$O(\alpha_s)$ calculation of the decays $b \to s+\gamma$ and $b \to s+g$

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We present an exact α_s calculation of the Wilson coefficients associated with the dipole moment operators. We also give an estimate of the branching ratio for $b \to s\gamma$. We find that higher dimensional effects are under control within 9% for $B(b \to s\gamma) = (4.3 \pm 0.37) \times 10^{-4}$.

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In spite of great sucesses of the standard model (SM) there are theoretical issues which demand further explanations, one of which pertains to the internal cancellation of various flavor-changing neutral current amplitudes, an understanding of which presumably has bearing on the origin of masses and *CP* violation. Thus, various theoretical constructs have been advanced and must be ipso facto constrained by these processes. It is also clear that they are very sensitive to the vagaries of masses present, enhanced by QCD. Impetus is further provided by the fact that rare processes such as $b \to s+\gamma$, $s+g$ are not just experimentally bounded but observed or on the verge of being observed. Thus, it has become an earnest enterprise to ascertain accurately the theoretical estimates of their rates, particularly from SM even as "backgrounds" for future extensions. While the methodology needs to be quite technical, the outcomes have wide implications.

In an article [1] by us recently, we presented a SM leading logarithmic analysis of the heavy particle effects on the process $b \to s\gamma$, which incorporates a complete operator mixing. Among the results, we find that our mixing matrix differs in some elements from the work by Misiak [2]. Also, without inclusion of their mixing with evanescent operators, [3] which lead to extra contributions (in unit of $g_s^2/8\pi^4$)

$$\Delta\gamma_{O_{67}O_{51}} = 1/3, \quad \Delta\gamma_{O_{67}O_{52}} = -8/3\ , \qquad \Delta\gamma_{O_{68}O_{51}} = -1/2, \quad \Delta\gamma_{O_{68}O_{52}} = -4\ , \tag{1}$$

the results by Ciuchini *et al.* [4] coincide with ours. (We thank Misiak for correspondence on this.) Whether one should or should not introduce these evanescent operators is far from being resolved. In our opinion, one does not need them to derive the effective theory in our approach. In this article, the results of Ref. [1] in the leading logarithmic approximation (LLA) will be used when needed. It may be remarked that the difference is not numerically significant for the processes under consideration. For earlier work, see [5,6].

More importantly, by choosing two different limits for extrapolation, $m_t = m_W$ and $m_t \gg m_W$, we estimated the effects due to higher orders in m_W^2/m_t^2 to be about 20%. In view of recent interest in the experimental branching ratio [7] for $B_s \to K^* + \gamma$ and an impending value for the inclusive rate $B_s \to X_s + \gamma$, together with the sensitivity of these processes as a short distance probe, an uncertainty in short distance analysis of 20% is hardly satisfactory or even acceptable. The purpose of this publication is to provide some remedy.

One can trace the uncertainty to the fact that $m_W^2/m_t^2 \approx 25\%$ for $m_t \approx 150$ GeV. It strongly suggests that one should calculate the order α_s diagrams exactly, wherever there are internal top and/or W-boson propagators. This is what we have done.

In all two-loop diagrams (and their attendant counterterms) which contribute to this calculation, we keep all orders in $x \equiv m_t^2/m_W^2$ but discard terms proportional to $O(m_b^2/m_{t,W}^2)$ or $O(m_s^2/m_{t,W}^2)$ (we have factored out the Fermi weak-coupling constant). The amount of algebra is highly nontrivial and is aided by SCHOONSCHIP [8].

Given a diagram in which there are top and/or $W-\phi$ internal lines, there are various sequences of operations one can follow to isolate its dependence on the heavy masses [9]. In any case, the underlying method is based on partitioning of the diagrams into heavy parts and operator inserted matrix elements. For the present situation, we treat the top and/or the $W-\phi$ as being correspondingly heavy, relative to other masses and external momenta. A heavy part always contains at least either a top or a $W-\phi$ or both internal lines. Vertices made of light particles and momenta acting on them and Wilson coefficients which contain all dependence of heavy masses are organized within this formalism. In this way, we obtain the effective Lagrangian

$$L_{\text{eff}} = \sum_i C_i O_i\ , \tag{2}$$

where C_i are the Wilson coefficients and O_i are sets of local operators, made of light fields.

Of particular interest in the present order $\alpha_w \alpha_s$ calculation are the coefficients $C_{O_{51}}$ and $C_{O_{52}}$, with the accompanying operators

$$O_{51} = i g_s \bar{s} G_{\mu\nu} \sigma_{\mu\nu} (m_s P_L + m_b P_R) b/2\ , \tag{3}$$

and $O_{52} = O_{51}(g_s G_{\mu\nu} \to -\frac{1}{3} e F_{\mu\nu})$.

Before we go on further, let us define

$\bar{C}_{O_{5i}} \equiv 16\pi^2 C_{O_{5i}}/G_t (i=1,2)$, and $G_t = 2\sqrt{2} G_F V_{ts}^* V_{tb}$, where G_F is the Fermi weak-coupling constant and V's are the Cabibbo-Kobayashi-Maskawa matrix elements. We expand $\bar{C}_{O_{5i}}$ in power series in α_s:

$$\bar{C}_{O_{5i}} = \sum_i^{\infty} \bar{C}_{O_{5i}}^{(n)} . \tag{4}$$

The α_s^0 results will be given in Eq. (7). Here we first report the exact α_s results (in the R^* scheme to be defined)

$$\begin{aligned}
\frac{\pi}{4\alpha_s}\bar{C}_{O_{51}}^{(1)} = & -\frac{4}{(x-1)^4} - \frac{5293}{576}\frac{1}{(x-1)^3} - \frac{21\,989}{3456}\frac{1}{(x-1)^2} - \frac{1817}{1728}\frac{1}{(x-1)} + \frac{247}{10\,368} \\
& + \ln\left[\frac{m_W^2}{\mu^2}\right]\left[-\frac{7}{16}\frac{1}{(x-1)^3} - \frac{21}{32}\frac{1}{(x-1)^2} - \frac{7}{48}\frac{1}{x-1} - \frac{35}{288}\right] \\
& + \ln\left[\frac{m_W^2}{\mu^2}\right]\ln x\left[\frac{7}{16}\frac{1}{(x-1)^4} + \frac{7}{8}\frac{1}{(x-1)^3} + \frac{7}{16}\frac{1}{(x-1)^2}\right] \\
& + \ln x\left[\frac{4}{(x-1)^5} + \frac{5509}{576}\frac{1}{(x-1)^4} + \frac{2893}{432}\frac{1}{(x-1)^3} + \frac{163}{192}\frac{1}{(x-1)^2} - \frac{11}{48}\frac{1}{x-1} - \frac{35}{288}\right] \\
& + \ln^2 x\left[+\frac{7}{16}\frac{1}{(x-1)^4} + \frac{7}{8}\frac{1}{(x-1)^3} + \frac{7}{16}\frac{1}{(x-1)^2}\right] \\
& + \mathrm{Sp}\left[1-\frac{1}{x}\right]\left[\frac{13}{8}\frac{1}{(x-1)^4} + \frac{187}{48}\frac{1}{(x-1)^3} + \frac{137}{48}\frac{1}{(x-1)^2} + \frac{1}{2}\frac{1}{x-1} - \frac{1}{12}\right] + \left[\frac{91}{2592} + \frac{1}{54}\ln\left[\frac{m_W^2}{\mu^2}\right]\right],
\end{aligned} \tag{5}$$

$$\begin{aligned}
\frac{\pi}{4\alpha_s}\bar{C}_{O_{52}}^{(1)} = & -\frac{4}{(x-1)^4} - \frac{335}{18}\frac{1}{(x-1)^3} - \frac{11\,959}{432}\frac{1}{(x-1)^2} - \frac{6347}{432}\frac{1}{(x-1)} - \frac{47}{648} \\
& + \ln\left[\frac{m_W^2}{\mu^2}\right]\left[-\frac{1}{(x-1)^3} - \frac{3}{(x-1)^2} - \frac{31}{12}\frac{1}{x-1} + \frac{17}{36}\right] \\
& + \ln\left[\frac{m_W^2}{\mu^2}\right]\ln x\left[\frac{1}{(x-1)^4} + \frac{7}{2}\frac{1}{(x-1)^3} + \frac{4}{(x-1)^2} + \frac{3}{2}\frac{1}{x-1}\right] \\
& + \ln x\left[\frac{4}{(x-1)^5} + \frac{643}{36}\frac{1}{(x-1)^4} + \frac{1337}{54}\frac{1}{(x-1)^3} + \frac{257}{24}\frac{1}{(x-1)^2} - \frac{19}{24}\frac{1}{x-1} + \frac{17}{36}\right] \\
& + \ln^2 x\left[\frac{1}{(x-1)^4} + \frac{7}{2}\frac{1}{(x-1)^3} + \frac{4}{(x-1)^2} + \frac{3}{2}\frac{1}{x-1}\right] \\
& + \mathrm{Sp}\left[1-\frac{1}{x}\right]\left[\frac{11}{4}\frac{1}{(x-1)^4} + \frac{139}{12}\frac{1}{(x-1)^3} + \frac{191}{12}\frac{1}{(x-1)^2} + \frac{31}{4}\frac{1}{x-1} + \frac{2}{3}\right] + \left[-\frac{713}{648} + \frac{1}{54}\ln\left[\frac{m_W^2}{\mu^2}\right]\right],
\end{aligned}$$

where the last terms in large square brackets in each of the two equations above are the contributions due to u and c quarks. Sp stands for Spence function. Equations (5) are to be contrasted with the asymptotic results ($m_t \gg m_W$)

$$\bar{C}_{O_{51}}^{(1)}(\mathrm{asym})\pi/4\alpha = \frac{611}{10\,368} - \frac{\pi^2}{72} - \frac{35}{288}\ln\left[\frac{m_t^2}{\mu^2}\right] + (1/54)\ln(m_W^2/\mu^2) ,$$

and

$$\bar{C}_{O_{52}}^{(1)}(\mathrm{asym})\pi/4\alpha = -\frac{95}{81} + \frac{\pi^2}{9} + \frac{17}{36}\ln\left[\frac{m_t^2}{\mu^2}\right] + (1/54)\ln(m_W^2/\mu^2) , \tag{6}$$

obtained by discarding all $O(m_W^2/m_t^2)$. One can use Eqs. (5) to check some of the mixing matrix elements of the renormalization group equations (RGE's). Unfortunately, $\gamma_{O_{67}O_{51}}$, $\gamma_{O_{67}O_{52}}$, $\gamma_{O_{68}O_{51}}$, and $\gamma_{O_{68}O_{52}}$ do not enter to this order. Their determination has to come from three loop diagrams. In our opinion, direct Feynman diagram computation of Green's functions for processes to extract mixing matrix elements should be the definitive procedure.

We plot exact α_s result Eqs. (5) in Fig. 1, together with the asymptotic result Eq. (6) for $\bar{C}_{O_{52}}^{(1)}\pi/4\alpha_s$. For $x=4$, the discrepancy is in fact about 50%. It is interesting to note that the exact $C_{O_{52}}^{(1)}$ is quite flat between $x=1$ to $x=10$. The dotted line is the α_s^0 exact result, also known as the Inami-Lim functions [10]:

$$\begin{aligned}
\bar{C}_{O_{51}}^{(0)} &= \frac{-2x-3x^2+6x^3-x^4-6x^2\ln x}{4(1-x)^4} , \\
\bar{C}_{O_{52}}^{(0)} &= \frac{7x-12x^2-3x^3+8x^4+(12x^2-18x^3)\ln x}{4(1-x)^4} .
\end{aligned} \tag{7}$$

We see that, below $x=6.9$, the second order QCD correction is bigger than the lowest-order result. This has been known for the approximate results for some time and in fact is an impetus for looking into rare decays of this genre.

Note that if one uses Eqs. (7) as (a part of) the boundary conditions, then (5) are the α_s solution of RGE for all values of x, insofar as the overall coefficients to $\ln\mu^2$ are concerned. We would like to stress that to renormalize the heavy graphs in the full theory, SM with all the quarks, we are using the R^* scheme. For the effective theory, we use the modified minimal subtraction ($\overline{\rm MS}$) scheme to renormalize the local operators. We also repeat that we do not introduce any evanescent operators in this analysis. If instead the $\overline{\rm MS}$ scheme is used throughout, the exact result to order α_s is obtained if one makes the replacement $m_t \to m_t[1-\alpha_2/\pi \ln(m_t^2/\mu^2)]$ in (7); this replacement is just a finite renormalization to go from the R^* scheme to the $\overline{\rm MS}$ scheme. Note that $\overline{C}^{(0)+(1)}_{O_{5i}}(\mu = m_{\rm heavy})$ will give the initial conditions if one is to perform a next to leading logarithm sum. For this, one has to compute the anomalous dimension matrix to order α_s^2, which requires another order of technical development.

We shall make the assumption that after α_s corrections, it is safe to add the leading logarithmic terms to complete the leading QCD sum. In other words, we assume that the higher-order QCD corrections can be approximated by LLA either in the limit $m_W^2/m_t^2 \ll 1$ or in the limit $m_W^2/m_t^2 = 1$. This assumption can be tested as in our previous publication. For $\overline{C}_{O_{52}}$, there entail two different extrapolations:

$$\overline{C}^{(0)+(1)}_{O_{52}}(\text{exact}) + \overline{C}^{\text{higher order}}_{O_{52}}(m_t = m_W) , \tag{8a}$$

$$\overline{C}^{(0)+(1)}_{O_{52}}(\text{exact}) + \overline{C}^{\text{higher order}}_{O_{52}}(m_t \gg m_W) , \tag{8b}$$

where $\overline{C}^{\text{higher order}}_{O_{52}}(m_t = m_W)$ and $\overline{C}^{\text{higher order}}_{O_{52}}(m_t \gg m_W)$ are the remaining LLA sums with the boundary conditions set at $m_t = m_W$ and $m_t \gg m_W$, respectively. One could, in principle, use Eq. (7) for the initial conditions and perform a LLA sum. The analytical result in this approximation would be valid if one matches Eq. (7) to the C_i at the scale $\mu = m_W$, or at the scale $\mu = m_t$, or at any other scale $\mu = O(m_W)$. However, the numerical results are very sensitive to the choice of the scale in the boundary conditions, and one loses control over the uncertainties due to a particular choice of the scale. These extrapolations we propose are used to estimate these errors: we first perform two LLA sums in the limits $m_W = m_t$ and $m_t \gg m_W$, for which boundary conditions are unambiguous. We then assume that for the physical case $m_t \sim 140$ GeV the corresponding numerical values for the C_i are somewhere in between the values of the C_i obtained in the two limits. Put differently, Eqs. (8a)–(8b) are intended as follows: We want to improve the calculation of the $C_i^{(n)}$ in Eq. (4) by taking into account some of the higher m_W/m_t effects, which are not given by LLA. A next to LLA sum to all orders in α_s is extremely difficult and practically a long wait. We will nevertheless assume that including all the m_W/m_t effects to order α_s gives a good approximation, together with the $C_i^{(n)}$ for $n>1$, calculated in LLA.

For the physical process $b \to s + \gamma$, some four quark operators also contribute, resulting in an effective coupling [21]

$$C^{\rm eff}_{O_{52}} = C_{O_{52}} + (1/8\pi^2) C_{O_{67}} + (3/8\pi^2) C_{O_{68}} . \tag{9}$$

Also, to remove the dependence on $|G_t|^2$, which is not accurately known experimentally, we normalize the $b \to s\gamma$ partial width to the well established semileptonic $b \to ce\overline{\nu}$ partial width, and use the relation [11] $|V_{ts}^* V_{tb}| \simeq |Vcb|$.

This ratio is given as

$$\frac{\Gamma(b\to s\gamma)}{\Gamma(b\to ce\overline{\nu})} \simeq \frac{\alpha_{\rm QED}}{6\pi g(m_c/m_b)} \times \left[1 - \frac{2\alpha_s(m_b)}{3\pi} f(m_c/m_b)\right]^{-1} \times |\overline{C}^{\rm eff}_{O_{52}}(m_b)|^2 , \tag{10}$$

where $g(m_c/m_b) \simeq 0.45$ and $f(m_c/m_b) \simeq 2.4$ correspond to the phase space factor and the one-loop QCD corrections to the semileptonic decay, respectively.

In Fig. 2, we have plotted $B(b \to s\gamma)$ as a function of

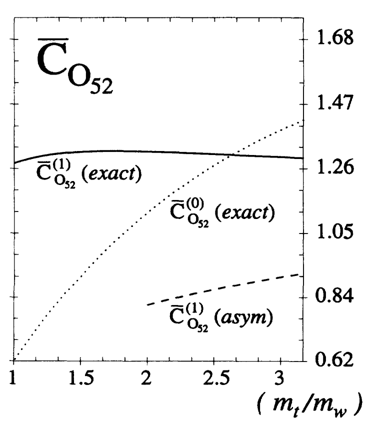

FIG. 1. $\overline{C}_{O_{52}}(m_b)$ dependence on m_t to order α_s with $m_b = 4.8$ GeV, $m_W = 81$ GeV, $\alpha_s(m_b) = 0.19$. See text for explanation of various curves.

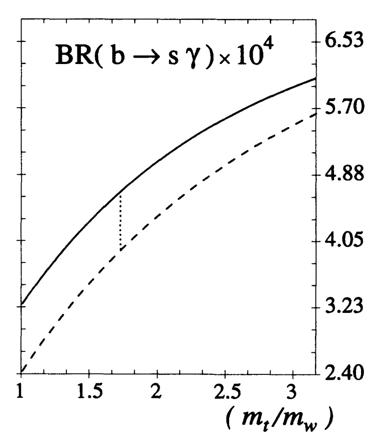

FIG. 2. Branching ratio for $b \to s\gamma$ as a function of m_t with QCD corrections. The solid line represents the interpolation given by Eq. (8a); the dashed line represents the interpolation given by Eq. (8b); the dotted line represents the values of the branching ratio for $m_t = 140$ GeV. We used $m_b = 4.8$ GeV, $m_W = 81$ GeV, $\alpha_s(m_b) = 0.19$.

m_t. The solid and dashed curves are obtained with the aid of the interpolation equations in Eq. (8), together with

$$B(b\to s\gamma)=[\Gamma(b\to s\gamma)/\Gamma(b\to ce\bar{\nu})]B(b\to ce\bar{\nu}) , \quad (11)$$

$$B(b\to ce\bar{\nu})\simeq 0.108 . \quad (12)$$

The vertical dotted line to guide the eyes intersects these curves at $m_t=140$ GeV and gives, respectively,

$$\begin{aligned} &B(b\to s\gamma)=4.66\times 10^{-4} ,\\ &B(b\to s\gamma)=3.93\times 10^{-4} . \end{aligned} \quad (13)$$

From Eq. (13), we obtain the mean value (this is our estimate)

$$B(b\to s\gamma)=(4.3\pm 0.37)\times 10^{-4} . \quad (14)$$

This is to be compared with an upper limit 5.4×10^{-4} given recently by the CLEO Collaboration [7].

The uncertainty due to subleading logarithmic and higher dimensional effects is about 9%, which is a big improvement and more reliable over what we gave before, where m_t^2/m_W^2 effects at $\alpha_s^{(1)}$ were not treated. This 9% is only an educated guess of the uncertainties due to short distance estimates of $|\bar{C}_i|^2$. Those due to long distance physics have been ignored. Also, Eq. (10) as a good approximation can be but is not questioned.

We now give some technical details. We use the general linear covariant gauges $(-1/2\alpha)(\partial_\mu G_\mu)^2$ for the gluons. The complete cancellation of α for $C_{O_{51}}$ and $C_{O_{52}}$ is a stringent confirmation on the correctness of the algebra. The gauge fixing for W fields is $-C^+C^-$ with $C^+=-\partial_\mu W_\mu^+ + m_W\phi^+ + ieA_\mu W_\mu^+$. For oversubtractons and renormalization, we use the R^* scheme. Thus, let Γ be a one light particle irreducible (1LPI) diagram, which contains the heavy particles, and let γ represent a 1LPI graph or subgraph ($\gamma\subseteq\Gamma$) with external generic momentum p. We define

$$\tau_\gamma^\epsilon=\text{pole part of } \gamma , \quad \epsilon=n-4 ,$$

$$\tau_\gamma^{(m)}=\gamma(p=0)+p\frac{\partial}{\partial p}\gamma(p=0) + \cdots + \frac{1}{m!}p^m\frac{\partial^m}{\partial p^m}\gamma(p=0) .$$

The R^* renormalization procedure is defined as

$$R^*(\gamma_{\text{heavy}})=(1-\tau_\gamma^{(m)})\gamma_{\text{heavy}}, \quad R^*(\gamma_{\text{light}})=(1-\tau_\gamma^\epsilon)\gamma_{\text{light}} ,$$

where m is so chosen that the neglected terms are genuinely of $O(1/m_{\text{heavy}}^2)$. It is important to repeat that the results of Eqs. (5) and (6) are given in the renormalized top mass under the R^* scheme, where $\beta_{m_t}=0$.

Except for trivial factorizable cases, all two loop integrals we need are related to [12]

$$\begin{aligned} I_{2,1,1,}(m_1^2,m_2^2,m_3^2) &= \int d^n k d^1 q \frac{1}{(k^2+m_1^2)^2[(k+q)^2+m_2^2](q^2+m_3^2)} \\ &= \pi^4\left[\frac{-2}{(n-4)^2}+\frac{1}{n-4}[1-2\gamma_E-2\ln(\pi m_1^2)]-\frac{1}{2}-\frac{1}{12}\pi^2\right] \\ &\quad +\pi^4[\gamma_E-\gamma_E^2+(1-2\gamma_E)\ln(\pi m_1^2)-\ln^2(\pi m_1^2)+f(a=m_2^2/m_1^2,\ b=m_3^2/m_1^2)] , \end{aligned}$$

where

$$f(a,b)=\frac{1}{2}\ln a\ \ln b+\frac{a+b-1}{\surd}\left[\text{Sp}\left[\frac{-y_2}{x_1}\right]+\text{Sp}\left[\frac{-x_2}{y_1}\right]+\frac{1}{4}\ln^2\frac{x_2}{y_1}+\frac{1}{4}\ln^2\frac{y_2}{x_1}+\frac{1}{4}\ln^2\frac{x_1}{y_1}-\frac{1}{4}\ln^2\frac{x_2}{y_2}+\frac{\pi^2}{6}\right] ,$$

$x_{1,2}=\frac{1}{2}(1-a+b\pm\surd)$, $y_{1,2}=\frac{1}{2}(1+a-b\pm\surd)$, and $\surd=\sqrt{(1-a+b)^2-4b}$.

Assuming $a,b\gg 1$ (i.e., we take $m_2=m_t$, $m_3=m_W\gg m_1=m_{b,s}$), we expand $f(a,b)$ in series of $m_{b,s}^2$, which must be retained to proper orders in intermediate steps. Details of this work are to be published [1].

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