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Nearly democratic mass matrices and flavor mixing

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We explore the phenomenologically acceptable parameter space for a class of mass matrices in both up and down sectors that are close to the purely democratic form (all matrix elements equal).

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I. INTRODUCTION

The idea that at some fundamental level the quark mass matrices have elements each exactly equal to one another, equivalent to an input with one and only one nonzero mass, is by now a familiar one [1], which we refer to as pure democracy. The full hierarchy of masses across the three families is then generated from the single large input mass by some as yet unknown perturbative process [2]. The resulting nearly democratic mass matrices have elements whose magnitudes differ only slightly from one another. To the extent that the dynamics generating the full mass matrices follows naturally from a democratic picture, it is worthwhile to make a direct translation from the nearly democratic form to measurables.

The experimental values by which any mass matrices must be constrained are the up and down sector masses and the Cabibbo-Kobayashi-Maskawa matrix elements. Both these quantities reflect a hierarchy. We take the following [3] for masses at 1 GeV: in the up sector, $m_1=5.1$ MeV, $m_2=1360$ MeV, and, for illustration,[1] $m_3=3\times10^5$ MeV, in the down sector, $m_1=8.9$ MeV, $m_2=145$ MeV, and $m_3=5700$ MeV. Let us define $w_i\equiv m_i^2$. The normalization we choose will make the sum of the masses squared in both the down and up sectors equal to 3. Thus the w's are given by dividing out the sum of the masses squared in each sector and multiplying by 3. We can then set w_3 equal to $3-w_2-w_2$. With this normalization, we have in the up sector $w_1=8.643\times10^{-10}$, $w_2=6.162\times10^{-5}$; for the down sector, $w_1=7.29\times10^{-6}$, $w_2=1.944\times10^{-3}$. It is also useful to work with the mass ratios w_1/w_3 and w_2/w_3. These are given, respectively, by 2.88×10^{-10} and 2.05×10^{-5} in the up sector and 2.43×10^{-6} and 0.64×10^{-3} in the down sector. If $w_2/w_3=\epsilon_d^2$ and ϵ_u^2 in the down and up sectors, then $\epsilon_d=0.0255$ and $\epsilon_u=0.0045$. In each case, w_1/w_3 is $O(\epsilon^4)$ (or smaller). We shall henceforth think of w_2 as $O(\epsilon^2)$ and w_1 as $O(\epsilon^4)$. The Cabibbo-Kobayashi-Maskawa matrix [3,4] similarly reveals a hierarchy, most easily seen in the Wolfenstein parametrization [5]:

$$V=\begin{bmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3[\rho-i\eta(1-\frac{1}{2}\lambda^2)] \\ -\lambda & 1-\frac{1}{2}\lambda^2-i\eta A^2\lambda^2 & A\lambda^2(1+i\eta\lambda^2) \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{bmatrix}, \tag{1.1}$$

where λ has the value 0.22.

This paper has several aims. We want to show in what way the hierarchy inherent in masses and in the Cabibbo-Kobayashi-Maskawa matrix reveals itself in a class of nearly democratic mass matrices. (That class is motivated by the possibility that the mass matrix is formed of elements that differ from one another only by phases [6]; we address this particular parametrization elsewhere.) We want to describe the parameter space of this class that is consistent with the current values of the masses and Cabibbo-Kobayashi-Maskawa matrix elements and prepare for improved data on the Cabibbo-Kobayashi-Maskawa matrix. In passing, we shall see an example in which mass matrices with texture are consistent with the class of nearly democratic mass matrices we study and with experimental data.

II. PARAMETRIZATION

Because the nearly democratic mass matrix is not necessarily Hermitian, it is simpler for our purposes to work not with M, but with the Hermitian matrix

[1]For a QCD scale of 150 MeV, a top quark mass of 3×10^5 MeV at 1 GeV corresponds to a physical mass of some 140 GeV.

$H \equiv MM^\dagger$. For the class of nearly democratic mass matrices we consider, H takes the form

$$H=\begin{bmatrix} 1 & (1-a)e^{ik} & (1-b)e^{il} \\ (1-a)e^{-ik} & 1 & (1-c)e^{in} \\ (1-b)e^{-il} & (1-c)e^{-in} & 1 \end{bmatrix}, \quad (2.1)$$

where a, b, c, k, l, and n are real. It is reasonable to expect the whatever dynamics generates the flavor mixing that results in the observed masses acts on the diagonal matrix elements only in higher order; this is what motivates our taking the diagonal elements equal.

We can reduce the problem with the unitary transformation Y, given by

$$Y=\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{ik} & 0 \\ 0 & 0 & e^{il} \end{bmatrix}. \quad (2.2)$$

This transformation leaves only a single phase in the mass matrix squared:

$$H' \equiv YHY^{-1}=\begin{bmatrix} 1 & (1-a) & (1-b) \\ (1-a) & 1 & (1-c)e^{i\Omega} \\ (1-b) & (1-c)e^{-i\Omega} & 1 \end{bmatrix}, \quad (2.3)$$

$$\Omega \equiv k-1+n . \quad (2.4)$$

The Cabibbo-Kabayashi-Maskawa matrix V is formed from the unitary transformations U_u and U_d that diagonalize the mass matrices in the up and down sectors, respectively. In particular, if $UH'U^{-1}$ is the diagonalized mass squared matrix, then $V=U_uU_d^{-1}$. The rows of U are the (normalized) eigenvectors of H.

Let us denote the jth eigenvector of $H'|_d$ by $\{v_{j1},v_{j2},v_{j3}\}$, and the complex conjugate of the jth eigenvector of $H'|_u$ by $\{u_{j1},u_{j2},u_{j3}\}$. Then V is

$$V=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}\begin{bmatrix} 1 & 0 & o \\ 0 & e^{-iK} & 0 \\ 0 & 0 & e^{-iL} \end{bmatrix} \times \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}. \quad (2.5)$$

The central diagonal matrix, containing the two independent parameters K and L, is the result of the successive diagonal transformations $Y^uY^{d\dagger}$. Note that $K=k|_d-k|_u$ and $L=1|_d-1|_u$. While it is straightforward to find the eigenvectors of H' in analytic form to the necessary order, these forms are not particularly enlightening and we leave them unspecified.

III. INCORPORATING THE MASS HIERARCHY

We wish to incorporate the mass hierarchy into the parameters of V. We do this by eliminating two of the four variables a, b, c , and Ω that appear in H' in favor of w_1 and w_2, using invariants of H'; these conditions will also limit the range of the remaining variables. We use

$$w_1w_2+w_1w_3+w_2w_3 =3-(1-a)^2-(1-b)^2-(1-c)^2 , \quad (3.1)$$

$$w_1w_2w_3=1-(1-a)^2-(1-b)^2-(1-c)^2 +2(1-a)(1-b)(1-c)\cos\Omega . \quad (3.2)$$

We use the trace only to replace w_3 by $3-w_1+w_2$. (Note that these expressions are symmetric in a, b, and c, and these quantities are therefore interchangeable in any numerical fit.) Combining these two equations, we can replace the second of them by

$$2(1-a)(1-b)(1-c)\cos\Omega =w_1w_2w_3-w_1w_2-w_1w_3-w_2w_3+2 . \quad (3.3)$$

We use Eqs. (3.1) and (3.3) to eliminate two of the four variables a, b, c, and Ω. It is simplest to start by replacing a, b, and c with the order 1 parameters A, B, and C, defined by

$$a \equiv Aw_2, \quad b \equiv Bw_2, \quad c \equiv Cw_2 . \quad (3.4)$$

We then solve Eq. (3.1) for B in terms of A and C; which root is chosen depends on whether we want B to be larger than or smaller than C, and this choice has no fundamental significance; in the following, we assume B is smaller than C. When this value is substituted in Eq. (3.3), we find $\cos\Omega$ expressed as a function of A, C, w_1, and w_2; this equation can be used to eliminate one of the three variables A, C, or $\cos\Omega$. The range of A and C is limited by the requirement that $|\cos\Omega|<1$. By fixing $\cos\Omega$ in Eq. (3.3), having eliminated B with Eq.(3.1), we can trace out a contour line on a curve of C vs A. Figure 1 shows this curve for the down system. The minimum value of A_d corresponds to $a=w_1(A_d \simeq 0.003\,75)$ and the maximum to $a=w_2(A_d=1)$; the minimum and maximum values of

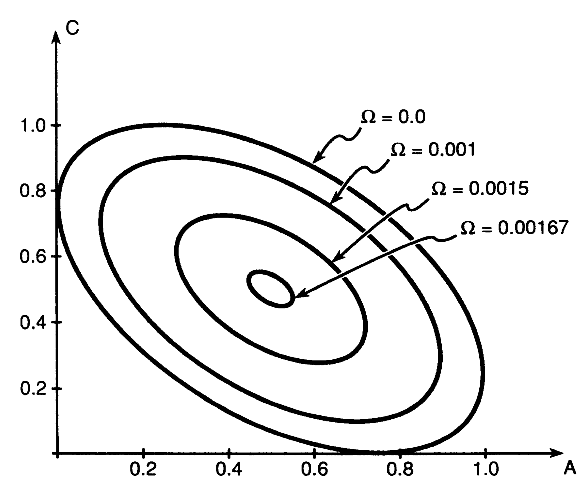

FIG. 1. Contours of allowed values of A_d and C_d for various values of Ω. The outermost contour corresponds to $\Omega=0$; outside that region, Ω is no longer a real angle. As Ω increases, the allowed region of A_d and C_d shrinks, until at a maximum value of $\Omega \simeq 0.001\,68$ there are no allowed values of A_d and C_d.

C_d are the same. The contour is such that values of $\cos\Omega$ outside the curve have magnitude larger than 1; thus, the only allowed values of A_d and C_d lie inside the curve. Moreover, this approach gives a minimum value of $\cos\Omega$, occurring roughly at $A_d = C_d = 0.5$; the corresponding (maximum) value of Ω at this point is $\Omega_{d,\max} = 1.68\times10^{-3}$. Figure 1 shows contours for several values of $\cos\Omega$. In the up sector, a and c similarly run between w_1 and w_2, and the maximum value of Ω is $\Omega_{u,\max} = 5.34\times10^{-5}$.

IV. ALLOWED PARAMETERS

We want to find those values of the remaining independent parameters that are consistent with what is currently known about V. For experimentally allowed ranges of the matrix elements (magnitudes only), we use Ref. [4]. These ranges assume a three-family structure and incorporate unitarity constraints on V. With the mass hierarchy incorporated, we have altogether six independent parameters: for example, A_d, A_u, C_d, C_u, K and L. [Given C and A in a given sector, B and Ω are each fixed in that sector through the invariant expressions (3.1) and (3.3).] We have made a systematic search of the parameter space; Fig. 2 illustrates a sample of our search technique. In this figure we have chosen the A and C values and vary K and L. The "holes" are regions of K and L for which the calculated value of V lies within the experimental values of that matrix. We summarize the results of our search for the allowed ranges for the variables as follows.

The minimum allowed value of A in either sector is w_1, and the maximum is w_2. Within this range, we can characterize the values consistent with the known values of V by saying that as A_d varies between its minimum and maximum values there are two possible solutions (branches) for A_u, one greater than and one less than A_d, plotted in Fig. 3. To leading order these values are asso-

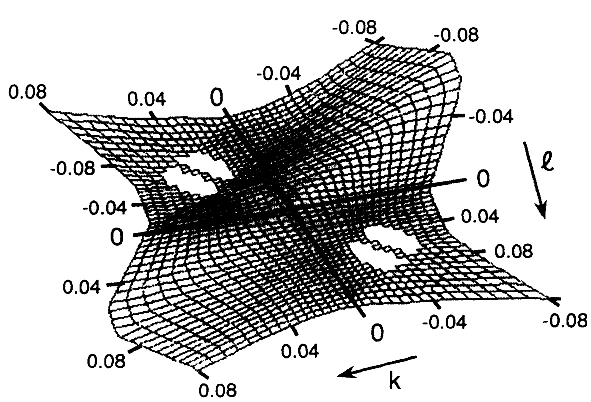

FIG. 2. Sample of our search technique for allowed values of K and L. We have chosen the parameters A_d, A_u, C_d, and C_u so that all parameters other than K and L are fixed. The gaps in the plot are regions in KL space where the values of the Cabibbo-Kobayashi-Maskawa matrix are consistent with experiment. Similar plots for other values of A_d, A_u, C_d, and C_u lead to a full picture of the allowed values of K and L.

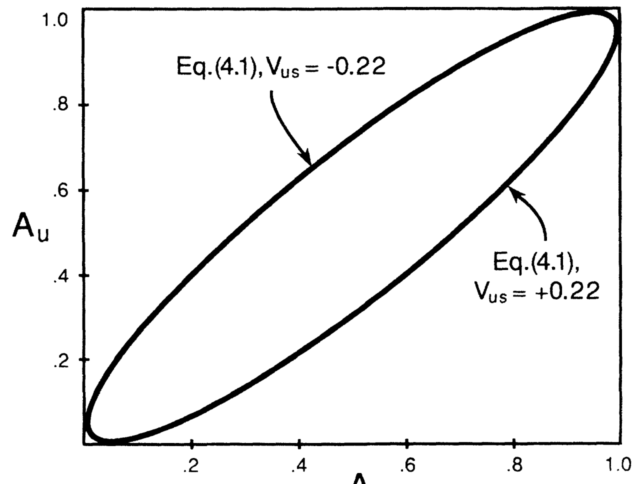

FIG. 3. Given A_d, A_u is essentially determined by the requirement that the us and ub elements of the Cabibbo-Kobayashi-Maskawa matrix have the magnitude 0.22, according to Eq. (4.1).

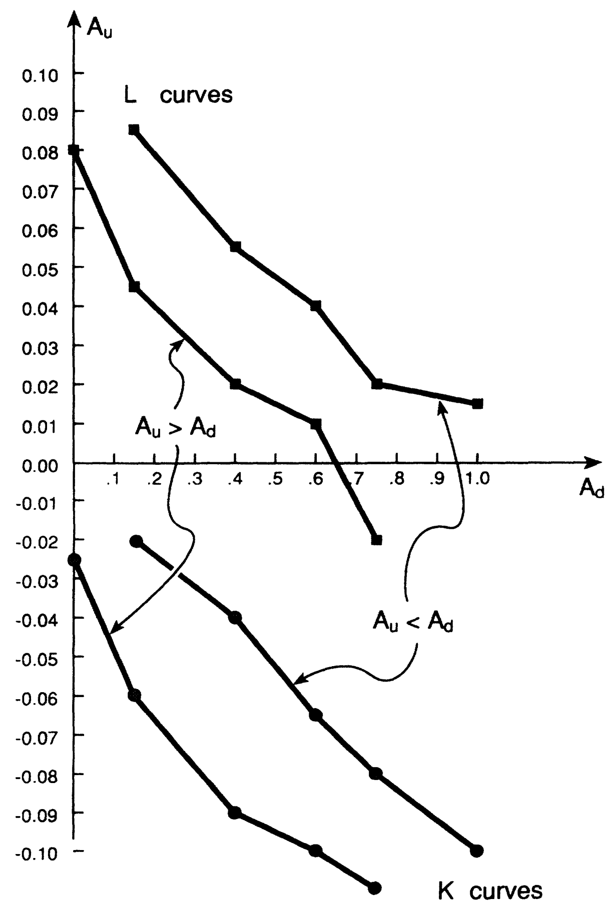

FIG. 4. Given A_d, the allowed values of K and L form an island whose center is at K_c and L_c. The curves show these values as a function of A_d. The A_u dependence is contained in the presence of curves for each of the two A_u branches. There is another set of KL values, not shown here, corresponding to the symmetry $K\to -K$, $L\to -L$.

ciated with the Cabibbo angle:

$$\pm V_{us} \simeq 0.22 \simeq \sqrt{A_d(1-A_u)} - \sqrt{A_u(1-A_d)} \ . \quad (4.1)$$

Given A_d and A_u, we characterize the values of C for which there are solutions as follows. We solve for B in terms of A, C, and Ω from Eq. (3.3); we use this value in Eq. (3.1), which we regard as a quadratic equation for C in terms of A and Ω, with two solutions. These solutions are extremums in terms of Ω when Ω is set equal to zero, with departures from the extreme values occurring as Ω increases from zero. The allowed values for C in each sector occur near the extreme values, meaning Ω for each sector is near zero. In other words, solutions occur only for Ω very close to zero, in fact, only for Ω as large as $O(\epsilon^2)$.

Once the other parameters have been chosen, we search for allowed values for the parameters K and L. Solutions have a symmetry; namely, for a given allowed pair (K,L), the pair $(-K,-L)$ is also allowed. This symmetry merely reflects the fact that the magnitudes of the matrix elements of V are even functions of (K,L). Thus we show solutions corresponding to only one of these pairs. The allowed (K,L) values form an island with a center (K_c, L_c) dependent on both A_d and A_u. Figure 4 is a plot of K_c and L_c as a function of the controlling parameter A_d. The dependence on A_u is contained in the presence of two curves for K_c and two curves for L_c, one for each of the A_u branches. For the currently known values of the Cabibbo-Kobayashi-Maskawa matrix, the size of the island is roughly 0.05×0.05 in KL space. Finally, as Fig. 2 indicates, the island is split into two regions by a narrow ridge.

V. HEAVY BASIS

Conversion of a nearly democratic mass matrix to a heavy basis, in which the mass hierarchy is directly reflected in a hierarchy of the diagonal elements of the mass matrix, is straightforward. The unitary transformation that makes the conversion between a heavy form and a "democratic" form, in which to leading order all matrix elements are equal, is

$$U_{dh} = \begin{bmatrix} \frac{\cos(\mu')}{\sqrt{2}} + \frac{\sin(\mu')}{\sqrt{6}} & \frac{\sin(\mu')}{\sqrt{6}} - \frac{\cos(\mu')}{\sqrt{2}} & \frac{-2\sin(\mu')}{\sqrt{6}} \\ \frac{\cos(\mu')}{\sqrt{6}} - \frac{\sin(\mu')}{\sqrt{2}} & \frac{\cos(\mu')}{\sqrt{6}} + \frac{\sin(\mu')}{\sqrt{2}} & -\frac{2\cos(\mu')}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} . \quad (5.1)$$

With this transformation,

$$H_{\text{heavy}} = U_{dh} H_{\text{dem}} U_{dh}^{-1} \ . \quad (5.2)$$

Here the presence of the angle μ' reflects the zeroth-order degeneracy in the two lowest masses. The resulting expression for H_{heavy} has the following six independent matrix elements:

$$H_{11} = \frac{w_2}{3}[(2A-B-C)\cos(2\mu') + \sqrt{3}(B-C)\sin(2\mu') + A+B+C] \ ,$$

$$H_{22} = \frac{w_2}{3}[-(2A-B-C)\cos(2\mu') - \sqrt{3}(B-C)\sin(2\mu') + A+B+C] \ ,$$

$$H_{33} = 9 - \tfrac{2}{3}w_2(A+B+C) \ ,$$

$$H_{12} = \frac{w_2}{3}[\sqrt{3}(B-C)\cos(2\mu') - (2A-B-C)\sin(2\mu')] \ ,$$

$$H_{13} = \frac{w_2}{6}[-\sqrt{6}(B-C)\cos(\mu') - \sqrt{2}(2A-B-C)\sin(\mu')] \ ,$$

$$H_{23} = \frac{w_2}{6}[\sqrt{6}(B-C)\sin(\mu') - \sqrt{2}(2A-B-C)\cos(\mu')] \ .$$

VI. EXAMPLE

We present here an example of nearly democratic mass matrices that both reproduces masses and the CKM matrix, and that, moreover, has texture in both the up and down sectors. In particular, the mass matrices for each sector contain zeros in a heavy basis. The numbers below are uniquely chosen, up to some discrete choices, out of the full parameter set described in the bulk of the paper to give these zeros.

We take Ω_d and Ω_u each equal to 0 and $A_d = 0.866$. The matrix element V_{us} fixes A_u, and within its experimental range we take $A_u = 0.684$. All B and C quantities are then determined, and

$$H^d_{\text{dem}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.00168 & 4.91\times10^{-5} \\ 0.00168 & 0 & 0.00120 \\ 4.91\times10^{-5} & 0.00120 & 0 \end{bmatrix} ,$$

$$H^u_{\text{dem}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 4.21\times10^{-5} & 3.35\times10^{-7} \\ 4.21\times10^{-5} & 0 & 5.00\times10^{-5} \\ 3.35\times10^{-7} & 5.00\times10^{-5} & 0 \end{bmatrix} .$$

Two K and L values that make the full V consistent with experiment are $K=0.09$, $L=-0.02$. With these parameters the matrix of absolute values of V is

$$\begin{bmatrix} 0.976 & 0.219 & 0.00468 \\ 0.219 & 0.975 & 0.0476 \\ 0.0150 & 0.0454 & 0.999 \end{bmatrix} .$$

The corresponding CP-violating parameter J is $J=4.74\times10^{-6}$.

Finally, by choosing the rotation parameter $\mu'=-0.376$, we have, for heavy basis mass matrices,

$$H^d_{\text{heavy}}=\begin{bmatrix} 0.0019 & 0 & 6.19\times10^{-4} \\ 0 & 7.32\times10^{-6} & -2.93\times10^{-4} \\ 6.19\times10^{-4} & -2.93\times10^{-4} & 3.00 \end{bmatrix},$$

$$H^u_{\text{heavy}}=\begin{bmatrix} 5.87\times10^{-5} & -1.32\times10^{-5} & 2.18\times10^{-5} \\ -1.32\times10^{-5} & 2.96\times10^{-6} & 0 \\ 2.18\times10^{-5} & 0 & 3.00 \end{bmatrix}.$$

The particular type of texture in the example above, namely, zeros in both the up and down sectors, is only one type of many different possibilities. We shall return to the more general question of texture in future work.

VII. COMMENTS

(1) A particular class of nearly democratic mass matrices is formed by mass matrices with every element on the unit circle, i.e., pure phases. In pure phase mass matrices [6], the absolute values are "purely" democratic. The fact that all our solutions have the property that k,l, and n are of order ϵ while a, b, and c are $O(\epsilon^2)$ is consistent with a pure phase mass matrix, as we shall discuss elsewhere [7]. Therefore, although we know of no dynamical model in which this occurs, we find it plausible that the mass matrices realized in nature can be expressed as pure phase mass matrices. We treat the pure phase mass matrices in separate work.

(2) We have concentrated on the simple fitting of masses and the elements of V_{km}. Within these fits we find possible solutions for every value of the d-sector a allowed by the mass hierarchy, namely, $w_1<a<w_2$, where w_1,w_2 are the appropriately normalized masses squared in the d sector. The value of the u-sector a is then restricted, as are all the other parameters. Additional constraints having to do with questions of texture, that is, a reduction in the number of independent parameters in the mass matrix, can be imposed on our fits. The example in Sec. VI illustrates how the imposition $H^d_{12}=H^u_{23}=0$ picks out a unique set of parameters. This example is only one possible realization of mass matrices with texture.

(3) The small size of the parameters K and L that appear in V [Eq. (2.5)] follows from the requirement that the CP-violating parameter J be small (or equivalently that the matrix elements of V have a certain hierarchy), not from the requirement that the masses have a hierarchy. In fact, as we have seen [in, for example, Eq. (2.3)], the masses do not have direct dependence on K and L.

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