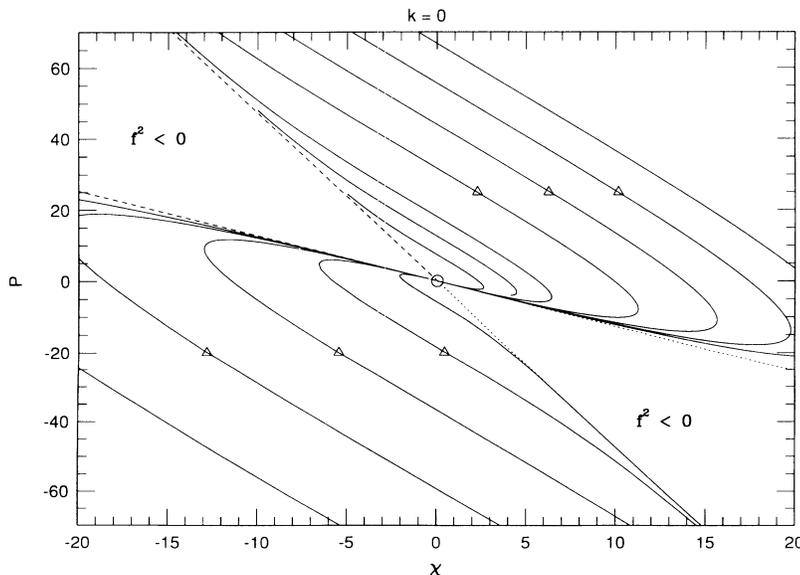


FIG. 1. A phase-plane diagram in the (χ, P) plane for $k=0$ and $c=0$. Evolution is in the direction of the arrows. The exact solutions (iii) and (iv) are identified by the half-dashed-half-dotted lines. The dashed part is a repeller and the dotted line is an attractor.

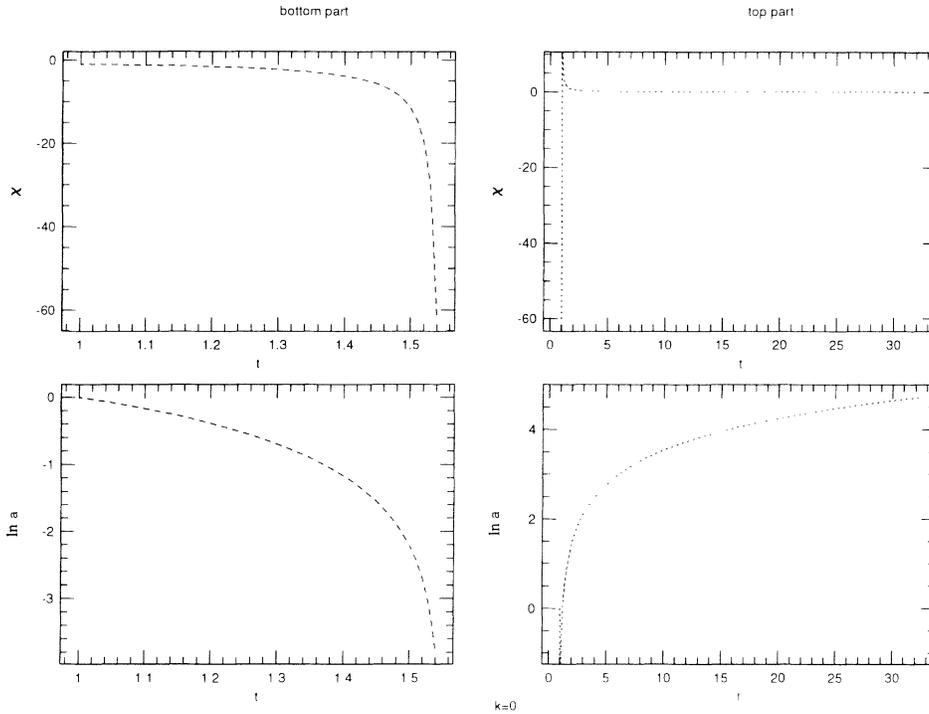


FIG. 2. Two typical solutions of $\ln a$ and χ as function of t .

We examine now the phase-plane properties derived from Eq. (25). First, we construct $dP/d\chi$ and find that

$$\frac{dP}{d\chi} = -3 + \frac{\chi P}{5\chi P + 4\chi^2 + 4k + P^2} . \tag{27}$$

The analytic solutions which bound the regions of the physical solutions are given by the requirement that $f^2=0$. From Eq. (26) it follows that

$$6\chi P + 6\chi^2 + P^2 + 6k = 0 . \tag{28}$$

Therefore, in the asymptotic regime where $\chi \gg k$, the solutions which bound the physical solutions are given as before by Eq. (22) for the new P and χ . However, the exact solution to Eq. (28) is

$$P = -3\chi \pm \sqrt{3\chi^2 - 6k} . \tag{29}$$

Those, there is radically different behavior in the case $k=1$ compared to $k=-1$ when one gets close to the origin of the χ - P plane. We denote by (ASiii) and (ASiv) those lines that are asymptotic to solution (iii) and (iv), and turn now to examine the properties of the trajectories in a closed universe.

For $k=1$, physically valid solutions penetrate into the region that was forbidden in the case that $k=0$. As in the flat case, there are two types of trajectories depending on which part of the phase plane shown in Fig. 3 they occupy. All solutions are asymptotic to the lines given by Eq. (28). However, all the trajectories end on the ex-

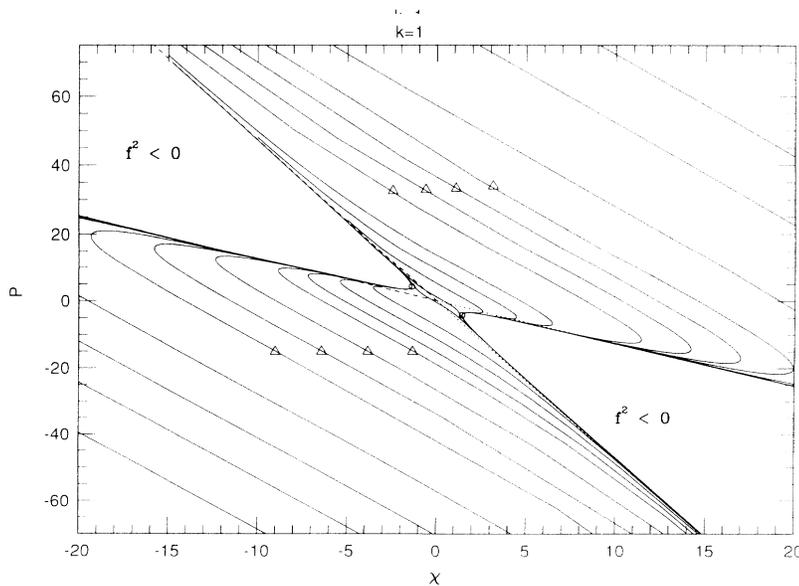


FIG. 3. The same as Fig. 1 for $k=1$ and $c=0$.

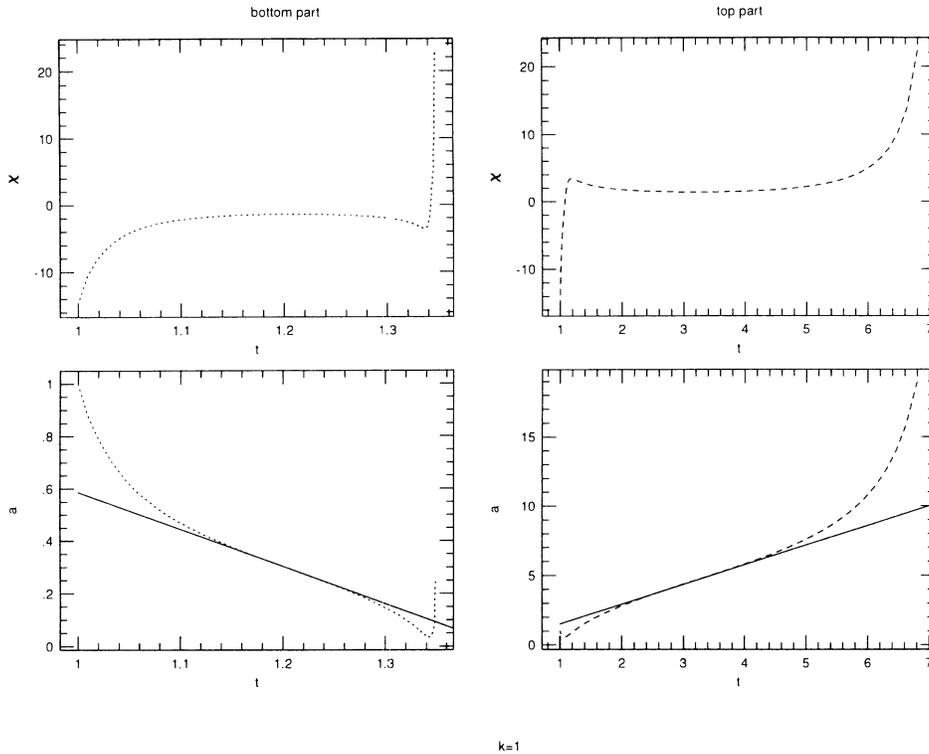


FIG. 4. Two typical solutions of a and χ as function of t . The straight line corresponds to the points marked by the circles in Fig. 3.

panding part of (ASiv). These on the upper part of the (χ, P) plane are repelled from the upper part of (ASiv), and are attracted to the collapsing part of (ASiii). The collapse slows down as the trajectories approach solution (ASiii). The trajectories route depend on whether they start below or above solution (iv). Trajectories that start from below go through the left-hand part of the phase plane in Fig. 3 and continue to collapse with different rate. The ones that start from above go through the right-hand part of the phase plane. In this case the collapse turns into a slow expansion along the expansion part of solution (ASiii). In both cases the further they

started up on (ASiv) the further they will go along solution (ASiii). In both cases the collapse or the slow expansion turn into rapid expansion as the trajectories are repelled from (ASiii) towards the expanding part of solution (iv). In Fig. 4 we show two typical sets of evolution for a , the scale factor, and χ to demonstrate each of these histories. One can see that the time scale on which changes take place is completely different but eventually both reach the point of fast expansion. In both cases the scale factor becomes infinite in a finite time.

Now, we turn our attention to an open universe by setting $k = -1$. There is an elementary exact solution with

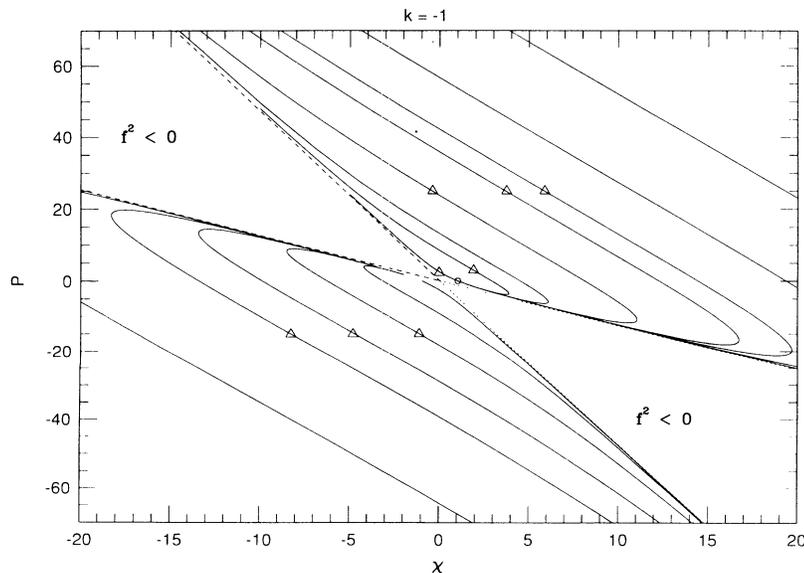


FIG. 5. The same as Fig. 1 for $k = -1$ and $c = 0$.

$P=0$ and $\chi=1$ (found already by Tseytlin) for which

$$a = t + t_0, \quad \phi = \phi_0. \tag{30}$$

This solution has the same form in the Einstein frame. All the trajectories on the right-hand side of Fig. 5 approach this solution marked by a circle on this figure. Tracing these trajectories to the past reveals that all the trajectories are repelled from the contracting part of the asymptotic solution (ASiv). Here since $f^2=0$ requires $6\chi P + 6\chi^2 + P^2 - 6 = 0$, all the trajectories start above the

exact solution (iv). Because of this, trajectories that occupy the left-hand side of Fig. 5 are repelled from below by the contracting part of (ASiv). These trajectories are attracted by the expanding part of (ASiii); i.e., these universes end up in rapid expansion. We see that in this case, unlike the case when $k=1$, there are two completely different types of behavior depending on the initial conditions. Figure 6 shows two of the line evolutions of typical example of these two different types of trajectory. In the first case, the Universe ends up growing linearly

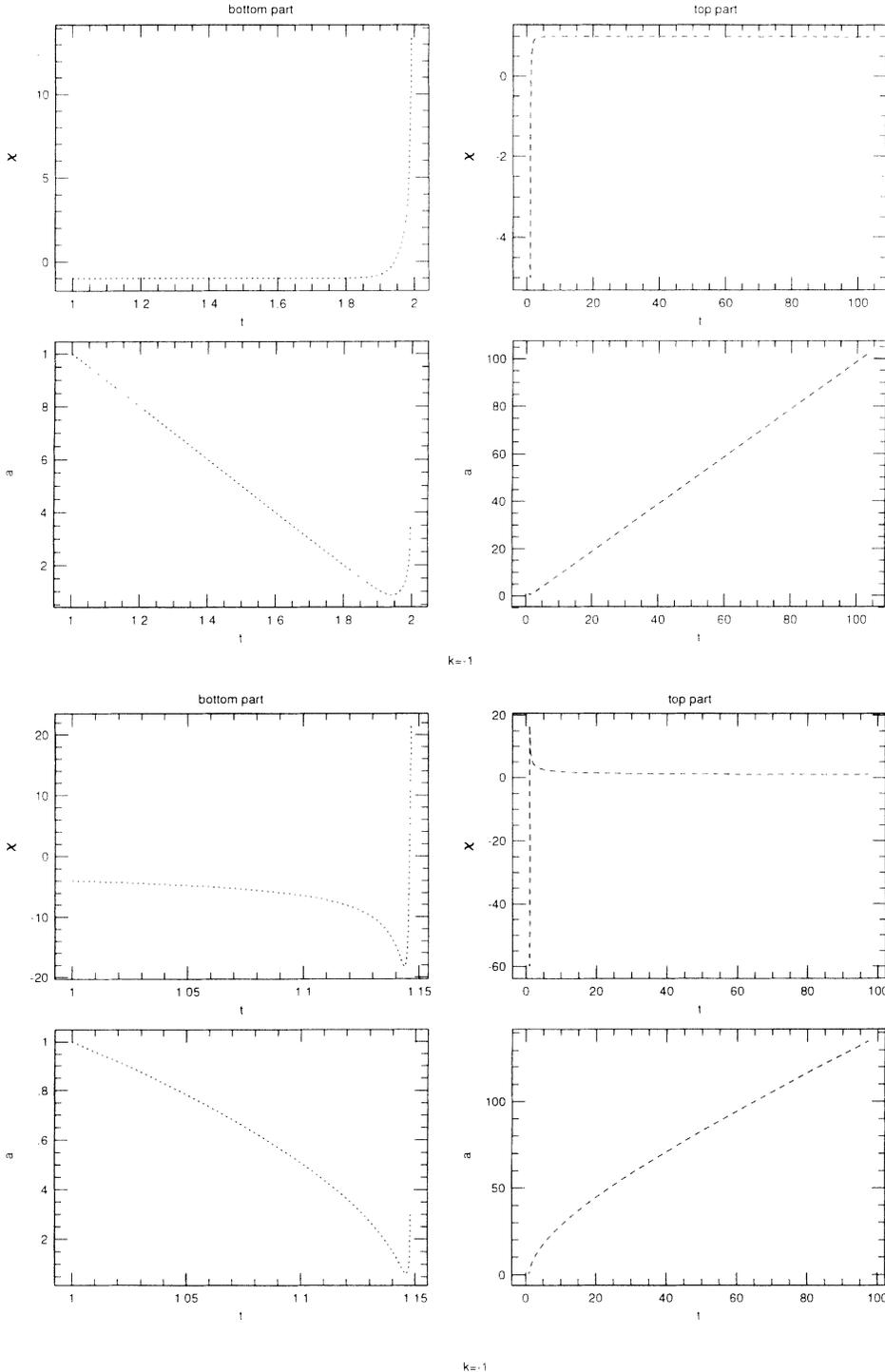


FIG. 6. Two typical solutions of a and χ as function of t (a) for trajectories that start near the origin, (b) for trajectories that start away from the origin.

while in the second case it starts from linear contraction and which then turns into rapid expansion.

We turn now briefly to the case of $c \neq 0$. The analogues of Eq. (17) are, for $k = 0$,

$$\begin{aligned} \dot{P} &= -12\chi^2 - 15P\chi - 3P^2 - 3c, \\ \dot{\chi} &= 3\chi^2 + 5P\chi + P^2 + c \end{aligned} \quad (31)$$

(now the overdot denotes d/dt as for the $k = 0$ case). As before the phase-plane properties can be explored from observing that

$$\frac{dP}{d\chi} = -3 + \frac{\chi^2}{3\chi^2 + 5P\chi + P^2 + c}. \quad (32)$$

The boundaries of the physical allowed regions are given by $f^2 = 0$, for which

$$6\chi^2 + 6P\chi + P^2 - c = 0. \quad (33)$$

The solutions to Eq. (33) are therefore

$$P = -3\chi \pm \sqrt{3\chi^2 + c}. \quad (34)$$

It seems therefore intuitively apparent that by comparing Eq. (34) with Eq. (29) that $c > 0$ should be qualitatively similar to $k < 0$ and the effect of $c < 0$ should be qualitatively similar to $k > 0$.

It is, in principle, possible to extend our method to deal with the cases $k \neq 0$, $c \neq 0$ simultaneously. We do not do this here because it is too complicated to be in good taste.

Our overall conclusions are that in some stringy era, the large-scale properties of the Universe are not without some interest. Let us describe the $c = 0$, $k = 0$ case first. The contracting part of the exact solutions, the dashed line in Fig. 1, repels all the trajectories. So the general evolution will be away from it, into the expanding phase. As one might have expected by time-reversal invariance, there are solutions which are the reverse of these. Crudely speaking, these occupy the left-hand half of Fig. 1 whereas the original ones occupy the right-hand half. These solutions start from the exact solution (iii) and are repelled towards the exact solution (iv) and so the Universe ends up in rapid expansion. If one starts in the upper part of the diagram, the evolution will be towards a universe that reaches constant size as $t \rightarrow \infty$, as given by the attractor shown by the dotted line, or analytically by Eq. (22). If, however, one starts in the lower part of the diagram, the expansion will always lead to hyperinflation. In our mathematical idealization, the radius of the Universe tends toward infinity in a finite proper time. However, it could be the case that in a realistic theory there is an exit regime when the effective temperature of

the Universe reduces below some critical scale, presumably the GUT scale. For $k = 1$, $c = 0$ (or equivalently for $c < 0$ and $k = 0$) then for large value of χ and P , the behavior is qualitatively similar. However, for small χ and P , there are significant differences. If one starts in a contracted era in the upper part of Fig. 3, provided one does not start too close to the repeller, an era of contraction will be followed by expansion, which can be very slow for a significant time. This is called a loitering era [8]. Eventually, the expansion will accelerate, again leading to hyperinflation unless some exit from the string-dominated era can occur. If one starts very close to the repeller, then there is again a loitering phase which occurs during the contraction. Eventually, however, expansion and then hyperinflation will set in, again subject to the caveats that we are still in the stringy era. For $k = -1$, $c = 0$ (or equivalently $c > 0$ and $k = 0$), if one starts in the upper part of the picture in a contracting phase, then this always turns into expansion, and at late times $a(t) \sim t$. This corresponds to the fixed point in the (χ, P) plane marked by an open circle in Fig. 5. Similarly, one can start at a point in the lower part of Fig. 5, and again one discovers that there is an attractive hyperinflating solution.

We see therefore that during a stringy era, one can get a large variety of different behavior depending on the initial conditions when the stringy era sets in. Furthermore, depending on when the stringy era ends, one can find a number of clues to how GUT era physics could have started off.

There is, in addition to the cosmological interest, the potential in this work to resolve a string theory puzzle. If string theory really does describe the physics of gravitation, it seems that there should be a massless scalar component to the gravitational field in the form of the dilaton. However, as has been emphasized by Damour and Nordvedt [9], in many theories that contain a dilaton, one is forced by the cosmological expansion to the general relativistic limit. However, their general results do not apply to the dilaton that emerges from string theory. Our explicit results, however, show that ϕ is often driven to infinity, which corresponds to weak coupling of the dilaton. The ultimate physical significance of this observation, however, remains obscure.

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