t-expansion calculation of $\langle \bar{\psi}\psi \rangle$ in the chiral limit

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We use the t-expansion method to calculate the $\bar{\psi}\psi$ expectation value in the framework of Hamiltonian lattice QCD with two massless dynamical quarks within the Kogut-Susskind formulation. Using the mass of the ω meson as input, we obtain a value of the order of $(625\ \mathrm{MeV})^3$, consistent with one recent Monte Carlo simulation.

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I. INTRODUCTION

In this paper we employ the t expansion to evaluate the $\bar{\psi}\psi$ expectation value on the lattice, using a Hamiltonian of two massless dynamical quarks within the Kogut-Susskind formulation. This work follows some recent publications [1–3] where we have investigated the spectrum of this model.

Our lattice QCD calculation differs from the widespread Monte Carlo (MC) simulations in two features. First, our calculations are analytical rather than numerical. Second, we work with two flavors, in contrast with the standard Lagrangian approach that deals with four flavors. The four-flavor model has an extra U(1) continuous chiral symmetry which, when spontaneously broken, yields a massless pion. In order to avoid finite size effects connected with the massless pion [4], one has to work with nonzero quark masses. In contrast, the two-flavor formulation has only a discrete chiral symmetry, and hence lacks a Goldstone boson. Nonetheless note that it allows working with massless quarks in a confining theory.

In Ref. [1] we have calculated the masses of the scalar state 0^{++}, the nucleon, and the lowest-lying mesons, i.e., ρ, ω, and π. On the basis of an H^7 expansion, we found the mesons to be completely degenerate. The high mass of the pion was connected to the lack of continuous chiral symmetry in our model. The ratio of N to ω turned out to be of the right magnitude, i.e., 1.2–1.5. One important feature of the model which was not included in Ref. [1] is the $\bar{\psi}\psi$ expectation value, and we present it here.

Current algebra and PCAC (partial conservation of axial vector current) imply the following relation between the pion mass and the expectation value of $\bar{\psi}\psi$:

$$f_\pi^2 m_\pi^2 = \left[\frac{m_u+m_d}{2}\right]\langle 0|\bar{u}u+\bar{d}d|0\rangle \ . \tag{1.1}$$

This formula is correct to order $O(m_\pi^4)$, which is small for the physical pion. We note that the expression for $\langle \bar{\psi}\psi \rangle$ which appears in Eq. (1.1) refers to the expectation value in the chiral limit. Since the physical ground state of QCD is not chirally symmetric, there is no reason for the expectation values $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ to vanish. Moreover, the symmetry of the vacuum under $\mathrm{SU}(3)_{L+R}$ implies that the expectation values of $\bar{u}u$ and $\bar{d}d$ coincide in the chiral limit. If both f_π and $\langle \bar{\psi}\psi \rangle$ tend to finite limits as $m_q \to 0$, then the pion mass must tend to zero in proportion to $(m_u+m_d)^{1/2}$.

On the lattice, there is an analogous relation to Eq. (1.1) [5]:

$$f_\pi^2 m_\pi^2 = (m_u+m_d)\frac{\langle \bar{\chi}\chi(m_{u,d}=0)\rangle}{N_f} \ , \tag{1.2}$$

where χ and $\bar{\chi}$ are the fermion independent variable of the standard Lagrangian approach, and where on a finite lattice $\langle \bar{\chi}\chi(m_{u,d}=0)\rangle$ is supposed to be found by extrapolation from nonzero $m_{u,d}$. Therefore, in MC simulations, one evaluates the quantity $\langle \bar{\chi}\chi \rangle$ with nonzero quark masses and then extrapolates to the chiral limit, assuming a linear dependence on m_q [6]. Quoting the results of Ref. [6] and using the experimental value for the ρ mass, we find there an estimate for $\langle \bar{\chi}\chi(m_q=0)\rangle$ to be of order $(740\ \mathrm{MeV})^3$ for two flavors per lattice site.

We note that since there is no Goldstone boson in the two-flavor model, there is no reason why an equation similar to (1.2) should be obeyed. Nonetheless it is interesting to compute the $\bar{\psi}\psi$ expectation value directly in the chiral limit.

II. THE t-EXPANSION CALCULATION OF $\langle \bar{\psi}\psi \rangle$

The SU(3) pure gauge theory as defined by the Kogut-Susskind Hamiltonian is [7]

$$H_G = \frac{g^2}{2}\left[\sum_l \mathbf{E}_l^2 + x\sum_p (6-\mathrm{tr}U_p-\mathrm{tr}U_p^\dagger)\right] \tag{2.1}$$

where g is the coupling constant and $x=2/g^4$. The link operators $\mathbf{E}_l$ and U_l which appear in Eq. (2.1) are conjugate quantum variables satisfying the commutation relations

$$[E_l^a, U_{l'}] = \frac{\lambda^a}{2}U_l\delta_{ll'} \ , \tag{2.2}$$

where λ^a are the eight Gell-Mann matrices of SU(3). E_l^a is the color electric flux operator associated with the link

l, and $\mathrm{tr}\,U_p$ is the color magnetic flux operator associated with the plaquette p.

For dynamical fermions we employ the Kogut-Susskind scheme [8–10] in which the fermions are represented by a single degree of freedom per site:

$$\{\chi_i^\dagger(\mathbf{r}),\chi_j(\mathbf{r}')\}=\delta_{\mathbf{r},\mathbf{r}'}\delta_{ij}\ , \tag{2.3}$$

where i,j are color indices. The fermionic part of the Hamiltonian is

$$H_F=\frac{i}{2}\sum_{\mathbf{r},\mu}\eta_\mu(\mathbf{r})[\chi^\dagger(\mathbf{r})U(\mathbf{r},\mu)\chi(\mathbf{r}+\mu)-\chi^\dagger(\mathbf{r}+\mu)U^\dagger(\mathbf{r},\mu)\chi(\mathbf{r})] \tag{2.4}$$

where

$$\eta_x(\mathbf{r})=(-1)^z\ ,\quad \eta_y(\mathbf{r})=(-1)^x\ ,\quad \eta_z(\mathbf{r})=(-1)^y\ . \tag{2.5}$$

$\mathbf{r}$ varies over the lattice sites and μ over the three directions of space. H_F describes QCD with two massless dynamical quarks (u and d).

Starting with the strong coupling vacuum $|0\rangle$, which is the state annihilated by the color electric field,

$$\mathbf{E}_l|0\rangle=0\ , \tag{2.6}$$

the full vacuum is chosen in a staggered form which divides the lattice into two sublattices, that of even r (i.e., even $x+y+z$) and that of odd r:

$$|v\rangle=\Pi_{\mathrm{odd}\ \mathbf{r}}|+\rangle\Pi_{\mathrm{even}\ \mathbf{r}}|-\rangle|0\rangle\ , \tag{2.7}$$

where $|-\rangle$ ($|+\rangle$) is a color singlet annihilated by χ_i ($\chi_i^\dagger$).

The t-expansion method has been reviewed extensively in the context of pure gauge theories [11–13]. Its application to lattice theories with dynamical quarks was described in [14,1]. Its main idea is that expressions such as

$$O(t)=\frac{\langle v|e^{-tH/2}Oe^{-tH/2}|v\rangle}{\langle v|e^{-tH}|v\rangle} \tag{2.8}$$

should tend to the ground-state expectation value in the $t\to\infty$ limit. Moreover, it can be expanded in the form [11]

$$O(t)=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\langle OH^n\rangle^c\ , \tag{2.9}$$

wherein all the coefficients $\langle OH^n\rangle^c$ have the same volume dependence as O, and are defined recursively by

$$\langle OH^n\rangle^c=\langle OH^n\rangle-\sum_{p=1}^{m}\begin{bmatrix}m\\p\end{bmatrix}\langle OH^{m-p}\rangle^c\langle v|H^p|v\rangle\ , \tag{2.10}$$

where the $\langle OH^m\rangle$ are defined to be

$$\langle OH^m\rangle=\frac{1}{2}\sum_{p=0}^{m}\begin{bmatrix}m\\p\end{bmatrix}\langle v|H^pOH^{m-p}|v\rangle\ . \tag{2.11}$$

Now we should translate the operator $\bar\psi\psi=\bar uu+\bar dd$ to its lattice analogue:

$$\bar\psi\psi\to\sum_i(-1)^{x+y+z}\chi_i^\dagger(\mathbf{r})\chi_i(\mathbf{r})\ , \tag{2.12}$$

where summation over the color indices is included. This operator has different values for even and odd sites; hence, we take its mean value over one even and one odd site:

$$O=\frac{1}{2}\sum_i[\chi_i^\dagger(0)\chi_i(0)-\chi_i^\dagger(\hat z)\chi_i(\hat z)]\ . \tag{2.13}$$

Using the symmetry of odd shifts combined with an interchange of χ with $\chi^\dagger$ [2] and the anticommutation relation of Eq. (2.3), we obtain the following expression for O:

$$O=\frac{3}{2}-\sum_i\chi_i(0)\chi_i^\dagger(0)\ . \tag{2.14}$$

The operator O represents the lattice $\bar\psi\psi$ expectation value for two flavors per one lattice site. Expanding O we have obtained the following series for $O(t)$ to order t^8:

$$\begin{aligned}\langle v|O|v\rangle=&\frac{3}{2}+\frac{9}{8}y^2t^2-\frac{3}{4}y^2t^3+\frac{1}{384}(-495y^4+112y^2)t^4+\frac{1}{192}(15y^6+312y^4-16y^2)t^5\\&+\frac{1}{207\,360}(6066y^8+331\,893y^6-231\,336y^4+3968y^2)t^6+\frac{1}{69\,120}(-21\,857y^8-217\,768y^6+37\,824y^4-256y^2)t^7\\&+\frac{1}{1\,254\,113\,280}(-67\,237\,128y^{10}-2\,641\,044\,798y^8+5\,708\,820\,357y^6-267\,546\,240y^4+130\,048y^2)t^8+O(t^9)\ .\end{aligned} \tag{2.15}$$

III. DISCUSSION

As usual, we are interested in the scaling behavior of dimensionless quantities. Such are the ratios between $\langle\bar\psi\psi\rangle^{1/3}$ and any mass series which we have previously calculated [1,2]. In the strong-coupling limit, all these masses, with the exception of the nucleon, which has a vanishing energy, tend to infinity. On the other hand, $\langle\bar\psi\psi\rangle$ tends to a constant in the same limit. Therefore, it is favorable to consider ratios of the form $\langle\bar\psi\psi\rangle^{1/3}/M$.

First, we examine the $N/\langle\bar\psi\psi\rangle^{1/3}$ ratio, as displayed in Fig. 1. Since both the nucleon and the $\langle\bar\psi\psi\rangle$ series were expanded to order t^8, we have more D-Padé approximants in our disposal than usual. In Fig. 1 we have plotted all the approximants which are stable and which more or less coincide with one another. The curves

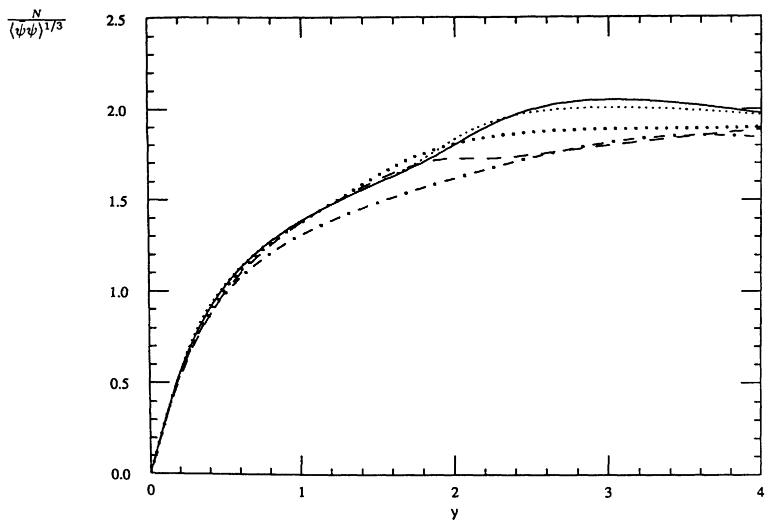

FIG. 1. Ratio between the nucleon mass and the $\langle \bar{\psi}\psi \rangle^{1/3}$. The approximants shown are the 0/5 (full line), 0/6 (dotted line), 0/7 (dot-dashed line), 1/5 (dashed line), and 1/6 (highest dotted line in the high y region).

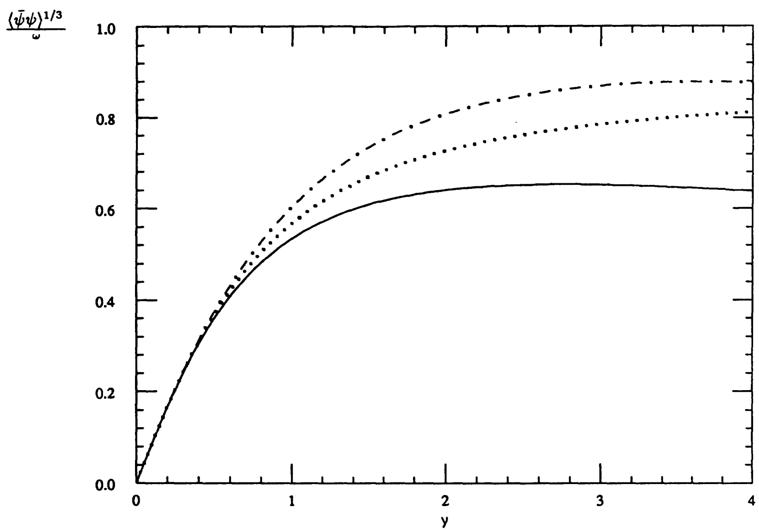

FIG. 2. Ratio between $\langle \bar{\psi}\psi \rangle^{1/3}$ and ω-meson mass. The full, dotted, and dot-dashed lines are the 0/3, 0/4, and 0/5 approximants, respectively.

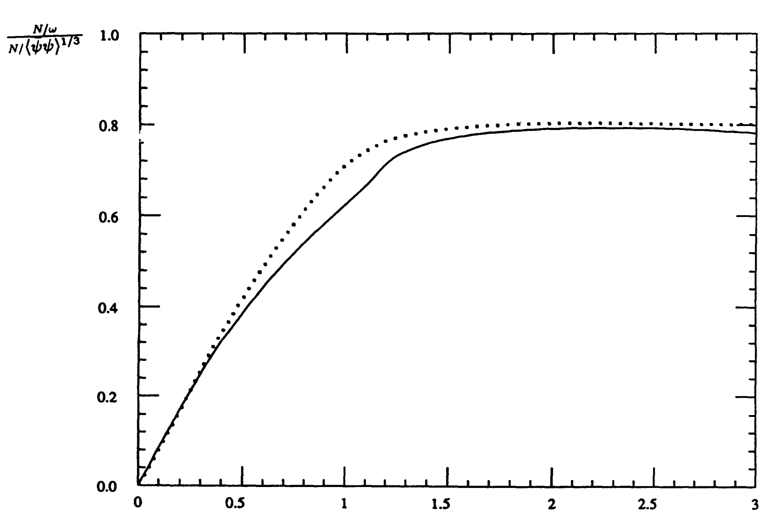

FIG. 3. Ratio between the N/ω and the $N/\langle \bar{\psi}\psi \rangle^{1/3}$ ratios. Shown are the 0/4 (full line) and 1/4 (dotted line) approximants.

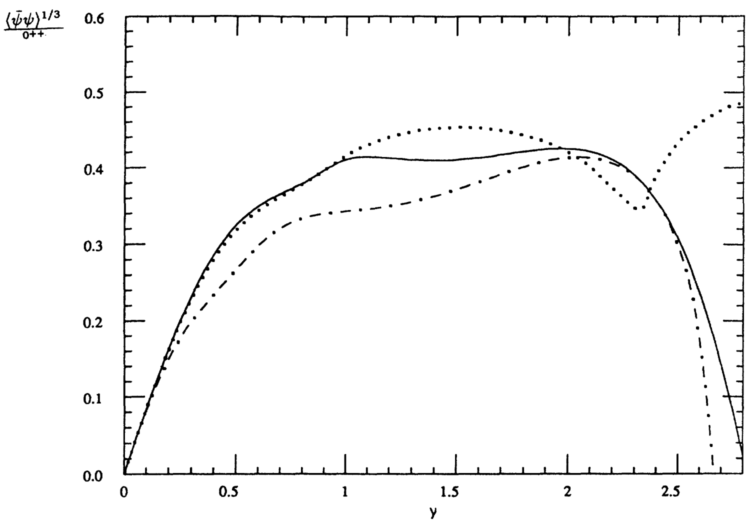

FIG. 4. 0/2 (full line), 0/3 (dotted line), and 1/3 (dot-dashed line) *D*-Padé approximants for the ratio between $\langle\bar\psi\psi\rangle^{1/3}$ and scalar mass.

displayed exhibit instability in the crossover region, and continue to rise beyond $y \sim 2$, where we read the physical results. Therefore it would be fair to say that near $y \sim 2$ we read the ratio to be in the range 1.6–2.0. This ratio corresponds to a $\langle\bar\psi\psi\rangle^{1/3}$ value in the range of 470–590 MeV. It is clear from Fig. 1 that we have obtained a value for the $\langle\bar\psi\psi\rangle^{1/3}$, which is too low due to the instability of the approximants.

Next, we study the $\langle\bar\psi\psi\rangle^{1/3}/\omega$ ratio, shown in Fig. 2. Again, we have a large dispersion of the approximants, and near $y \sim 2$ we have a ratio in the rangle 0.64–0.82, indicating a value for the $\langle\bar\psi\psi\rangle^{1/3}$ in the rangle 500–640 MeV. The approximants shown do not exhibit a clear asymptotic behavior and therefore we try, in Fig. 3, to extract a better estimate by plotting the ratio between the N/ω mass ratio and the $N/\langle\bar\psi\psi\rangle^{1/3}$ ratio. The only two approximants, namely 0/5 and 1/4, which are stable, coincide with one another onto a clear asymptotic value of 0.8. This yields a $\langle\bar\psi\psi\rangle^{1/3}$ value of 625 MeV. We trust this result due to its clear asymptotic trend.

In Fig. 4 we display the ratio of $\langle\bar\psi\psi\rangle^{1/3}$ to $M(0^{++})$. As explained in [1,2], ratios which involve the 0^{++} state break down right after $y \sim 2$. However, we can estimate from the peak in the crossover region a value in the range 0.37–0.46. This value gives an estimate for the $\langle\bar\psi\psi\rangle^{1/3}$ value in the range 480–600 MeV.

Finally, we have also examined the ratio between the $\langle\bar\psi\psi\rangle^{1/3}$ and the heavier baryons, the Δ and the N^*, which were calculated in Ref. [2]. As we expected, the ratio that involves these baryons do not show any scaling tendency, and trying to ready the ratios near $y \sim 2$ we obtain lower values for the $\langle\bar\psi\psi\rangle^{1/3}$. These values are consistent with the results of Ref. [2] where we found these baryons to be about 25 percent heavier than their observed values.

Thus we conclude that the most trustworthy result is that of Fig. 3, which indicates a value of (625 MeV)^3 for the $\langle\bar\psi\psi\rangle$. This is in accordance with our working hypothesis [1–3], which is to plot all possible ratios, but to regard as reliable only those which show scaling behavior. The existence of scaling, for us, is an indication for the correctness of the results. We note, however, that we see no physical reason why the ratio which involves the ω mass should yield a better result. Moreover, the existence of scaling and the consistency of our result with the result reported by MC simulations [6] both indicate that our model, indeed, should give a good physical picture of all hadrons made out of u and d quarks, but for the pion.

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