Weinberg-type sum rules at zero and finite temperature

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We consider sum rules of the Weinberg type at zero and nonzero temperatures. On the basis of the operator product expansion at zero temperature we obtain a new sum rule which involves the average of a four-quark operator on one side and experimentally measured spectral densities on the other. We further generalize the sum rules to finite temperature. These involve transverse and longitudinal spectral densities at each value of the momentum. Various scenarios for the relation between chiral symmetry restoration and these finite temperature sum rules are discussed.

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I. INTRODUCTION

In a famous 1967 paper [1] Weinberg asked the question: "What relations are imposed by current algebra upon the spectra of the 1^+ and 1^- mesons?" Under certain conditions the answer was two sum rules involving the vector and axial-vector spectral densities. They are known as the Weinberg sum rules:

$$\text{(I)} \qquad \int_0^\infty \frac{ds}{s}\left[\rho_V(s)-\rho_A(s)\right] = F_\pi^2 \qquad (1)$$

$$\text{(II)} \qquad \int_0^\infty ds\left[\rho_V(s)-\rho_A(s)\right] = 0. \qquad (2)$$

Assuming vector meson dominance and the Kawarahayashi-Suzuki-Fayyazuddin-Riazuddin (KSFR) relation [2] these sum rules lead to the prediction that the ρ and a_1 masses are related by $m_{a_1}=\sqrt{2}m_\rho$, which is approximately valid. In this paper we ask two questions. The first one is as follows: Given that QCD is now known to be the theory of the strong interactions, what extra information can we get from sum rules of the Weinberg type?

The last 15 years have seen a great deal of activity surrounding QCD at finite temperature. Of particular interest are the issues of deconfinement and chiral symmetry restoration at temperatures of the order of 160 MeV. Therefore, we are led to consider a second question: What are the implications of the approach to chiral symmetry restoration at finite temperature for sum rules of the Weinberg type?

The status of the original Weinberg sum rules in the context of QCD sum rules was discussed by Shifman, Vainshtein, and Zakharov [3] and then by Narison [4], while a more up-to-date phenomenological analysis was performed by Peccei and Sola [5]. The two sum rules derived by Weinberg are very general, as he showed, and do not depend on specific details of the QCD Lagrangian. Higher-order sum rules (involving more powers of s in the integrand) do depend on the dynamics of chiral symmetry breaking in the vacuum. In Sec. II we derive a third sum rule of the type of Eqs. (1) and (2). This new sum rule involves the vacuum expectation value (VEV) of a certain local four-quark operator. It can be obtained from the sum rule if we know the vector and axial-vector spectral densities accurately enough from experiment. It can also be obtained from lattice QCD; the chirality-violating structure of the operator helps here because its VEV has no short-distance perturbative contribution. We perform a detailed analysis of all sum rules in Sec. III. We will see that they are restrictive enough to fill in gaps in the experimental data, allowing us to determine the spectral densities with quite some accuracy.

There has been a lot of discussion in the literature and at conferences about the temperature dependence of hadron masses. Some calculations yield increasing masses, some yield decreasing masses, and still others yield masses that either increase or decrease depending on the quantum numbers of the hadron; see [6–10] and the review [11]. Clearly all these calculations are only approximate. In addition, the very notion of a mass at finite temperature must be very clearly defined, such as the screening mass or the pole mass corresponding to collective excitations.

A common denominator of all studies of this type is the temperature dependence of correlation functions. It would be good if some general statements about these correlation functions could be made which rely on the fundamental properties of QCD at finite temperature. This is the aim in Sec. IV. We generalize the original Weinberg sum rules, and the new one, to finite temperature. The first one [Eq. (1)] generalizes to a sum rule involving only the longitudinal spectral density and depends on the three-momentum. The second one [Eq. (2)] generalizes to two separate sum rules, one involving the longitudinal spectral density and the other involving the transverse spectral density, both depending on momentum. At zero three-momentum they collapse to the same expression. In the vacuum there is no dependence on

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momentum because of Lorentz invariance, but at finite temperature there is a preferred rest frame, hence a dependence on momentum and on polarization. We would like to point out here that probably the first discussion of Weinberg sum rules at finite temperature was given by Bochkarev and Shaposhnikov [6] in the context of QCD sum rules and for zero momentum.

The finite temperature sum rules can be used to constrain models or approximations to QCD, and can help us to understand the approach to chiral symmetry restoration. Various possibilities will be considered in Sec. V. We should reference here the early paper on phenomenology of the chiral phase transition in heavy ion collisions by Pisarski [12]. For a recent discussion of the topic one can see [13].

We remark that throughout this paper we assume that the up and down quark masses are identically zero so that chiral symmetry is exact. Consideration of the impact of nonzero quark masses on the original Weinberg sum rules within perturbative QCD was done by Floratos, Narison, and de Rafael [14].

II. DERIVATION OF ZERO TEMPERATURE SUM RULES FROM QCD

We define the vector and axial-vector currents:

$$V_\mu^a = \bar{q}\gamma_\mu(\tau^a/2)q\,, \tag{3}$$

$$A_\mu^a = \bar{q}\gamma_\mu\gamma_5(\tau^a/2)q\,, \tag{4}$$

where $\tau^a/2$ is the isospin generator. With this normalization the current algebra of charges obeys the equal time commutation relations

$$\left[Q_V^a, Q_V^b\right] = i\varepsilon^{abc}Q_V^c\,, \tag{5}$$

$$\left[Q_V^a, Q_A^b\right] = i\varepsilon^{abc}Q_A^c\,, \tag{6}$$

$$\left[Q_A^a, Q_A^b\right] = i\varepsilon^{abc}Q_V^c\,. \tag{7}$$

We define the vector and axial-vector spectral densities in the usual way. They are positive definite quantities defined for positive s:

$$\langle 0|V_a^\mu(x)V_b^\nu(0)|0\rangle = -\frac{\delta^{ab}}{(2\pi)^3}\int d^4p\,\theta(p^0)\,e^{ip\cdot x}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\rho_V(s)\,, \tag{8}$$

$$\langle 0|A_a^\mu(x)A_b^\nu(0)|0\rangle = -\frac{\delta^{ab}}{(2\pi)^3}\int d^4p\,\theta(p^0)\,e^{ip\cdot x}\left[\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\rho_A(s) + F_\pi^2\delta(s)p^\mu p^\nu\right]. \tag{9}$$

The dimension of the spectral densities is energy squared. Note the explicit contribution of the pion to the axial-vector correlator.

In this paper we work in imaginary time so that all distances are spacelike or Euclidean: $x^2 = t^2 - r^2 = -\tau^2$. In this domain the spectral representation of the correlation functions looks like [11]

$$\begin{aligned}\Delta D_\mu^{ab\mu}(\tau) &\equiv \langle 0|\mathcal{T}\left[V^{a\mu}(x)V_\mu^b(0) - A^{a\mu}(x)A_\mu^b(0)\right]|0\rangle \\ &= -\frac{\delta^{ab}}{4\pi^2\tau}\int_0^\infty ds\,\sqrt{s}\,\left[3\rho_V(s) - 3\rho_A(s) - s\,F_\pi^2\delta(s)\right]K_1(\sqrt{s}\tau)\end{aligned} \tag{10}$$

and

$$\begin{aligned}\Delta D_{ab}^{00}(\tau) &\equiv \langle 0|\mathcal{T}\left[V_a^0(x)V_b^0(0) - A_a^0(x)A_b^0(0)\right]|0\rangle = -\frac{\delta_{ab}}{4\pi^2\tau}\int_0^\infty ds\sqrt{s} \\ &\times\left[\rho_V(s) - \rho_A(s) - s\,F_\pi^2\delta(s)\right]\left[\frac{K_0(\sqrt{s}\tau)}{\sqrt{s}\tau} + \left(\frac{2}{s\tau^2}+1\right)K_1(\sqrt{s}\tau)\right].\end{aligned} \tag{11}$$

Notice that the integrands essentially involve the standard Feynman propagator for a particle of mass m which, in the Euclidean domain, is

$$D(m,\tau)_{\text{free scalar}} = \frac{m}{4\pi^2\tau}K_1(m\tau)\,. \tag{12}$$

Exponential decay of the Bessel function K_1 at large argument ensures convergence of such integrals for any QCD correlation functions, except probably at $\tau = 0$. In this sense, there is no difference between the Euclidean time representation [11] and the Borel-transformed sum rules [3], in which the propagator is replaced with $\exp(-s/M^2)$, with the Borel parameter M replacing Euclidean time τ.

The coordinate representation is more transparent and accessible to numerical methods, such as lattice calculations. Recent studies based on the instanton liquid model [15] and lattice QCD [16] have reported on the calculation of a set of Euclidean correlation functions, including vector and axial-vector ones. Unfortunately, none of them has focused on their *difference* with sufficiently high accuracy, and therefore they are not discussed in the present work.

We now come to the central idea behind the derivation of the sum rules: *each* sum rule corresponds to a *particular term* in the small-distance asymptotic expansion of the correlation function.

In the limit $\tau \to 0$ the product of currents can be expanded according to the operator product expansion (OPE), a very powerful means for connecting VEV's of quark and gluon operators to experimentally observable hadronic properties. The first terms in this expansion were first computed in [3]. For the contracted polarization tensor the result is

$$D_\mu^{ab\mu}(\tau) \equiv \langle 0|\mathcal{T}\left[V^{a\mu}(x)V_\mu^b(0)\right]|0\rangle = -\frac{3\delta^{ab}}{\pi^4\tau^6}\left[1+\frac{\alpha_s(\tau)}{\pi}-\frac{\langle 0|\left(gF_{\mu\nu}^c\right)^2|0\rangle\tau^4}{3\times 2^7} -\frac{\pi^2\tau^6}{8}\ln(\mu\tau)\langle 0|\mathcal{O}_\rho|0\rangle+\cdots\right], \quad (13)$$

where, in the argument of the logarithm, $\mu \ll 1/\tau$ is the renormalization scale, and $\mathcal{O}_\rho$ is a complicated four-quark operator. There is a similar expression for the correlator of two axial-vector currents but with a different four-quark operator $\mathcal{O}_{a_1}$. For our purposes we only need their difference, which is given below.

Since chiral symmetry breaking is a long-wavelength phenomenon, at very short distances, or at very high energies, the difference between vector and axial-vector correlators should go to zero. Indeed, taking this difference one finds that all terms except for the four-quark operators in Eq. (13) drop out.

One can now look for consequences of this statement for the spectral density. Expanding the Bessel function in Eq. (10) for small values of τ we get

$$\Delta D_\mu^{ab\mu}(\tau) = -\frac{3\delta^{ab}}{4\pi^2}\int_0^\infty ds\,[\rho_V(s)-\rho_A(s)] \times\left[\frac{1}{\tau^2}+\frac{s}{2}\ln\left(\frac{\sqrt{s}\tau}{2}e^{C-1/2}\right) +O\left(\tau^2,\tau^2\ln\tau\right)\right], \quad (14)$$

where C is Euler's constant. The OPE has no power divergence in τ in the *difference* $\Delta D_\mu^{ab\mu}$. Therefore the coefficient of $1/\tau^2$ in Eq. (14) must vanish. This is just the second Weinberg sum rule [Eq. (2)]. In the OPE framework it simply follows from the observation that the first covariant operators which are not chirality blind are four-quark ones which have dimension 6 or more. Similarly expanding Eq. (11) for small τ and applying the observation of chirality blindness we get

$$\int_0^\infty \frac{ds}{s}\left[\rho_V(s)-\rho_A(s)-s\,F_\pi^2\delta(s)\right]\left[\frac{1}{\tau^4}+\frac{s}{4\tau^2}\right] = 0. \quad (15)$$

The first and second terms in the last square brackets reproduce the first and second Weinberg sum rules, respectively.

The next term in the small τ expansion is logarithmic. In Eq. (14) we multiply the argument of the logarithm by μ/μ which we must do to match the OPE. Equating the coefficients of $\ln(\mu\tau)$ in $\Delta D_\mu^{ab\mu}(\tau)$ we obtain the third sum rule:

$$\text{(III)} \qquad \int_0^\infty ds\,s\,[\rho_V(s)-\rho_A(s)] = -2\pi\langle 0|\alpha_s\mathcal{O}_\mu^\mu|0\rangle\,. \quad (16)$$

Here

$$\mathcal{O}^{\mu\nu} = (\bar{u}_L\gamma^\mu t^a u_L - \bar{d}_L\gamma^\mu t^a d_L)(\bar{u}_R\gamma^\nu t^a u_R - \bar{d}_R\gamma^\nu t^a d_R), \quad (17)$$

where t^a are the color SU(3) matrices and R, L stand for right- and left-handed quarks. Note the appearance of the renormalization scale μ on the right side of this sum rule. Since the other side of the equation is expressed in terms of physical observables, it must be that $\alpha_s(\mu)$ times the four-quark operator is a renormalization group invariant.

The numerical value of the VEV of this operator is unknown. The estimate suggested in [3] is based on the so-called "vacuum dominance" hypothesis, which leads to

$$\langle 0|\mathcal{O}_\mu^\mu|0\rangle = \frac{16}{9}\langle 0|\bar{u}u|0\rangle^2\,. \quad (18)$$

The accuracy of this estimate should of course be questioned, and various models of chiral symmetry breaking [17] and lattice numerical calculations can be used for that purpose. Let us only add a comment on μ dependence here. If the vacuum dominance hypothesis is correct, then the VEV should be proportional to $[\ln(\mu/\Lambda_{\rm QCD})]^{8/b}$, the anomalous dimension of the quark condensate. (Here $b=\frac{11}{3}N_c-\frac{2}{3}N_f$ comes from the Gell-Mann–Low function.) Since the power is close to 1, after being multiplied by $\alpha_s(\mu)\sim 1/\ln(\mu/\Lambda_{\rm QCD})$ the right side of the third sum rule is nearly μ independent. Thus, at least concerning the μ dependence, this approximation can approximately hold.

The regular (τ-independent) term was not considered in the QCD sum rule context; it was first discussed in connection with point-to-point correlators in the coordinate representation by one of us [11]. It is interesting to express it in terms of an integral over the difference in spectral densities, and it may be useful for lattice calculations. Dropping terms which vanish in the limit, we find

$$\Delta D_\mu^\mu(\tau\to 0) = -\ln(\mu\tau)\frac{3}{8\pi^2}\int_0^\infty ds\,s\,[\rho_V(s)-\rho_A(s)] - \frac{3}{8\pi^2}\int_0^\infty ds\,s\,\ln\left(\frac{\sqrt{s}}{\tilde{\mu}}\right)[\rho_V(s)-\rho_A(s)]\,, \quad (19)$$

where $\tilde{\mu}=2\mu e^{1/2-C}=1.85\mu$. The use of μ here is just for convenience; ΔD is actually independent of it.

III. PHENOMENOLOGY AT ZERO TEMPERATURE

Phenomenological analysis of the Weinberg sum rules was originally made in a very simple approximation using only the contributions of ρ, a_1, π mesons. In other words, Weinberg *assumed* that contributions from all excited states other than the lowest resonances mentioned canceled out. Together with the KSFR relation it leads to the famous prediction $m_{a_1} = \sqrt{2} m_\rho$ which looked excellent from the point of view of data available at the time. However, now we know that this prediction, as well as predictions for coupling constants, agrees with experiment only up to the level of 10–20%.

The sum rules are exact in the chiral limit, and so one should be willing to verify them as accurately as possible. If the complete spectral densities were measured, one could simply evaluate the integrals and check whether the sum rules are indeed satisfied, up to the accuracy of the data. Unfortunately the situation is not that straightforward because there are *no* meaningful measurements of the nonresonance contribution in the axial channel. Therefore we first have to close this hole using the sum rules themselves.

Let us first discuss how well the spectral densities are determined experimentally. In the pole-plus-continuum approximation one would write

$$\rho_V(s) = \frac{m_\rho^4}{g_\rho^2}\delta(s - m_\rho^2) + \frac{s}{8\pi^2}\left[1 + \frac{\alpha_s(s)}{\pi} + \cdots\right]\theta(s - E_V^2) \qquad (20)$$

and

$$\rho_A(s) = \frac{m_{a_1}^4}{g_a^2}\delta(s - m_{a_1}^2) + \frac{s}{8\pi^2}\left[1 + \frac{\alpha_s(s)}{\pi} + \cdots\right]\theta(s - E_A^2)\,. \qquad (21)$$

The continuum is the same in the vector and axial-vector channels, according to perturbative QCD, but the phenomenological threshhold is in general different. Note that the individual integrals over s for the vector and axial-vector channels which enter the sum rules are actually divergent because of the continuum, but the *difference* is finite. The coupling constants are the same ones used in a vector dominance approximation to the currents as expressed in the current-field identities of Sakurai [18]:

$$V_\mu^a = \frac{m_\rho^2}{g_\rho}\rho_\mu^a\,, \qquad (22)$$

$$A_\mu^a = \frac{m_{a_1}^2}{g_a}a_\mu^a + \text{pion}\,. \qquad (23)$$

We do not use these approximations for the spectral densities because the three sum rules involve integrations of the spectral densities with different powers of s and so it is likely important to incorporate the finite widths of the resonances.

The vector spectral density is very well measured in $e^+e^- \to \rho \to \pi^+\pi^-$. An s-wave relativistic Breit-Wigner form is not a good representation because the ρ meson is a p-wave resonance. A much better representation is given by the Gounaris-Sakurai formula [19,20]. It turns out that this complicated formula can be approximated by a relativistic Breit-Wigner form with an effective width $\Gamma'_\rho = 118$ MeV and an effective pole mass $m'_\rho = 761$ MeV:

$$\rho_V(s) = \frac{m_\rho^4}{g_\rho^2}\frac{1}{\pi}\frac{m_\rho\Gamma'_\rho}{(s - m_\rho'^2)^2 + m_\rho^2\Gamma_\rho'^2} + \frac{s}{8\pi^2}\frac{1}{1 + \exp[(E_V - \sqrt{s})/\delta_V]} \times\left[1 + \frac{0.22}{\ln(1 + \sqrt{s}/0.2\,\text{GeV})}\right]. \qquad (24)$$

We take g_ρ from the KSFR relation

$$g_\rho^2 = \frac{m_\rho^2}{2F_\pi^2}\,. \qquad (25)$$

With $m_\rho = 768$ MeV and $F_\pi = 94.5$ MeV one gets from this $g_\rho^2/4\pi = 2.63$. The second term in Eq. (24), corresponding to the continuum from 2n-pion states ($n = 2, 3, \ldots$), has $E_V = 1.3$ GeV and $\delta_V = 0.2$ GeV [11].

The coupling of the a_1 to the current can be determined from the measured branching ratios of $\tau \to \nu_\tau +$ hadrons. According to the Particle Data Group [21], the branching into the two three-pion channels dominated by the a_1 is $11.2 \pm 1.4\%$ while the ρ-dominated two-pion channel is $24.0 \pm 0.6\%$. The first number gives rise to the main uncertainty in our numerical analysis below. Using these numbers and the theoretical expression for the branching ratios (which follows from the narrow width approximation)

$$\frac{B(\tau \to \nu_\tau + a_1)}{B(\tau \to \nu_\tau + \rho)} = \frac{m_{a_1}^2}{m_\rho^2}\frac{g_\rho^2}{g_{a_1}^2}\frac{(1 - m_{a_1}^2/m_\tau^2)^2}{(1 - m_\rho^2/m_\tau^2)^2}\frac{(1 + 2m_{a_1}^2/m_\tau^2)}{(1 + 2m_\rho^2/m_\tau^2)}\,, \qquad (26)$$

one can get the coupling $g_{a_1} = 10.5 \pm 0.7$.

For the axial-vector spectral density we use an expression analogous to the vector one but with the following differences. First, we use a constant width of 400 MeV and a constant mass of 1260 MeV for the a_1 contribution (for more details about this see Ref. [22]). We have, however, cut off this resonance below the threshold $m_\rho + m_\pi$.

The large width of the a_1 and its proximity to the τ lepton causes a significant correction to Eq. (26). Numerically integrating the differential decay rate [5] with the realistic shape of the resonance we get finally a value $g_{a_1} = 9.1 \pm 0.7$.

The available data for the nonresonant axial states are very poor so that the continuum threshhold E_A and the width δ_A are unknown. The reason is partly statistical. More importantly, since the data about the axial-vector spectral density come from the τ lepton decay, there are fundamental limitations due to the τ mass which is not big enough to provide sufficient phase space for three- and five-pion final states with the needed invariant mass. Therefore, some authors (for example [5]) have analyzed the Weinberg sum rules without the axial-vector continuum.

In Fig. 1 we show our spectral density with axial-vector continuum using the same width as the vector con-

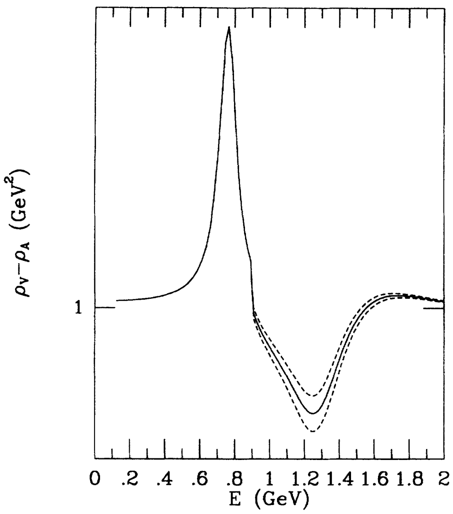

FIG. 1. The difference between the spectral densities of the vector and axial-vector currents versus s at zero temperature. The two dashed curves show the uncertainty due to the experimental determination of the a_1 coupling constant, described in the text. The abrupt change at $s = 0.8$ GeV^2 corresponds to the sharp onset of the a_1 contribution at $(m_\rho + m_\pi)^2$.

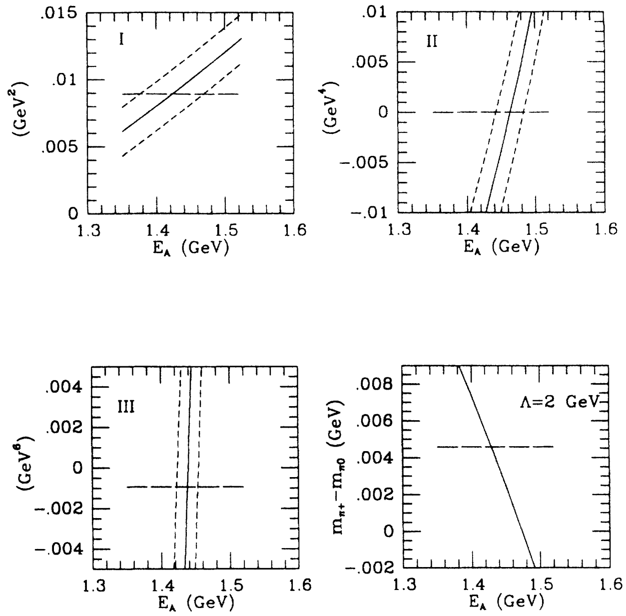

FIG. 2. Dependence of the zero temperature sum rules I–III on the effective perturbative threshhold E_A in the axial-vector channel. The last panel shows the π^+ - π^0 mass difference. As in Fig. 1, the solid and the two dashed curves correspond to the central value and uncertainty in the a_1 coupling constant. In all cases, the expected magnitude of the corresponding sum is shown by the horizontal long-dashed line.

tinuum [11] and with a threshhold value to be determined below. In this figure the dashed curves correspond to the experimental uncertainty in the branching ratio into a_1. One can see that this is a rather nontrivial, sign-changing function, which should obey the sum rules under consideration. Naturally, the new sum rule we consider is more sensitive to the large s behavior of the difference of the spectral densities. Thus we may at least ask whether *all* sum rules are consistent with one common value of the parameter E_A.

In Figs. 2(a)–2(c) we have plotted sum rules I – III as functions of E_A. The horizontal dashed line shows in all cases the right side of the sum rule which depends on the vacuum quark condensate or F_π as appropriate. The intersection of the lines should occur at the same value of E_A. As explained in the previous section, we do not know exactly the VEV of the relavent operators; therefore we use the vacuum dominance estimate, with

$$|\langle 0|\bar{u}u|0\rangle|^{1/3} = 240\ \mathrm{MeV} \quad (\mu = 1\ \mathrm{GeV})\,. \tag{27}$$

Fortunately, there seems to be very little sensitivity to the value of the quark condensates. One can clearly see that sum rules II and III are quite consistent with the common value of $E_A = 1.45$ GeV. This observation is nontrivial.

Now we can come back to the first sum rule, use this value of E_A as input, and compare the numerical value of the integral to the right-hand side. This procedure predicts F_π about 5% higher than the experimental value.

Finally, let us comment on a closely related integral of the spectral densities under consideration. It was shown in [23] that the electromagnetic mass difference of pions can be expressed as

$$m_{\pi^+}^2 - m_{\pi^0}^2 = \frac{3e^2}{16\pi^2 F_\pi^2} \int ds \ln\left(\frac{\Lambda^2 + s}{s}\right) [\rho_V(s) - \rho_A(s)], \tag{28}$$

where Λ is some cutoff parameter used to regulate the divergent integral over virtual momentum in the loop. The result obviously depends on it; only in one particular limit, namely, for $\Lambda \gg m_\rho, m_{a_1}$ and for the original Weinberg values of the ρ and a_1 parameters without continuum, can one get rid of it and recover the original result $m_{\pi^+}^2 - m_{\pi^0}^2 = (3\ln 2\alpha/2\pi) m_\rho^2$ of [23]. However, for the parameters extracted from data as explained above, it is no longer true. The integral does depend on the cutoff Λ.

In Fig. 2(d) we show this sum rule with $\Lambda = 2$ GeV as a function of E_A. Note that the value of the pion mass splitting is very sensitive to E_A, and can even change sign if it is only 40 MeV above the suggested value. However, at $E_A = 1.45$ GeV it agrees with the experimental value (horizontal line) reasonably well. Fine-tuning could be accomplished by adjusting the cutoff Λ, but we shall not do this.

IV. FINITE TEMPERATURE SUM RULES

In this section we first generalize Weinberg's two sum rules to finite temperature using essentially the same methods as he used without any specific reference to

QCD. Then we verify the generalizations by using the OPE, which also allows us to obtain the finite temperature extension of sum rule III. Finally, we investigate the behavior of these sum rules at low temperature.

A. Derivation of Weinberg-type sum rules at fixed momentum

Consideration of Weinberg-type sum rules at finite temperature (or chemical potential) is more involved than at zero temperature. Lorentz invariance is not manifest because there is a preferred frame of reference, the frame in which the matter is at rest. Thus spectral densities and other functions may depend on energy and momentum separately and not just on their invariant s. Also, the number of Lorentz tensors is greater because there is a new vector available, namely, the vector $u_\mu = (1,0,0,0)$ which specifies the rest frame of the matter.

For a given four-momentum p it is useful to define two projection tensors. The first one $P_T^{\mu\nu}$ is both three- and four-dimensionally transverse:

$$P_T^{ij} \equiv \delta^{ij} - \frac{p^i p^j}{\mathbf{p}^2}\,, \tag{29}$$

with all other components zero. The second one $P_L^{\mu\nu}$ is only four-dimensionally transverse:

$$P_L^{\mu\nu} \equiv -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + P_T^{\mu\nu}\right)\,. \tag{30}$$

The notation is L for longitudinal and T for transverse with respect to $\mathbf{p}$. There are no other symmetric second-rank tensors which are four-dimensionally transverse.

We now define the longitudinal and transverse spectral densities for the vector current as

$$\langle V_a^\mu(x) V_b^\nu(0)\rangle = \frac{\delta^{ab}}{(2\pi)^3}\int d^4p\,\theta(p^0)\,e^{ip\cdot x}\left[\rho_V^L P_L^{\mu\nu} + \rho_V^T P_T^{\mu\nu}\right] \tag{31}$$

and for the axial vector current as

$$\langle A_a^\mu(x) A_b^\nu(0)\rangle = \frac{\delta^{ab}}{(2\pi)^3}\int d^4p\,\theta(p^0)\,e^{ip\cdot x}\left[\rho_A^L P_L^{\mu\nu} + \rho_A^T P_T^{\mu\nu}\right]\,. \tag{32}$$

In these expressions the angular brackets refer to the thermal average. In general the spectral densities depend on p^0 and $\mathbf{p}$ separately as well as on the temperature (and chemical potential). These definitions are standard and insure that both spectral densities are non-negative. In the vacuum we can always go to the rest frame of a massive particle, and in that frame there can be no difference between longitudinal and transverse polarizations, so that $\rho_L = \rho_T = \rho$. Since $P_L^{\mu\nu} + P_T^{\mu\nu} = -(g^{\mu\nu} - p^\mu p^\nu/p^2)$ these equations collapse to Eqs. (8) and (9). The pion, being a massless Goldstone boson, is special. It contributes to the longitudinal axial-vector spectral density and not to the transverse one. In fact, we could write

$$F_\pi^2 \delta(p^2) p^\mu p^\nu = F_\pi^2 p^2 \delta(p^2) P_L^{\mu\nu}\,. \tag{33}$$

This should not be done at finite temperature because the contribution of the pion to the longitudinal spectral density cannot be assumed to be a δ function in p^2. In general the pion's dispersion relation will be more complicated and will develop a width at nonzero momentum. Therefore, we do not try to separate out the pionic contribution but subsume it in the spectral density ρ_A^L, without any loss of generality.

Following Weinberg, we define a three-point function by

$$-i\epsilon_{abc} M^{\mu\nu\lambda}(q,p) = \int d^4x\, d^4y\, e^{-i(q\cdot x + p\cdot y)} \times \langle \mathcal{T}\left[A_a^\mu(x) A_b^\nu(y) V_c^\lambda(0)\right]\rangle\,. \tag{34}$$

We multiply both sides by q_μ. On the right side we can use

$$q_\mu e^{-i(q\cdot x+p\cdot y)} = i\frac{\partial}{\partial x^\mu} e^{-i(q\cdot x + p\cdot y)}\,. \tag{35}$$

Both the vector and axial-vector currents are conserved. We assume that we can integrate by parts and that the surface term is zero. The nonzero contribution comes from

$$\frac{\partial}{\partial x^\mu}\left\{\mathcal{T}\left[A_a^\mu(x)A_b^\nu(y)V_c^\lambda(0)\right]\right\} = \delta(x^0 - y^0)\left\{\theta(x^0)\left[A_a^0(x), A_b^\nu(y)\right]V_c^\lambda(0) + \theta(-x^0)V_c^\lambda(0)\left[A_a^0(x), A_b^\nu(y)\right]\right\} + \delta(x^0)\left\{\theta(y^0)A_b^\nu(y)\left[A_a^0(x), V_c^\lambda(0)\right] + \theta(-y^0)\left[A_a^0(x), V_c^\lambda(0)\right]A_b^\nu(y)\right\}\,. \tag{36}$$

From this expression we see the need for knowledge of the equal time commutators. Consistent with the normalization of Eqs. (5)–(7) we have

$$\delta(z^0)\left[A_a^0(x), A_b^\nu(y)\right] = i\epsilon_{abd} V_d^\nu(x)\delta(\mathbf{z}) + S_{Vab}^{\nu j}(\mathbf{x})\frac{\partial}{\partial z^j}\delta(\mathbf{z})\,, \tag{37}$$

$$\delta(z^0)\left[A_a^0(x), V_b^\nu(y)\right] = i\epsilon_{abd} A_d^\nu(x)\delta(\mathbf{z}) + S_{Aab}^{\nu j}(\mathbf{x})\frac{\partial}{\partial z^j}\delta(\mathbf{z})\,.$$

Here $z = x - y$, and the S's denote the Schwinger terms.

Consider now the contribution of the Schwinger terms to the thermal average. Generically they will be of the form

$$\langle SJ\rangle = Z^{-1}\sum_{m,n} e^{-K_n/T}\langle n|S|m\rangle\langle m|J|n\rangle\,, \tag{38}$$

where $K = H - \mu N$ is the Hamiltonian minus the chemical potential times conserved particle number, the states are chosen to be eigenstates of H, N, and isospin, and J is either the vector or the axial-vector current. J has isospin 1, and so we get zero if either (i) S is a c number or (ii) S is an operator with no isospin 1 component. We shall assume that one of these holds. Then

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\langle \mathcal{T}\left[A_a^\mu(x)A_b^\nu(y)V_c^\lambda(0)\right]\rangle \\ = i\epsilon_{abd}\delta(x-y)\langle\mathcal{T}\left[V_d^\nu(x)V_c^\lambda(0)\right]\rangle \\ +i\epsilon_{acd}\delta(x)\langle\mathcal{T}\left[A_b^\nu(y)A_d^\lambda(0)\right]\rangle\,.\end{aligned} \tag{39}$$

It is now a simple matter to show that

$$\frac{1}{2}q_\mu M^{\mu\nu\lambda}(q,p) = D_V^{\nu\lambda}(q+p) - D_A^{\nu\lambda}(p)\,, \tag{40}$$

where the D's are the propagators for the currents, such as

$$\delta_{ab}D_A^{\nu\lambda}(p) = \int d^4y\, e^{-ip\cdot y}\langle\mathcal{T}\left[A_a^\nu(y)A_b^\lambda(0)\right]\rangle\,. \tag{41}$$

Similarly, one can show that

$$\frac{1}{2}(q+p)_\lambda M^{\mu\nu\lambda}(q,p) = D_A^{\mu\nu}(q) - D_A^{\mu\nu}(p)\,. \tag{42}$$

These Ward identities have exactly the same form as at zero temperature [1].

With a similar consideration of the three-point function

$$\begin{aligned}-i\epsilon_{abc}N^{\mu\nu\lambda}(q,p) = \int d^4x\, d^4y\, e^{-i(q\cdot x+p\cdot y)} \\ \times\langle\mathcal{T}\left[V_a^\mu(x)V_b^\nu(y)V_c^\lambda(0)\right]\rangle\,,\end{aligned} \tag{43}$$

one can prove two more Ward identities:

$$\frac{1}{2}q_\mu N^{\mu\nu\lambda}(q,p) = D_V^{\nu\lambda}(q+p) - D_V^{\nu\lambda}(p) \tag{44}$$

and

$$\frac{1}{2}(q+p)_\lambda N^{\mu\nu\lambda}(q,p) = D_V^{\mu\nu}(q) - D_V^{\mu\nu}(p)\,. \tag{45}$$

Multiply Eq. (42) by $(q+p)_\lambda$ and Eq. (44) by q_μ. Do the same for the other two Ward identities. One obtains the constraints

$$\begin{aligned}(q+p)_\lambda D_V^{\nu\lambda}(q+p) &= q_\lambda D_V^{\nu\lambda}(q) + p_\lambda D_V^{\nu\lambda}(p) \\ &= q_\lambda D_A^{\nu\lambda}(q) + p_\lambda D_A^{\nu\lambda}(p)\,.\end{aligned} \tag{46}$$

This implies linearity in the momentum:

$$k_\lambda D_V^{\nu\lambda}(k) = k_\lambda D_A^{\nu\lambda}(k) = C^{\nu\lambda}k_\lambda\,, \tag{47}$$

where $C^{\nu\lambda}$ is momentum independent (but can depend on temperature) and is the same for the vector and axial-vector channels. By taking the Fourier transform of these relations we can find the thermal average of the equal time commutators:

$$\begin{aligned}\delta(x^0)\langle[V_a^\nu(x),V_b^0(0)]\rangle &= \delta(x^0)\langle[A_a^\nu(x),A_b^0(0)]\rangle \\ &= \delta_{ab}C^{\nu\lambda}\frac{\partial}{\partial x^\lambda}\delta(x)\,.\end{aligned} \tag{48}$$

The commutators above can be expressed in terms of the spectral densities from Eqs. (31) and (32). Taking their *difference* one obtains the finite temperature generalization of the first Weinberg sum rule:

$$\text{(I)}\qquad \int_0^\infty \frac{d\omega\,\omega}{\omega^2-\mathbf{p}^2}\left[\rho_V^L(\omega,\mathbf{p}) - \rho_A^L(\omega,\mathbf{p})\right] = 0\,. \tag{49}$$

Notice that this sum rule involves only the longitudinal spectral densities and not the transverse ones. At zero temperature the spectral densities depend only on $p^2 = s = \omega^2 - \mathbf{p}^2$. Then this equation reduces to Eq. (1) once we remember to separate out the pion piece of ρ_A^L, namely, $sF_\pi^2\delta(s)$. At finite temperature, the spectral densities in general will depend on ω and $\mathbf{p}$ separately and not just on the combination s. Then this sum rule must be satisfied at *each* value of the momentum.

At this point, Weinberg made an additional assumption in order to obtain the second sum rule [Eq. (2)]: The currents behave like free fields as $p^2 \to \infty$. He also related the difference between the vector and axial-vector propagators to the matrix element of a particular operator between the vacuum and a one-pion state. This is difficult to generalize to an ensemble average. To obtain the finite temperature generalization of the second sum rule we follow the arguments of Das, Mathur, and Okubo [24] instead.

Deleting the index V or A the explicit expressions for the propagator and the Schwinger term are

$$D^{00}(p^0,\mathbf{p}) = \mathbf{p}^2 D_L(p^0,\mathbf{p}), \tag{50}$$

$$D^{0j}(p^0,\mathbf{p}) = p^0p^jD_L(p^0,\mathbf{p}), \tag{51}$$

$$D^{ij}(p^0,\mathbf{p}) = \left(\delta^{ij} - \frac{p^ip^j}{\mathbf{p}^2}\right)D_T(p^0,\mathbf{p}) + \frac{p^ip^j}{\mathbf{p}^2}D_L'(p^0,\mathbf{p}), \tag{52}$$

where

$$D_L(p^0,\mathbf{p}) = 2i\int_0^\infty \frac{d\omega\,\omega}{\omega^2-\mathbf{p}^2}\left[\frac{\rho^L(\omega,\mathbf{p})}{\omega^2-p_0^2+i\epsilon}\right], \tag{53}$$

$$D_L'(p^0,\mathbf{p}) = 2i\int_0^\infty \frac{d\omega\,\omega^3}{\omega^2-\mathbf{p}^2}\left[\frac{\rho^L(\omega,\mathbf{p})}{\omega^2-p_0^2+i\epsilon}\right], \tag{54}$$

$$D_T(p^0,\mathbf{p}) = 2i\int_0^\infty d\omega\,\omega\left[\frac{\rho^T(\omega,\mathbf{p})}{\omega^2-p_0^2+i\epsilon}\right], \tag{55}$$

and

$$C^{00} = C^{0j} = C^{j0} = 0\,,\qquad C^{ij}(\mathbf{p}) = \delta^{ij}D_S(\mathbf{p})\,, \tag{56}$$

where

$$D_S(\mathbf{p}) = 2i\int_0^\infty \frac{d\omega\,\omega}{\omega^2-\mathbf{p}^2}\,\rho^L(\omega,\mathbf{p})\,. \tag{57}$$

The first observation we can make concerns the thermally averaged Schwinger term C. Since it is the same for the vector and the axial-vector correlators, by Eq. (47), the $D_S(\mathbf{p})$ must be the same as well. Equating them reproduces the first finite temperature sum rule [Eq. (49)].

The essence of the argument of Das, Mathur, and Okubo is that spontaneous chiral symmetry breaking is a low-energy phenomenon. At very high energy it must disappear, at least in the limit that quark masses are zero and chiral symmetry is exact. Thus the difference between the vector and axial-vector propagators should go to zero at very high energy:

$$\lim_{p^0\to\infty,\ \mathbf{p}\ \text{fixed}} \left[D_V^{\mu\nu}(p^0,\mathbf{p}) - D_A^{\mu\nu}(p^0,\mathbf{p})\right] = 0\,. \quad (58)$$

If we do this for the time-time or time-space components of the propagators, that is, for the D_L, we again reproduce the first finite temperature sum rule. Expanding to the next order in $1/p_0^2$ we obtain a finite temperature generalization of the second zero temperature sum rule, which is

$$\text{(II-L)} \qquad \int_0^\infty d\omega\,\omega\left[\rho_V^L(\omega,\mathbf{p}) - \rho_A^L(\omega,\mathbf{p})\right] = 0\,. \quad (59)$$

Like the first, this sum rule involves only the longitudinal spectral densities, and we call it II-L. Also like the first, it reduces to the original Weinberg sum rule as the temperature and/or chemical potential go to zero.

Next we consider the space-space components of the propagators. Examination of the D_L' in the infinite energy limit gives us the sum rule II-L and nothing new. Examination of the D_T in the infinite energy limit gives us another sum rule which we call II-T because it involves the transverse spectral densities,

$$\text{(II-T)} \qquad \int_0^\infty d\omega\,\omega\left[\rho_V^T(\omega,\mathbf{p}) - \rho_A^T(\omega,\mathbf{p})\right] = 0\,. \quad (60)$$

The finite temperature sum rules II-L and II-T should become degenerate at $\mathbf{p} = \mathbf{0}$ because there ought not to be any difference between longitudinal and transverse excitations at rest. The sum rule II-T also then reduces to the original second sum rule in the vacuum.

We want to emphasize that the sum rules derived in this section, I, II-L, and II-T, must be satisfied for *every* value of the momentum. Furthermore, our derivation is more general than QCD; any theory which satisfies the assumptions we made must obey these sum rules. Perhaps they would be useful in the context of models of the electroweak interactions where the Higgs particle is a composite of other fields or for technicolor theories.

B. Sum rules and the operator product expansion

Application of the OPE to finite temperature has a peculiar history. In the first papers ([6] and several later ones) the authors considered only the T dependence of average values of the same operators as at $T = 0$, the Lorentz scalars. However, the rest frame of the heat bath selects a four-vector; thus symmetric tensors should also be included. In fact, the situation is completely analogous to that in deep-inelastic scattering, for which one also has a preferred frame, that of the target. Thus, one can simply use formulae derived in that context (see discussion in [11]). The finite temperature sum rules were recently reexamined along these lines in [9].

The fact that we are not going to discuss vector and axial-vector channels as such, but only concentrate on their *difference*, brings in significant simplifications. Most operators describing the interaction of a quark with the gluonic field are chirality blind and therefore cancel. In the chiral limit, the difference appears only starting with the four-quark operators.

To leading order in the momentum the difference between the vector and axial-vector correlators is given by the OPE to be

$$\Delta D^{\mu\nu} = -i\frac{4\pi\alpha_s}{(p^2)^3}\left[p^2\langle\mathcal{O}^{\mu\nu}\rangle - p^\mu p_\alpha\langle\mathcal{O}^{\alpha\nu}\rangle - p^\nu p_\alpha\langle\mathcal{O}^{\mu\alpha}\rangle + g^{\mu\nu}p_\alpha p_\beta\langle\mathcal{O}^{\alpha\beta}\rangle\right] + O(1/p^6)\,, \quad (61)$$

where the operator $\mathcal{O}$ was defined in Sec. II. This structure first appeared in the OPE analysis of the next-twist correction to deep inelastic scattering in [25]. Observe that this quantity is transverse: $p_\mu\Delta D^{\mu\nu} = p_\nu\Delta D^{\mu\nu} = 0$. This is consistent with Eq. (47), the equality of the Schwinger terms, and therefore with the assumptions made to derive it.

First, consider ΔD^{00}. In terms of the spectral densities it is given by Eqs. (50) and (53). Expand it in inverse powers of p_0^2 in the limit that $|p_0| \to \infty$. Since the coefficients of $1/p_0^2$ and $1/p_0^4$ in Eq. (61) are zero, it must be that the corresponding coefficients in Eq. (50) are also zero. This gives us the finite temperature sum rules I and II-L immediately. We can say nothing about the next term without knowledge of higher dimension operators in the OPE, which would contribute to order $1/p_0^6$.

Next, consider ΔD^μ_μ. From Eqs. (50)–(55) it is

$$\Delta D^\mu_\mu = 2i\int_0^\infty \frac{d\omega\,\omega}{p_0^2 - \omega^2 - i\epsilon}\left[2\Delta\rho^T(\omega,\mathbf{p}) + \Delta\rho^L(\omega,\mathbf{p})\right]\,. \quad (62)$$

Again, expand in inverse powers of p_0^2. The term of order $1/p_0^2$, when combined with the just derived sum rule II-L, gives us the sum rule II-T. The term of order $1/p_0^4$ gives us the finite temperature version of sum rule III:

$$\text{(III)} \qquad \int_0^\infty d\omega\,\omega^3\left[2\Delta\rho^T(\omega,\mathbf{p}) + \Delta\rho^L(\omega,\mathbf{p})\right] = -2\pi\alpha_s\left[\langle\mathcal{O}^\mu_\mu\rangle + 2\langle\mathcal{O}^{00}\rangle\right]\,. \quad (63)$$

We can make two observations about this sum rule. In the limit of vanishing temperature, Lorentz covariance says that

$$\langle\mathcal{O}^{00}\rangle_{T=0} = \frac{1}{4}\langle\mathcal{O}^\mu_\mu\rangle_{T=0}\,. \quad (64)$$

This reduces Eq. (63) to the previously derived zero temperature sum rule Eq. (16). At finite temperature, the right side of Eq. (63) depends on T but not on $\mathbf{p}$. Therefore, the integral on the left side must be momentum independent. If the integral is known at zero momentum, for example, then it must have the same value for any momentum.

C. Low-temperature limit

As we are taking the zero quark mass limit in this work, the pion is massless below any critical temperature for chiral symmetry restoration and/or deconfinement, and thus at parametrically low temperature the heat bath is dominated by pions. In [26] the so-called Dey-Eletsky-Ioffe mixing theorem was proved, which says that, to order T^2, there is no change in the masses of vector and axial-vector mesons. What changes are the couplings to the currents. The finite temperature correlators can be described by a mixing between the vector and axial-vector $T = 0$ correlators with a temperature-dependent coefficient:

$$D_V^{\mu\nu}(p,T) = (1-\epsilon)D_V^{\mu\nu}(p,0) + \epsilon D_A^{\mu\nu}(p,0)\,, \tag{65}$$
$$D_A^{\mu\nu}(p,T) = (1-\epsilon)D_A^{\mu\nu}(p,0) + \epsilon D_V^{\mu\nu}(p,0)\,. \tag{66}$$

These are valid to first order in $\epsilon \equiv T^2/6F_\pi^2$. This implies the same mixing of the spectral densities: namely,

$$\rho_V(p^0,\mathbf{p},T) = (1-\epsilon)\rho_V(s,0) + \epsilon\rho_A(s,0)\,, \tag{67}$$
$$\rho_A(p^0,\mathbf{p},T) = (1-\epsilon)\rho_A(s,0) + \epsilon\rho_V(s,0)\,, \tag{68}$$

with the appropriate longitudinal and transverse subscripts. The temperature dependence of the pion decay coupling was thus proven to be $F_\pi^2(T) = (1-\epsilon)F_\pi^2$ for small T consistent with the prediction of chiral perturbation theory [27]. Therefore, the finite temperature sum rules I [Eq. (49)], II-L [Eq. (59)], and II-T [Eq. (60)] reduce to the original, zero temperature sum rules but with both sides of Eqs. (1) and (2) multiplied by the factor $(1-2\epsilon)$.

One may ask whether the third sum rule also obeys the Dey-Eletsky-Ioffe mixing theorem. A general formula describing the thermal average of any four-quark operator using soft pion methods was derived in [28]. For an operator $\mathcal{O}_{AB} = \bar{q}Aq\bar{q}Bq$ the expression is

$$\langle \mathcal{O}_{AB}\rangle = \frac{\langle\bar{u}u\rangle^2}{144}\left(1-\frac{T^2}{4F_\pi^2}\right)[\mathrm{Tr}(A)\mathrm{Tr}(B)-\mathrm{Tr}(AB)] - \frac{\langle\bar{u}u\rangle^2}{144}\frac{T^2}{12F_\pi^2}[\mathrm{Tr}(\gamma_5\tau^a A)\mathrm{Tr}(\gamma_5\tau^a B)-\mathrm{Tr}(\gamma_5\tau^a A\gamma_5\tau^a B)]\,, \tag{69}$$

where it is assumed that at $T = 0$ one can use the vacuum dominance approximation.

The average of the four-quark operator appearing in sum rule III gets multiplied by the correct factor $(1-2\epsilon)$, as shown by Eletsky [28]. This is not a trivial result: The average value of an arbitrarily chosen four-quark operator will not have the same temperature dependence. As already emphasized by Eletsky, a simplistic application of factorization at nonzero temperature, which would suggest the same behavior as for the quark condensate squared,

$$\langle\bar{u}u\rangle^2 = \left(1-\frac{T^2}{4F_\pi^2}\right)\langle 0|\bar{u}u|0\rangle^2, \tag{70}$$

would be wrong, and in fact violates the sum rule.

In summary, at low temperature the sum rules under discussion satisfy the Dey-Eletsky-Ioffe mixing theorem exactly.

V. SCENARIOS FOR CHIRAL SYMMETRY RESTORATION

Chiral transformations are rotations of the quark field with γ_5, and they may or may not have the $\mathrm{SU}(N_f)$ (isospin) generators. The corresponding $\mathrm{U}(1)_A$ and $\mathrm{SU}(N_f)_A$ have different fates in QCD; the former is explicitly violated by the anomaly, and the latter is broken spontaneously at low temperature and is restored at some critical temperature T_c, provided the quark mass is strictly zero as it is assumed in this paper. A minireview of the subject elaborates on this [13]. The ρ and a_1 currents are both unchanged by the $\mathrm{U}(1)_A$ transformation but are mixed under $\mathrm{SU}(N_f)_A$. Therefore, if this symmetry is restored at high temperatures, then there should be no difference between the vector and the axial-vector correlators, and the object of our considerations is zero.

In this section we speculute on exactly how this difference goes to zero with increasing temperature. Generally, one may suggest many different scenarios. Let us discuss the following three.

A. Mixing of vector and axial-vector spectral densities

The simplest scenario is that the T dependence factorizes. It means that the vector and axial-vector spectral densities mix, without changing their shape, as in the low-temperature limit considered in the previous section, only with a more general function $\epsilon(T)$. When the mixing becomes maximal, $\epsilon = 1/2$, chiral symmetry is restored.

It is amusing to see at what temperature this occurs using the lowest-order formula, $\epsilon = T^2/6F_\pi^2$. This estimate gives $T_{\text{complete mixing}} = \sqrt{3}F_\pi \approx 164$ MeV, which is indeed roughly equal to the expected critical temperature T_c.

B. Shift in meson pole masses and residues

In this scenario we assume that the ρ and a_1 mesons retain their identities and dominate the correlation function. However, their parameters change with temperature. In particular, the masses may move towards each other [10] or go to zero [7]. At T_c they become degenerate, and chiral symmetry is restored.

It is instructive then to look at the sum rules. Let us assume that vector meson dominance is a good approximation for the spectral densities and not worry about the continuum contribution for the time being. Let us focus on zero momentum for the sake of simplicity. When a pole mass is defined at finite temperature, it is usually defined as the energy of the excitation at zero momentum.

The vector spectral density is (there is no difference between longitudinal and transverse at zero momentum)

$$\rho_V(\omega) = \frac{1}{\pi}\frac{m_\rho^4}{g_\rho^2}\,\mathrm{Im}\,\frac{1}{\omega^2 - m_\rho^2 - \Pi_R^\rho(\omega) - i\Pi_I^\rho(\omega)}\,, \quad (71)$$

where Π_R^ρ and Π_I^ρ are the real and imaginary parts of the ρ self-energy at temperature T. In the narrow width approximation this becomes

$$\rho_V(\omega) = \frac{m_\rho^4}{g_\rho^2}\delta(\omega^2 - m_\rho^2 - \Pi_R^\rho(\omega))\,. \quad (72)$$

The pole mass is determined self-consistently from $m_\rho^2(T) = m_\rho^2 + \Pi_R^\rho[m_\rho(T)]$. Then the spectral density can be rewritten as

$$\rho_V(\omega) = Z_\rho(T)\frac{m_\rho^4}{g_\rho^2}\delta(\omega^2 - m_\rho^2(T))\,, \quad (73)$$

where the temperature-dependent residue is

$$Z_\rho^{-1}(T) = \left|1 - \frac{d}{d\omega^2}\Pi_R^\rho(\omega)\right|\,. \quad (74)$$

The normalization is $Z_\rho(0) = 1$. Similarly

$$\rho_A(\omega) = Z_a(T)\frac{m_{a_1}^4}{g_a^2}\delta(\omega^2 - m_{a_1}^2(T)) + Z_\pi(T)F_\pi^2\omega^2\delta\left(\omega^2\right)\,. \quad (75)$$

Substituting these spectral densities into the finite temperature sum rules I, II-L, and II-T tells us that the ρ and a_1 residues are equal,

$$Z_\rho(T) = Z_a(T)\,, \quad (76)$$

and that the pion residue is

$$Z_\pi(T) = 2Z_\rho(T)\left[\frac{m_\rho^2}{m_\rho^2(T)} - \frac{m_\rho^2}{m_{a_1}^2(T)}\right]\,. \quad (77)$$

We expect that $m_{a_1}^2(T) - m_\rho^2(T) \to 0$ as the temperature increases. Three types of behavior can be distinguished: Both the ρ and the a_1 masses decrease with T, both masses increase with T, or the ρ mass increases while the a_1 mass decreases with T. The sum rules do not appear to rule out any of these possibilities. In any case, the result is that $Z_\pi(T) \to 0$ unless $Z_\rho(T) \to \infty$, which seems rather unphysical.

C. Resonance broadening and downward shift of the continuum

As distinct from the previous scenarios, it may be that particles are not well defined as we approach a chiral symmetry restoring phase transition. That is, the imaginary part of the self-energy may become larger with increasing temperature. This broadening would also decrease the maximum peak value of the spectral density. Euphemistically, the vector and axial-vector mesons melt away. There may be also a decrease in the threshholds $E_V(T)$ and $E_A(T)$ of the continuum. The continuum would merge with the broadened particle poles to give a very broad distribution of strength in the spectral densities. The difference of spectral densities shown for $T = 0$ in Fig. 1 would become flatter and decrease everywhere towards zero, effectively restoring chiral symmetry.

Concluding this section, we say once more that the sum rules by themselves cannot of course tell which scenario is preferable. However, the sum rules can be used to significantly restrict the parametrization of the spectral densities at nonzero temperature.

VI. CONCLUSION

In this paper we studied Weinberg-type sum rules at zero and at nonzero temperature. All considerations were made in the exact chiral limit of QCD, $m_q \to 0$. In the former case, we derived a new sum rule of the Weinberg type. Although it belongs to an infinite series of sum rules, one for each type of OPE term at small distances, we think it is special in several respects. First, it is relatively simple theoretically because it is related to the VEV of a specific four-quark operator. Sum rules of higher order than the third are much more complicated. Second, it is related to the leading nonzero $\ln(\tau)$ term of the correlators, while others can be related to subleading terms which are much more difficult to single out, especially in lattice simulations.

Continuing the zero temperature analysis, we reexamined the experimental data together with all relevant sum rules. We found that, although we do not have sufficient information on the VEV of the operator for sum rule III, we still can use it, together with sum rule II, to fix the numerical value of E_A, the continuum threshold in the axial-vector channel. This essentially closes the gap in the experimental data, and allows one to test the original Weinberg sum rules without any *ad hoc* assumptions. Good agreement with the experimental values of F_π and the electromagnetic mass difference of pions provides a nontrivial consistency check of the data used.

Our finite temperature analysis consists of several different parts. First, following Weinberg's original derivation, one can find generalizations of his sum rules to nonzero temperature. Sum rule I involves only the difference of the longitudinal spectral densities, while sum rule II bifurcates into two sum rules, one involving the longitudinal spectral densities, and the other involving the transverse ones. *These sum rules must be satisfied at each value of the momentum.* These new features arise because of the appearance of a preferred reference frame at nonzero temperature. These sum rules were derived without specific reference to QCD, and so they are applicable to other theories satisfying the assumptions made. We were also able to derive them from the OPE. Furthermore, we used the OPE to obtain the finite temperature generalization of the new sum rule III, which makes specific reference to the dynamics of QCD.

We also considered very low temperatures at which chiral perturbation methods predict the general behavior of the correlators. We showed that these results are in exact agreement with all sum rules under consideration.

We would like to emphasize that the average value of the four-quark operator which appears in sum rule III is of great theoretical interest. It shows correlation between densities and currents made of left- and right-handed quarks and is another order parameter for restoration of SU(N_f) chiral symmetry. The average value in the QCD vacuum and at finite temperature can and should be studied in lattice numerical simulations. This task is facilitated by the fact that it does not have any perturbative contributions.

Finally, we speculated on possible scenarios of chiral symmetry restoration. We have no preferences among them, and only future work, including especially lattice numerical simulations, can clarify which of them (if any) is realized in QCD. However, the derived sum rules should hold in any case, thus providing some relations among parameters of the vector and axial-vector spectral densities.

Note added in proof. A detailed analysis of the sum rules and their phenomenology at zero temperature has been carried out independently by J. F. Donoghue and E. Golowich, Phys. Rev. D **49**, 1513 (1994).

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