# Analysis of the photon spectrum in inclusive $B \to X_s\gamma$ decays

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(Received 20 December 1993)

Using a combination of the operator product and heavy quark expansions, we resum the leading nonperturbative contributions to the inclusive photon spectrum in $B \to X_s\gamma$ decays. The shape of the spectrum is determined by a universal structure function, which describes the distribution of the light-cone momentum of the $b$ quark inside the $B$ meson. The moments of this function are proportional to forward matrix elements of higher-dimension operators. As a by-product, we obtain the bound $\lambda_1 < 0$ for one of the parameters of the heavy quark effective theory. An integral over the $B \to X_s\gamma$ structure function is related to the shape function that governs the fall off of the lepton spectrum close to the end point in $B \to X_u\,\ell\,\bar\nu$ decays. A measurement of the photon spectrum in rare $B$ decays can therefore help to obtain a model-independent determination of $V_{ub}$.

PACS number(s): 13.40.Hq, 12.39.Hg, 14.40.Nd

## I. INTRODUCTION

The rare decays of $B$ mesons are of great importance, since they are sensitive probes of new physics beyond the standard model (see, e.g., Refs. [1, 2]). Such processes are induced by the exchange of heavy particles, which manifests itself at low energies in the appearance of local operators multiplied by small coefficient functions. These coefficients depend on the masses and quantum numbers of the heavy particles. In the standard model, rare decays of the type $b \to s\gamma$ are induced by penguin diagrams with virtual top or charm quark exchange. Recently, the first such decay mode, $B \to K^*\gamma$, has been observed by the CLEO Collaboration. The reported branching ratio is $(4.5 \pm 1.9 \pm 0.9)\%$ [3]. The interpretation of this result is difficult, however, due to the lack of reliable theoretical methods to calculate the low-energy hadronic matrix element of the penguin operator between meson states. Existing estimates of this matrix element rely on quark models [4, 5] or QCD sum rules [6–8] and are thus model dependent.

For several reasons, one expects that the theoretical analysis is more reliable for inclusive $B \to X_s\gamma$ decays, where one sums over all possible final states containing a strange particle. Assuming quark-hadron duality, the inclusive decay rate was traditionally calculated using the free quark decay model and including short-distance corrections from virtual and real gluons [9–17]. Recently, however, it has been observed that inclusive decays of hadrons containing a heavy quark $Q$ allow for a systematic expansion in powers of $\Lambda/m_Q$, where $\Lambda$ is a characteristic low energy scale of the strong interactions [18–23]. The parton model emerges as the leading term in this QCD-based expansion, and the nonperturbative corrections to it are suppressed by a factor $\Lambda^2/m_Q^2$. The fact that there are no first-order power corrections relies on a particular definition of $m_Q$, which is provided in a natural way by requiring that there be no residual mass term for the heavy quark in the heavy quark effective theory [24, 25]. This definition is unique and can be regarded as a nonperturbative generalization of the concept of a pole mass.

The availability of a systematic expansion of the inclusive $B \to X_s\gamma$ decay rate raises the hope for a better understanding of rare decays, which is necessary to increase the sensitivity to new physics. For practical reasons, however, it is not sufficient to have a reliable calculation of the total decay rate. In fact, the distribution of the photon energy will be affected by various experimental cuts, and it is thus the spectrum $d\Gamma/dE_\gamma$ that needs to be calculated. In the free quark decay model, the photon in $b \to s\gamma$ decays is monochromatic. Corrections to this simple picture arise from two sources: Real gluon emission produces three-body final states, leading to a continuous energy spectrum. These effects have been calculated in perturbation theory [17]; they will not be discussed here. In addition, bound-state corrections in the initial state, in particular the "Fermi motion" of the $b$ quark, lead to a dispersion of the spectrum. These nonperturbative effects can Doppler shift the spectrum above the parton model end point. So far, such effects have been estimated [26] using the phenomenological model of Altarelli *et al.* [27].

In this paper, we present a more rigorous treatment of bound-state corrections to the free quark decay model. In particular, we show that QCD provides a natural framework to account for the "Fermi motion." Extending our previous analysis of the end point region of the lepton spectrum in $B \to X_u\,\ell\,\bar\nu$ decays [28], we resum the leading nonperturbative contributions to the photon spectrum in $B \to X_s\gamma$ decays to all orders in the $1/m_b$ expansion. We show that the spectrum is determined by a fundamental structure function, which describes the light-cone residual momentum distribution of the heavy quark inside the $B$ meson. Quite remarkably, an integral over this structure function is related to the shape function that governs the end point region of the lepton spectrum in $B \to X_u\,\ell\,\bar\nu$ decays. In Sec. II, we generalize the results of Refs. [19, 22] and construct the operator product expansion for the photon spectrum in $B \to X_s\gamma$ decays, including the leading nonperturbative corrections. In Sec. III, we perform a resummation of the

0556-2821/94/49(9)/4623(11)/$06.00  

most singular terms in the photon spectrum to all orders in $1/m_b$. The moments of the spectrum are related to forward matrix elements of local, higher-dimension operators in the heavy quark effective theory. We show that the characteristic width of the spectrum is determined by the kinetic energy of the $b$ quark inside the $B$ meson. From this relation, we derive the bound $\lambda_1 < 0$ for one of the parameters of the effective theory. In Sec. IV, we illustrate our results using a toy model, which is a simplified version of the approach of Ref. [27]. In Sec. V, we derive the relation between the photon spectrum and the $b$-quark structure function of the $B$ meson, and we relate the Fourier transform of the spectrum to the forward matrix element of a gauge-invariant, bilocal operator. Section VI is devoted to a discussion of the connection between rare $B$ decays and inclusive semileptonic decays. We show that the shape function [28], which describes the falloff of the lepton spectrum close to the end point in $B \to X_u\,\ell\,\bar\nu$ decays, is given by an integral over the photon spectrum in rare $B \to X_s\,\gamma$ decays, up to corrections of order $1/m_b$. This connection may help to obtain a model-independent determination of $V_{ub}$. In Sec. VII, we summarize our results and give some conclusions.

## II. OPERATOR PRODUCT EXPANSION

In this section, we discuss the application of the operator product expansion to the inclusive rare decays $B \to X_s\,\gamma$. We derive expressions for the photon spectrum and the total decay rate, to order $1/m_b^2$ in the heavy quark expansion and to leading logarithmic order in $\alpha_s(m_b)$. In the limit where the mass of the strange quark is neglected, the total decay rate and the average photon energy have been calculated in Refs. [19, 22]. We will generalize the results presented there.

In leading logarithmic approximation, the rare decays of interest are mediated by an effective Hamiltonian containing a local penguin operator:

$$\begin{aligned}\mathcal{H}_{\rm eff} = &-\frac{4G_F}{\sqrt{2}}\,V_{tb}V_{ts}^*\,c_7(m_b)\\ &\times\frac{e}{16\pi^2}\,\bar s\,\sigma^{\mu\nu}\,(m_b\,P_R + m_s\,P_L)\,b\,F_{\mu\nu}\,,\end{aligned}\qquad(1)$$

where $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$ are left- and right-handed projection operators, and $F_{\mu\nu}$ is the electromagnetic field strength tensor. The Wilson coefficient $c_7(m_b)$ describes the evolution from high-energy scales $\mu \sim m_t$ or $m_W$ to low-energy scales $\mu \sim m_b$ [9–17]. It is sensitive to the mass of the top quark and, more generally, to any kind of new physics beyond the standard model.

In terms of the effective Hamiltonian, the inclusive differential decay rate is given by

$$\begin{aligned}d\Gamma = &\frac{d^3\mathbf{p}_\gamma}{(2\pi)^3\,2E_\gamma}\sum_{X_s,\rm pol}(2\pi)^4\,\delta^4(m_B v - p_\gamma - p_X)\\ &\times\Big|\langle X_s(p_X)\,\gamma(p_\gamma)|\,\mathcal{H}_{\rm eff}\,|B(v)\rangle\Big|^2\,,\end{aligned}\qquad(2)$$

where we sum over the two transverse polarization states of the photon. We use a mass-independent normalization of states such that

$$\langle B(v)|B(v)\rangle = v^0\,(2\pi)^3\,\delta^3(\mathbf{0})\,,\qquad(3)$$

where $v$ is the four-velocity of the $B$ meson. The decay rate can be written in terms of the imaginary part of a correlator of two local currents, which contains all dependence on hadronic dynamics. We define

$$\begin{aligned}T(v,p_\gamma) = &-i\int dx\,e^{i(m_b v - p_\gamma)\cdot x}\\ &\times\langle B(v)|\,\mathrm{T}\left\{J^\mu(x), J_\mu^\dagger(0)\right\}|B(v)\rangle\,,\end{aligned}\qquad(4)$$

where

$$\begin{aligned}J^\mu(x) &= \bar b_v(x)\,[\gamma^\mu,\not p_\gamma]\,(m_b\,P_L + m_s\,P_R)\,s(x)\,,\\ b_v(x) &= e^{i m_b v\cdot x}\,b(x)\,.\end{aligned}\qquad(5)$$

This leads to

$$d\Gamma = \frac{d^3\mathbf{p}_\gamma}{(2\pi)^3\,2E_\gamma}\,\frac{G_F^2\,\alpha}{8\pi^3}\,|\,V_{tb}V_{ts}^*|^2\,|c_7(m_b)|^2\,\mathrm{Im}\,T(v,p_\gamma)\,.\qquad(6)$$

Note that the Fourier components of the rescaled heavy quark field $b_v(x)$ in (5) contain the "residual" momentum $k = p_b - m_b v$. In contrast with the full heavy quark momentum $p_b$, the residual momentum is of order $\Lambda$. It is then appropriate to construct an expansion in powers of $k/m_b$.

To this end, it is convenient to introduce dimensionless variables

$$\hat p = \frac{p_\gamma}{m_b}\,,\qquad \hat m = \sqrt{\rho} = \frac{m_s}{m_b}\,,\qquad y = 2v\cdot\hat p = \frac{2E_\gamma}{m_b}\,,\qquad(7)$$

where $E_\gamma$ denotes the photon energy in the rest frame of the $B$ meson. Since $v^2 = 1$ and $p_\gamma^2 = 0$, the function $T(v,p_\gamma)$ only depends on the kinematic variable $y$. The correlator $T(y)$ is analytic in the complex $y$ plane, with discontinuities on the real axis. The physical region corresponding to the decay $B \to X_s\,\gamma$ is

$$0 \le y \le \frac{m_B}{m_b}\left(1 - \frac{m_{K^*}^2}{m_B^2}\right) \simeq 1.09\,,\qquad(8)$$

where we have assumed $m_b \simeq 4.7$ GeV for the purpose of illustration. In addition, there is a cut starting at $y \simeq 4.16$ corresponding to the process $\gamma + B \to X_{sbb}$, where $X_{sbb}$ contains two $b$ quarks and an $s$ quark. This unphysical cut is separated from the physical one by a large energy gap $\Delta E_\gamma = m_B\,(1 + m_{K^*}/m_B)^2 \simeq 7.2$ GeV. Because of this analytic structure, phase-space integrals of $T(y)$ with smooth weight functions can be deformed from the physical region into contour integrals far away from the physical singularities. It is then possible to construct an operator product expansion of the correlator in terms of local operators $\mathcal{O}_i$, which contain the $b$-quark fields and have dimension $d \ge 3$ [18–23]. These operators are multiplied by short-distance coefficient functions $C_i(y)$, which can be computed in perturbation theory using free quark states. The leading contributions in $\alpha_s$

come from a tree diagram with an intermediate $s$ quark carrying the momentum $(p_b - p_\gamma) = (m_b v + k - p_\gamma)$. The operator product expansion is obtained by replacing the residual momentum $k$ by a covariant derivative $iD$. This gives the propagator in the background field of the light degrees of freedom in the decaying $B$ meson. Next, one expands the propagator in powers of $iD/m_b$. This gives

$$\frac{1}{m_b(\not{v} - \not{\hat{p}} - \hat{m} + i\epsilon) + i\not{D}} = \frac{\not{v} - \not{\hat{p}} + \hat{m}}{m_b\,\Delta} \sum_{n=0}^{\infty} \left[ -i\not{D}\,\frac{(\not{v} - \not{\hat{p}} + \hat{m})}{m_b\,\Delta} \right]^n , \quad (9)$$

where

$$\Delta = (v - \hat{p})^2 - \hat{m}^2 + i\epsilon = 1 - y - \rho + i\epsilon . \quad (10)$$

Once the Wilson coefficients have been determined, one has to evaluate the forward matrix elements of the local operators $\mathcal{O}_i$ between $B$-meson states. The leading operators have dimension 3 and can be related to matrix elements of vector and axial vector currents. The vector current matrix element is normalized by current conservation,

$$\langle B(v) |\, \bar{b}_v\, \gamma^\mu\, b_v\, | B(v) \rangle = v^\mu , \quad (11)$$

whereas the matrix element of the axial vector current vanishes by parity invariance. These leading-order contributions reproduce the parton model. The nonperturbative corrections to it are described by the matrix elements of higher-dimension operators. They can be evaluated using the powerful formalism of the heavy quark effective theory [29]. The heavy quark field $b_v$ is split into "large" and "small" two-component spinors $h_v = \frac{1}{2}(1+\not{v})\, b_v$ and $H_v = \frac{1}{2}(1-\not{v})\, b_v$, and the field $H_v$ is integrated out to obtain an effective Lagrangian [30–32]. Any operator in the full theory has an expansion in terms of operators in the effective theory, which only contain the field $h_v$. Using techniques [21, 22, 33] that are standard by now, we obtain, to leading order in $1/m_b$,

$$\langle B(v) |\, \bar{b}_v\, \Gamma\, iD^\mu\, b_v\, | B(v) \rangle = \frac{\lambda_1 + 3\lambda_2}{12 m_b}\, \mathrm{Tr} \left\{ \Gamma\, (\gamma^\mu + v^\mu - 5 v^\mu P_+) \right\} ,$$

$$\langle B(v) |\, \bar{b}_v\, \Gamma\, iD^\mu\, iD^\nu\, b_v\, | B(v) \rangle = \frac{\lambda_1}{6}\, (g^{\mu\nu} - v^\mu v^\nu)\, \mathrm{Tr} \left\{ \Gamma\, P_+ \right\} + \frac{\lambda_2}{4}\, \mathrm{Tr} \left\{ \Gamma\, P_+\, i\sigma^{\mu\nu} P_+ \right\} , \quad (12)$$

where $P_+ = \frac{1}{2}(1 + \not{v})$, and $\Gamma$ denotes an arbitrary combination of Dirac matrices. The low-energy parameters $\lambda_1$ and $\lambda_2$ are related to the kinetic energy $K_b$ of the heavy quark inside the $B$ meson, and to the mass splitting between $B$ and $B^*$ mesons. They are defined as [33]

$$K_b = -\frac{\lambda_1}{2 m_b} , \qquad m_{B^*}^2 - m_B^2 = 4 \lambda_2 . \quad (13)$$

Using these results, we find

$$T(y) = -\frac{4 m_b^3\, (1+\rho)\, y^2}{\Delta} \left\{ 1 - \frac{\lambda_1}{6 m_b^2} \left( \frac{2y^2}{\Delta^2} + \frac{7y}{\Delta} + 5 \right) - \frac{\lambda_2}{2 m_b^2} \left( \frac{5y - 8}{\Delta} + 5 \right) + O(m_b^{-3}) \right\} , \quad (14)$$

with $\Delta$ as given in (10). In the limit $\rho = 0$, this agrees with Ref. [22]. For the inclusive photon spectrum, we obtain from (6)

$$\frac{d\Gamma}{dy} = \frac{G_F^2\, \alpha\, m_b^5}{32 \pi^4}\, |\, V_{tb} V_{ts}^* |^2\, |c_7(m_b)|^2 \times (1+\rho)(1-\rho)^3\, \eta_b\, s(y, \rho) , \quad (15)$$

where

$$\eta_b = 1 + \frac{\lambda_1 - 9\kappa\lambda_2}{2 m_b^2} + O(m_b^{-3}) , \qquad \kappa = \frac{3 + 5\rho}{3 - 3\rho} . \quad (16)$$

The spectral function $s(y, \rho)$ is given by

$$s(y,\rho) = \delta(1 - y - \rho) - (1-\rho)\, \frac{\lambda_1 + 3\kappa\lambda_2}{2 m_b^2}\, \delta'(1 - y - \rho) - (1-\rho)^2\, \frac{\lambda_1}{6 m_b^2}\, \delta''(1 - y - \rho) + O(m_b^{-3}) . \quad (17)$$

The interpretation of the singular structure of this function is the main subject of this paper. Recall that the operator product expansion, which leads to (15), is only justified when the spectrum is integrated with a smooth weight function. Hence, one should understand the singular expression (17) in the sense of distributions. Integrated quantities such as the total decay rate and the average photon energy obey a well-defined $1/m_b$ expansion. We find

$$\Gamma = \frac{G_F^2\, \alpha\, m_b^5}{32 \pi^4}\, |\, V_{tb} V_{ts}^* |^2\, |c_7(m_b)|^2\, (1+\rho)(1-\rho)^3\, \eta_b , \quad (18)$$

$$\langle\, y\, \rangle = \frac{2}{m_b}\, \langle\, E_\gamma\, \rangle = (1-\rho) \left\{ 1 - \frac{\lambda_1 + 3\kappa\lambda_2}{2 m_b^2} + O(m_b^{-3}) \right\} .$$

In the limit $\rho = 0$, we confirm the results of Refs. [19, 22].

In order to obtain an estimate of the magnitude of the nonperturbative corrections, we use the quark masses $m_b = 4.7$ GeV and $m_s = 0.2$ GeV, corresponding to $\rho \simeq 2 \times 10^{-3}$. From the observed value of the $B^*$-$B$ mass splitting, one obtains $\lambda_2 \simeq 0.12\ \mathrm{GeV}^2$. The parameter $\lambda_1$ is not directly related to an observable. The field-theory analogue of the virial theorem relates the kinetic energy of a heavy quark inside a hadron (and thus $\lambda_1$) to a matrix element of the gluon field strength tensor [34]. This theorem makes explicit an "intrinsic smallness" of $\lambda_1$, which was not taken into account in existing QCD sum rule calculations of this parameter [35–37]. As a consequence, we expect that $(-\lambda_1)$ is smaller than predicted in these analyses. Here we shall use the range $-\lambda_1 = 0.1$–$0.3\ \mathrm{GeV}^2$. We then obtain $\eta_b \simeq 0.97$, corresponding to a 3% decrease of the parton model decay rate. The correction to the average photon energy is below 1%.

## III. RESUMMATION OF THE MOST SINGULAR TERMS

The numerical analysis presented above shows that the nonperturbative corrections to integrated quantities such as the total decay rate are very small. Were it just for these corrections, one could say that the most important result of the formalism presented in the previous section would be to give a theoretical justification for the parton model. However, in this section we will show that much more interesting information can be extracted from this analysis. This information is encoded in the coefficients of the singular terms in the spectral function $s(y,\rho)$.

The singularities at $y = 1-\rho$ in (17) arise from the fact that the operator product expansion becomes singular when the $s$-quark propagator is almost on shell [22]. Although it is legitimate to evaluate phase-space averages of smooth functions using the singular theoretical expression, one cannot trust the shape of the spectrum as given in (17). The true spectrum will be different. A similar situation is encountered in inclusive semileptonic $B \to X_u\,\ell\,\bar\nu$ decays, where the lepton spectrum obtained from the operator product expansion becomes singular when the lepton energy approaches the parton model end point [19–21]. In Ref. [28], we have suggested that the deviation of the true lepton spectrum from the parton model prediction could be described by a shape function $S(y)$, which has a support only in a small region close to the end point. The singularities in the theoretical spectrum can be identified with the first few terms in a moment expansion of the shape function.

We can adopt a similar point of view in the case of $B \to X_s\,\gamma$ decays. Let us identify the function $s(y,\rho)$ in (15) with the *physical* photon spectrum subject to the normalization condition

$$\int_0^\infty dy\, s(y,\rho) = 1\,, \tag{19}$$

so that the total rate is given by the first equation in (18). We will show below that the characteristic width $\sigma_y$ of the spectrum is proportional to $1/m_b$. When integrated with a smooth function that is slowly varying on scales of order $1/m_b$, the spectral function $s(y,\rho)$ can be replaced by a singular expansion [28]:

$$s(y,\rho) = \sum_{n=0}^\infty \frac{M_n(\rho)}{n!}\,\delta^{(n)}(1-y-\rho)\,. \tag{20}$$

The moments $M_n(\rho)$ are defined as

$$M_n(\rho) = \int_0^\infty dy\,(y-1+\rho)^n\, s(y,\rho)\,. \tag{21}$$

By definition, $M_0(\rho)=1$. We can now identify the singular expression in (17) with the first few terms in this expansion and relate the moments $M_n(\rho)$ to nonperturbative parameters. Neglecting terms of order $1/m_b^3$, we obtain

$$M_1(\rho) = -(1-\rho)\,\frac{\lambda_1+3\kappa\lambda_2}{2m_b^2}\,,$$

$$M_2(\rho) = -(1-\rho)^2\,\frac{\lambda_1}{3m_b^2} \equiv \sigma_y^2\,. \tag{22}$$

From dimensional analysis, it follows that the moments obey an expansion of the form

$$M_n(\rho) = \frac{a_n(\rho)}{m_b^n} + \frac{b_n(\rho)}{m_b^{n+1}} + \cdots\,, \tag{23}$$

with coefficients $a_n(\rho)$ and $b_n(\rho)$ that are independent of $m_b$ (up to logarithms arising from radiative corrections). Hence, the QCD prediction that $M_1(\rho)$ is of order $1/m_b^2$ indicates a nontrivial cancellation: The shift in the average value of $y$ due to bound-state effects is of order $1/m_b^2$, corresponding to a shift of order $\Lambda^2/m_b$ in the average photon energy. Naively, one would expect this shift to be of order $\Lambda$.

The second moment is a measure of the width of the photon spectrum. As stated above, we find that $\sigma_y \propto 1/m_b$. In Ref. [28], we showed that the quantity $\sigma_y$ evaluated for $\rho=0$ describes the width of the end point region of the lepton spectrum in $B\to X_u\,\ell\,\bar\nu$ decays. The connection between these two cases will be discussed in more detail in Sec. VI. Note that the width is determined by the parameter $(-\lambda_1)$, which is proportional to the kinetic energy of the $b$ quark inside the $B$ meson [see (13)]. Since, by definition, the second moment is positive, we obtain the bound $\lambda_1<0$. Although this result is certainly not surprising, it is not trivial, since operator renormalizations could spoil the positive definiteness of the kinetic energy operator that defines $(-\lambda_1)$ [38].

For an understanding of the properties of the spectrum close to the end point, it is not sufficient to truncate the $1/m_b$ expansion at order $1/m_b^2$ [28]. What is relevant are the rescaled moments

$$\int_0^\infty dE_\gamma\,\Big[2E_\gamma - m_b\,(1-\rho)\Big]^n\, s(E_\gamma,\rho) = a_n(\rho) + \frac{b_n(\rho)}{m_b} + \cdots\,, \tag{24}$$

where $s(E_\gamma,\rho) \equiv (2/m_b)\,s(y,\rho)$, such that $\int_0^\infty dE_\gamma\, s(E_\gamma,\rho)=1$. These moments are all equally important in the limit $m_b\to\infty$. Hence, it is necessary to resum the operator product expansion. However, with the exception of the first moment, for which the coefficient $a_1(\rho)$ vanishes, one may argue that it is a good approximation to keep the leading coefficient $a_n(\rho)$ for each moment. The corrections involving $b_n(\rho)$ change the moments by small amounts.

The aim is thus to construct a partial resummation of the operator product expansion, in which one keeps the leading term in each moment but neglects $1/m_b$ corrections. A crucial observation is that, at any order in the $1/m_b$ expansion, the coefficient $a_n(\rho)$ receives contributions only from the most singular terms in the theoretical expression for $s(y,\rho)$. For instance, at order $1/m_b^2$ the coefficient of the $\delta''$ function in (17) determines $a_2(\rho)$; the coefficient of the $\delta'$ function, however, determines $b_1(\rho)$. A resummation of the most singular terms can be constructed by using the following alternative way of writing the $s$-quark propagator in (9):

$$\frac{1}{m_b(\not{p} - \not{\hat{q}} - \hat{m} + i\epsilon) + i\not{D}} = \frac{\not{p} - \not{\hat{q}} + \hat{m}}{m_b\Delta + 2(v - \hat{p}) \cdot iD} \sum_{n=0}^{\infty} \left[ (\not{p} - \not{\hat{q}} - \hat{m})\, i\not{D}\, \frac{1}{m_b\Delta + 2(v - \hat{p}) \cdot iD} \right]^n . \tag{25}$$

Note that all terms but the first one are multiplied by a factor $\Delta = (\not{p}-\not{\hat{q}}+\hat{m})(\not{p}-\not{\hat{q}}-\hat{m})$. Since $\Delta$ vanishes at the end point, it follows that the first term is more singular than the other ones. Hence, the leading singularities can be resummed by using the replacement

$$\frac{1}{m_b(\not{p} - \not{\hat{q}} - \hat{m} + i\epsilon) + i\not{D}} \to \frac{\not{p} - \not{\hat{q}} + \hat{m}}{m_b\Delta + 2(v - \hat{p}) \cdot iD} \tag{26}$$

for the $s$-quark propagator. The imaginary part of this expression is given by a $\delta$ function, and it is straightforward to find that

$$s(y, \rho) = \left\langle \delta\left[1 - y - \rho + \frac{2}{m_b}(v - \hat{p}) \cdot iD\right] \right\rangle + \text{less singular terms}, \tag{27}$$

where we define the expectation value of an operator $\mathcal{O}$ as

$$\langle\, \mathcal{O}\, \rangle = \frac{\langle B(v)|\, \bar{h}_v\, \mathcal{O}\, h_v\, |B(v)\rangle}{\langle B(v)|\, \bar{h}_v\, h_v\, |B(v)\rangle} . \tag{28}$$

Here, $h_v$ is the velocity-dependent heavy quark field in the heavy quark effective theory [31, 32], and the states are the eigenstates of the corresponding effective Lagrangian.[1] Equation (27) is a formal definition of the spectral function, which is valid to all orders in the $1/m_b$ expansion. The "less singular terms" omitted here do not contribute to the leading-order coefficients in the expansion of the moments. Expanding our result in powers of $1/m_b$, and comparing with (20), we find

$$a_n(\rho) = \left\langle \left[2(v - \hat{p}) \cdot iD\right]^n \right\rangle\Big|_{y=1-\rho} . \tag{29}$$

In order to extract further information from this relation, it is necessary to investigate the structure of forward matrix elements of higher-dimension operators in the heavy quark effective theory. Using the equation of motion, $iv \cdot D\, h_v = 0$, one can show that [28]

$$\begin{aligned}
&\langle\, iD^\mu \,\rangle = 0 , \\
&\langle\, iD^\mu\, iD^\nu \,\rangle = A_2\, (v^\mu v^\nu - g^{\mu\nu}) , \\
&\langle\, iD^\mu\, iD^\nu\, iD^\alpha \,\rangle = A_3\, (v^\mu v^\alpha - g^{\mu\alpha})\, v^\nu , \\
&\langle\, iD^{\mu_1} \cdots iD^{\mu_n} \rangle = A_n\, v^{\mu_1} \cdots v^{\mu_n} + \text{ terms with } g^{\mu_i\mu_j} ,
\end{aligned} \tag{30}$$

where $A_2 = -\lambda_1/3$ [33]. For $n \geq 4$, the matrix elements can no longer be parametrized by a single parameter $A_n$. However, terms involving the metric tensor give only small contributions, of order $\rho$, to the coefficients $a_n(\rho)$ in (29). Hence, we obtain $a_0(\rho) = 1$, $a_1(\rho) = 0$, as well as

$$\begin{aligned}
&a_2(\rho) = -(1 - \rho)^2\, \frac{\lambda_1}{3} , \\
&a_3(\rho) = (1 - \rho)^2 (1 + \rho)\, A_3 , \\
&a_n(\rho) = A_n + O(\rho) , \quad n \geq 4 .
\end{aligned} \tag{31}$$

Let us summarize the main results of this section: Apart from radiative corrections, the inclusive photon spectrum in $B \to X_s\gamma$ decays can be described by a spectral function $s(y, \rho)$, which is a genuinely nonperturbative form factor that accounts for bound-state effects in the decaying meson. The moments of this function obey a well-defined $1/m_b$ expansion. The leading terms in this expansion are related to forward matrix elements of higher-dimension operators in the heavy quark effective theory. In the limit $\rho = 0$, these matrix elements are described by a set of fundamental parameters $A_n$. Since the even moments of the spectral function are positive, it follows that $A_{2n} > 0$. In particular, this gives the bound $\lambda_1 < 0$.

## IV. TOY MODEL

Before we discuss in more detail the physics of the results derived in the previous section, we find it instructive to illustrate them in the framework of a toy model, which is a simplified version of the phenomenological approach of Altarelli *et al.* (ACM) [27]. In the ACM model, the validity of the parton model is assumed, and bound-state effects are incorporated by assigning a momentum distribution $\phi(|\mathbf{p}_b|)$ to the heavy quark. In addition, the heavy quark mass is treated as a momentum-dependent parameter $m_b(|\mathbf{p}_b|)$. For simplicity, we shall not consider this aspect of the model. It is then appropriate to replace the covariant derivative by the spatial components of the heavy quark momentum $\mathbf{p}_b$. The gluon field and the time component of the covariant derivative are neglected. Accordingly, in the rest frame of the $B$ meson, one makes the replacement

$$2(v - \hat{p}) \cdot iD\Big|_{y=1-\rho} \to (1 - \rho)\, p_\parallel , \tag{32}$$

where $p_\parallel = \mathbf{p}_b \cdot \mathbf{p}_\gamma / |\mathbf{p}_\gamma|$ denotes the component of the $b$-quark momentum in the photon direction. The matrix elements in (27) and (29) are replaced by integrals over the momentum distribution of the heavy quark. In the ACM model, one assumes a Gaussian momentum distribution:

$$\phi(|\mathbf{p}_b|) = \frac{4}{\sqrt{\pi}\, p_F^3}\, \exp\left( -\frac{|\mathbf{p}_b|^2}{p_F^2} \right) , \tag{33}$$

[1] It is sufficient to evaluate the matrix elements in the effective theory, since we are not interested in the higher-order corrections in (23).

where $p_F$ is the Fermi momentum. It is straightforward to calculate the hadronic matrix elements $a_n(\rho)$ in this toy model. We find

$$a_n^{\rm toy}(\rho) = (1-\rho)^n \langle p_\parallel^n \rangle = \begin{cases} (1-\rho)^n \dfrac{(n-1)!!}{2^{n/2}}\, p_F^n \,, & n \text{ even}, \\ 0 & , \; n \text{ odd}. \end{cases} \tag{34}$$

Comparing (34) with (31) for $n = 2$, one obtains the relation $-\lambda_1 = \frac{3}{2} p_F^2$ between the low-energy parameter $\lambda_1$ and the Fermi momentum. Finally, it is possible to calculate the leading term in the spectral function $s(y,\rho)$ in (27). The result is again a Gaussian distribution:

$$s_{\rm toy}(y,\rho) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left\{ -\frac{(y-1+\rho)^2}{2\sigma_y^2} \right\}. \tag{35}$$

The width $\sigma_y$ has been defined in (22).

At this point, we have to stress that our toy model is presented for pedagogical purposes only; we do not claim that it provides a realistic description of the spectral function. In fact, this simple model of the "Fermi motion" is inconsistent with QCD. Note that the vanishing of the odd moments of the distribution function, which implies the symmetry of the spectral function around $y = 1-\rho$, is a consequence of rotational invariance. As such, it is unavoidable in a model where the time component of the heavy quark momentum is neglected. In QCD, however, there is no reason why any of the coefficients $a_n$ (except $a_1$) should vanish. Hence, we expect that the physical spectral function is asymmetric. In the following section, we shall discuss in more detail the correct generalization of the toy model in the context of QCD. This will lead us to the concept of a universal light-cone structure function, which replaces the distribution function $\phi(|\mathbf{p}_b|)$.

## V. LIGHT-CONE STRUCTURE FUNCTION

The alert reader will have realized the close analogy of our discussion in Sec. III with deep inelastic scattering. In this section, we will exploit this relationship. From now on, we will neglect the mass of the strange quark and set $\rho = 0$. We expect this to be an excellent approximation. For instance, the coefficients $a_2(\rho)$ and $a_3(\rho)$ in (31) change by less than 1.5% when $m_s$ is varied between 0 and 0.4 GeV.

For $\rho = 0$, the vector

$$n_\mu = 2(v-\hat{p})_\mu \Big|_{y=1} \tag{36}$$

is a null vector on the forward light cone satisfying $n^2 = 0$ and $n\cdot v = 1$. Let us denote the scalar product of a four-vector $p$ with $n$ by $n\cdot p \equiv p_+$. In the rest frame of the $B$ meson, we are free to choose $n_\mu = (1,0,0,1)$, such that $p_+ = p^0 + p^3$. Moreover, we can simplify expressions by using the light-cone gauge (LCG) $n\cdot A = 0$. From (29) and (31), it then follows that

$$a_n(0) = A_n = \langle (iD_+)^n \rangle \stackrel{\rm LCG}{=} \langle (i\partial_+)^n \rangle . \tag{37}$$

This is the correct generalization of (34). Note that $i\partial_+$ is the operator corresponding to the light-cone residual momentum $k_+$ of the $b$ quark in the $B$ meson.

Using this notation, the spectral function $s(y) \equiv s(y,0)$ in (27) takes the form

$$s(y) = \left\langle \delta\Big(1 - y + \frac{iD_+}{m_b}\Big) \right\rangle + \text{ less singular terms} = \int dk_+\, \delta\Big(1-y+\frac{k_+}{m_b}\Big) \Big\{ f(k_+) + O(1/m_b) \Big\} , \tag{38}$$

where

$$f(k_+) = \langle \delta(iD_+ - k_+) \rangle = \frac{\langle B(v)|\, \bar{h}_v\, \delta(iD_+ - k_+)\, h_v\, |B(v)\rangle}{\langle B(v)|\, \bar{h}_v\, h_v\, |B(v)\rangle} \tag{39}$$

is a universal *structure function*, which determines the probability to find a $b$ quark with light-cone residual momentum $k_+$ inside the $B$ meson. Since this function is defined in terms of a matrix element in the heavy quark effective theory, it is independent of the $b$-quark mass. The moments of $f(k_+)$ are given directly in terms of the hadronic matrix elements $A_n$ defined in (30):

$$A_n = \int dk_+\, k_+^n\, f(k_+) . \tag{40}$$

The corresponding moment expansion reads

$$f(k_+) = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, A_n\, \delta^{(n)}(k_+) = \delta(k_+) - \frac{\lambda_1}{6}\, \delta''(k_+) - \frac{A_3}{6}\, \delta'''(k_+) + \cdots . \tag{41}$$

Equation (38) shows very nicely the structure of the $1/m_b$ expansion proposed in this paper: The photon spectrum is a convolution of the free quark decay spectrum (i.e., a $\delta$ function) with a nonperturbative distribution function. The leading term in the $1/m_b$ expansion of this function is given by the universal structure function $f(k_+)$. This is the correct generalization of the toy model discussed in the previous section. The important distinction is that the light-cone momentum $k_+$ contains the time component of the residual momentum $k$. Thus, the odd moments of the structure function are not forced to vanish by rotational invariance, and $f(k_+)$ is in general not symmetric around $k_+ = 0$.

One can consider (38) as the recipe for a systematic and consistent implementation of the leading bound-state corrections, even when one goes beyond the leading order in perturbation theory. Needless to say, however, the structure function $f(k_+)$ cannot be calculated from first principles. One option is to extract this universal function from experimental data. The photon spectrum in rare $B \to X_s\,\gamma$ decays is an ideal place for this purpose. Alternatively, one may try to obtain a QCD-based prediction for $f(k_+)$ using nonperturbative techniques such as lattice gauge theory or QCD sum rules. To this end, it may be useful to relate the structure function to a forward matrix element of a gauge-invariant, bilocal operator. Let us define the Fourier transform of $f(k_+)$ as

$$f(t) = \int dk_+\, f(k_+)\, e^{-ik_+ t}\,. \tag{42}$$

It then follows that

$$f(t) \equiv \frac{\langle B(v)|\,\bar h_v(0)\, P\, \exp[-i\int_0^z dx_\mu A^\mu(x)]\, h_v(z)\,|B(v)\rangle}{\langle B(v)|\,\bar h_v(0)\, h_v(0)\,|B(v)\rangle}\,, \tag{43}$$

where $z = t\,n$ is a four-vector on the light cone satisfying $z^2 = 0$ and $z \cdot v = t$, $P$ denotes path ordering, and the integral is along a straight line. In light-cone gauge, the phase factor equals unity. The function $f(t)$ describes the spatial distribution of the $b$ quark inside the $B$ meson.

Finally, we note that the residual momentum structure function $f(k_+)$ obeys a simple relation to the usual structure function $b_B(x)$, which determines the probability to find in the $B$ meson a $b$ quark with total light-cone momentum fraction $x$. Using that $p_b = m_b v + k$ and $n \cdot v = 1$, we have

$$x \equiv \frac{(p_b)_+}{(p_B)_+} = \frac{m_b + k_+}{m_B} \tag{44}$$

and hence

$$b_B(x)\, dx = \Big\{f(k_+) + O(1/m_b)\Big\}\, dk_+\,, \qquad k_+ = m_B\, x - m_b\,. \tag{45}$$

From the requirement that $0 \le x \le 1$, it follows that the allowed range for the light-cone residual momentum is $-m_b \le k_+ \le m_B - m_b$. In the limit $m_b \to \infty$, this becomes

$$-\infty < k_+ \le \bar\Lambda\,, \tag{46}$$

where $\bar\Lambda$ denotes the asymptotic value of the mass difference $m_B - m_b$ and can be identified with the effective mass of the light degrees of freedom in the $B$ meson [24, 25]. An interesting consequence of (46) is that it determines the kinematic end point for the photon energy in $B \to X_s\,\gamma$ decays. From (38), we find that

$$y_{\max} = 1 + \frac{\bar\Lambda}{m_b} + O(m_b^{-2}) = \frac{m_B}{m_b} + O(m_b^{-2})\,. \tag{47}$$

This is in fact consistent with the physical endpoint given in (8).

Knowing the moments of $f(k_+)$, it is straightforward to calculate the moments of the structure function $b_B(x)$. We find

$$\int_0^1 dx\, (1-x)^n\, b_B(x) = \frac{1}{m_B^n} \sum_{k=0}^n (-1)^k \binom{n}{k} A_k\, \bar\Lambda^{n-k} + O(m_B^{-n-1})\,. \tag{48}$$

In particular, this leads to the sum rules

$$\begin{aligned} &\int_0^1 dx\, b_B(x) = 1\,, \\ &\int_0^1 dx\, (1-x)\, b_B(x) = \frac{\bar\Lambda}{m_B} + O(m_B^{-2})\,, \\ &\int_0^1 dx\, (1-x)^2\, b_B(x) = \frac{1}{m_B^2}\left(\bar\Lambda^2 - \frac{\lambda_1}{3}\right) + O(m_B^{-3})\,. \end{aligned} \tag{49}$$

The second relation has been derived previously, in a different context, in Refs. [39,40]. Note that it implies the lower bound $\bar\Lambda > 0$, which in view of the recent criticism [38] of the Guralnik-Manohar bound $\bar\Lambda > 237$ MeV [41] seems less trivial than one may think.

## VI. RELATION TO $B \to X_u\,\ell\,\bar\nu$ DECAYS

In this section, we discuss an interesting relation between the leading nonperturbative corrections in rare and semileptonic inclusive $B$ decays. This connection may help to obtain a model-independent determination of the element $V_{ub}$ of the Kobayashi-Maskawa matrix. In the limit $m_u = m_\ell = 0$, the inclusive lepton spectrum in $B \to X_u\,\ell\,\bar\nu$ decays can be written as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y} = \frac{G_F^2\,|V_{ub}|^2}{96\pi^3}\, m_b^5 \Big[F(y)\,\Theta(1-y) + F(1)\,S(y)\Big]\,, \qquad y = \frac{2E_\ell}{m_b}\,. \tag{50}$$

In this case, the kinematic variable $y$ denotes the rescaled lepton energy, and $F(y)$ is a slowly varying function of $y$. Apart from small nonperturbative corrections of order $1/m_b^2$ [19–21], it is given by parton model kinematics: $F(y) \simeq y^2(3-2y)$. Close to the end point, one can replace $F(y) \simeq F(1) \simeq 1$, up to corrections of order $1/m_b^2$. The shape function $S(y)$, on the other hand, is a rapidly varying, genuinely nonperturbative object [28]. It is nonzero only in a small region around the end point of the spectrum. Using the resummation technique developed in Sec. III, we find that

$$\begin{aligned} \Theta(1-y) + S(y) &= \left\langle \Theta\!\left(1 - y + \frac{in\cdot D}{m_b}\right)\right\rangle + \text{ less singular terms} \\ &= \int dk_+\, \Theta\!\left(1 - y + \frac{k_+}{m_b}\right) \Big\{f(k_+) + O(1/m_b)\Big\}\,. \end{aligned} \tag{51}$$

Here, $n = 2(v - p_\ell/m_b)$ is again a null vector when $y = 1$, and hence $f(k_+)$ coincides with the universal function defined in (39). We observe that, close to the end point region, the lepton spectrum in semileptonic $B \to X_u\,\ell\,\bar\nu$ decays can again be written as a convolution of the free quark decay distribution (i.e., a step function) with the structure function $f(k_+)$. Comparing the above relation with (38), we obtain

$$s(y) = -\frac{\partial}{\partial y}\Big[\Theta(1-y) + S(y)\Big] + \text{ less singular terms.} \tag{52}$$

Let us integrate this equation to obtain

$$\Theta(1-y)+S(y)\simeq\int_y^\infty dy'\,s(y')\,,$$
$$\int_y^\infty dy'\left[\Theta(1-y')+S(y')\right]\simeq\int_y^\infty dy'\,(y'-y)\,s(y')\,. \tag{53}$$

These relations are exact up to corrections of order $1/m_b$ or $\alpha_s$.

The main goal of the study of $B\to X_u\,\ell\,\bar\nu$ decays is to extract the element $V_{ub}$ of the Kobayashi-Maskawa matrix. The experimental analysis is very complicated, as there is only a small window close to the end point region where the signal is not overshadowed by the large background from $B$ decays into charmed particles. What can be measured is an integral over the end point region:

$$\widehat\Gamma_u(E_0)\equiv\int_{E_0}^\infty dE_\ell\,\frac{d\Gamma(B\to X_u\,\ell\,\bar\nu)}{dE_\ell}\,, \tag{54}$$

where $E_0$ is above the kinematic end point for $B\to D\,\ell\,\bar\nu$ transitions, i.e., $E_0>2.3$ GeV. The traditional way to extract a value of $V_{ub}$ from such a measurement is to compare $\widehat\Gamma_u(E_0)$ with the predictions of various quark models. Since the end point region of the lepton spectrum is strongly affected by nonperturbative effects, this procedure suffers from a considerable amount of model dependence. Currently, the value of $V_{ub}$ obtained following this strategy has a theoretical uncertainty of at least a factor 2 [42, 43].

Based on the results obtained in this paper and in Ref. [28], we propose a new strategy to extract $V_{ub}$ with little model dependence. The idea is to use the second equation in (53) to relate the integral $\widehat\Gamma_u(E_0)$ to a weighted integral over the photon spectrum in rare decays. Defining

$$\widehat\Gamma_s(E_0)\equiv\frac{2}{m_B}\int_{E_0}^\infty dE_\gamma\,(E_\gamma-E_0)\,\frac{d\Gamma(B\to X_s\,\gamma)}{dE_\gamma}\,, \tag{55}$$

and using that $|\,V_{tb}V_{ts}^*|\simeq|\,V_{cb}\,|$, we find the remarkable relation

$$\left|\frac{V_{ub}}{V_{cb}}\right|^2\simeq\left|\frac{V_{ub}}{V_{tb}V_{ts}^*}\right|^2 =\frac{3\alpha}{\pi}\,|c_7(m_b)|^2\,\eta_{\rm QCD}\,\frac{\widehat\Gamma_u(E_0)}{\widehat\Gamma_s(E_0)}+O\left(\frac{\Lambda}{m_b}\right)\,, \tag{56}$$

where $\eta_{\rm QCD}$ contains radiative corrections, which have so far been neglected in this paper. The Wilson coefficient $c_7(m_b)$ can be calculated in perturbation theory. In the standard model, it is known to leading logarithmic accuracy [15, 16]. Hence, from a measurement of the integrated quantities $\widehat\Gamma_u(E_0)$ and $\widehat\Gamma_s(E_0)$ one obtains a direct determination of $|V_{ub}/V_{cb}|$. The fact that the right-hand side in (56) must be independent of $E_0$ provides a constraint, which can help in the analysis of the data. By taking the ratio of the integrated decay rates, we are able to reduce hadronic uncertainties to the level of power corrections, the leading ones being of order $\Lambda/m_b\simeq0.1$, where we take $\Lambda\simeq500$ MeV as a typical low-energy scale of the strong interactions. It is thus not inconceivable that the theoretical uncertainties in (56) can be controlled to a level of, say, 10–30 %, corresponding to an uncertainty in $|V_{ub}/V_{cb}|$ of 5–15 %. This would be a major improvement over the present situation.

To achieve such an accuracy, it is necessary to study in detail the QCD correction factor $\eta_{\rm QCD}$ in (56), which arises when radiative corrections are included in the operator product expansion of the inclusive decay rates. We shall briefly discuss the qualitative structure of these corrections; a more complete treatment will be presented elsewhere [44]. In the free quark decay model, the one-loop radiative corrections to $B\to X_s\,\gamma$ and $B\to X_u\,\ell\,\bar\nu$ decays have been investigated by several authors [17, 45–47]. For the integrated quantities $\widehat\Gamma_i(E_0)$, they have the general structure ($i=u$ or $s$)

$$\widehat\Gamma_i^{\rm parton}(E_0)\propto(1-y_0)\,\Theta(1-y_0)\left\{1-\frac{2\alpha_s}{3\pi}\left[\ln^2(1-y_0)+a_i\,\ln(1-y_0)+b_i\right]\right\}+O\Big[(1-y_0)^2\Big]\,, \tag{57}$$

where $y_0=2E_0/m_b$. The Sudakov-type double logarithms are universal and enter both quantities with the same coefficient. This statement is true to all orders in perturbation theory [48, 49]. Note that the result for $\widehat\Gamma_s(y_0)$ is still of this form when one takes into account the finite strange-quark mass. This effect is known to modify the end point behavior of the spectrum in the region $1-y_0\sim\rho=O(1/m_b^2)$ [17]. However, since the integral in (55) extends over a larger region of order $1/m_b$, the corrections induced by $\rho\neq0$ are subleading. They are of the same magnitude as terms of order $(1-y_0)^2$, which we neglect in (57).

Since integrations over $y$ and $k_+$ commute, we can incorporate bound-state corrections by convoluting the parton model result (57) with the structure function $f(k_+)$, as we did for the tree-level expressions in (38) and (51). When we then take the ratio of $\widehat\Gamma_u(E_0)$ and $\widehat\Gamma_s(E_0)$, the large double-logarithmic corrections cancel. We find

$$\eta_{\rm QCD}=1-\frac{2\alpha_s}{3\pi}\Big[(a_u-a_s)\,\ln r+(b_u-b_s)\Big]\,. \tag{58}$$

Here

$$\ln r = \frac{\displaystyle\int_{m_b(y_0-1)}^{\bar\Lambda} dk_+\, f(k_+)\left(1-y_0+\frac{k_+}{m_b}\right)\ln\left(1-y_0+\frac{k_+}{m_b}\right)}{\displaystyle\int_{m_b(y_0-1)}^{\bar\Lambda} dk_+\, f(k_+)\left(1-y_0+\frac{k_+}{m_b}\right)} \tag{59}$$

is a nonperturbative parameter of order $\Lambda/m_b$. Using the fact that the logarithm is a monotonic function over the range of integration, and that $1-y_0+\bar\Lambda/m_b = 1-2E_0/m_B$ up to corrections of order $1/m_b^2$, we find the bound

$$-\ln r > -\ln\left(1-\frac{2E_0}{m_B}\right). \tag{60}$$

A precise calculation of $\ln r$, however, requires some knowledge of the nonperturbative structure function $f(k_+)$. The coefficients $a_s$ and $b_s$ in (58) can be extracted from the analysis of Ali and Greub [17]. We obtain $a_s = 3/2$ and $b_s = 2\pi^2/3+1$. In $b_s$, we have neglected small contributions from operators other than $O_7$ in the short-distance expansion of the effective Hamiltonian (1). Unfortunately, the existing analytical calculations of the radiative corrections to the lepton spectrum in $B \to X_u\,\ell\,\bar\nu$ decays disagree on the value of $b_u$. We find $a_u = 19/6$ and $b_u = \pi^2 - 23/12 + \delta$, where $\delta = 0$ according to Jezabek and Kühn [47], while $\delta \simeq 3/8$ according to Corbò [46]. For the correction factor $\eta_{\rm QCD}$, we find

$$\begin{aligned}\eta_{\rm QCD} &= 1-\frac{2\alpha_s}{9\pi}\left(5\ln r+\pi^2+3\delta-\frac{35}{4}\right)\\ &\simeq 1+\left[2.31-1.11\,\ln\left(\frac{r}{0.1}\right)-0.67\,\delta\right]\frac{\alpha_s}{\pi}.\end{aligned} \tag{61}$$

Since the controversial quantity $\delta$ enters with a small coefficient, the corresponding uncertainty in $\eta_{\rm QCD}$ is small. Using $\alpha_s/\pi \simeq 0.08$ and $r \simeq 0.1$, we expect $\eta_{\rm QCD} \simeq 1.18$. We conclude that the perturbative corrections are not unexpectedly large, but they deserve further investigation. In particular, the nature of the single-logarithmic corrections should be clarified. It is tempting to interpret the $\ln r$ term in (58) as a renormalization-group logarithm arising from the scaling from large scales $\mu^2 \simeq m_b^2$ down to smaller scales $\mu^2 \simeq m_b^2(1-y_0) \sim m_b\,\Lambda$, which are characteristic of the end point region. If this interpretation is correct, it should be possible to resum these logarithms using renormalization-group techniques. At present, however, we cannot prove this assertion.

Let us finally compare our new strategy to alternative model-independent determinations of $V_{ub}$ from exclusive $B$ decays. Using heavy quark flavor symmetry, one can in principle extract $V_{ub}$ from a comparison of exclusive decays $B \to h\,\ell\,\bar\nu$ and $D \to h\,\ell\,\bar\nu$, where $h$ is a light hadron such as $\pi$ or $\rho$ (see, e.g., Refs. [50–52]). The problem is that the comparison must be done close to the zero recoil limit. Thus, one is restricted to a small fraction of phase space. Presently, there is no convincing experimental evidence for exclusive charmless $B$ decays. But even if such an analysis becomes feasible as high-statistics data from a $B$ factory become available, the theoretical uncertainties will be of order $1/m_c$ instead of $1/m_b$. We thus believe that our new approach, based on a comparison of inclusive decay spectra, is preferable from the point of view of both theoretical uncertainty and experimental feasibility.

## VII. SUMMARY AND CONCLUSIONS

We have presented a systematic, QCD-based analysis of bound-state corrections to the photon spectrum in inclusive $B \to X_s\,\gamma$ decays. Using the operator product expansion and the heavy quark effective theory, we are able to resum the leading nonperturbative effects into a universal structure function $f(k_+)$, which describes the distribution of the light-cone residual momentum of the $b$ quark inside the $B$ meson. This formalism provides the generalization of the phenomenological concept of the "Fermi motion" in the context of QCD. We find that the moments of the structure function are given by a set of universal forward matrix elements of higher-dimension operators. The characteristic width of the photon spectrum is related to the expectation value of the kinetic energy of the heavy quark inside the $B$ meson. As a by-product, we obtain the bound $\lambda_1 < 0$ for one of the parameters of the heavy quark effective theory.

The formalism presented here is rather general. So far, it has been applied to the photon spectrum in rare decays and to the lepton spectrum in $B \to X_u\,\ell\,\bar\nu$ transitions [28]. Applications to other processes such as $B \to X_c\,\ell\,\bar\nu$ [53] or purely hadronic decays are possible, too. The fact that the leading nonperturbative corrections to inclusive decay spectra can be traced back to a universal structure function leads to interesting relations between different processes. In Sec. VI, we have shown that a weighted integral over the photon spectrum in $B \to X_s\,\gamma$ decays is related to an integral over the end point region of the lepton spectrum in $B \to X_u\,\ell\,\bar\nu$ decays. Based on this connection, we have proposed a model-independent way to extract the ratio $|V_{ub}/V_{cb}|$ of elements of the Kobayashi-Maskawa matrix. Hadronic uncertainties enter this determination only at the level of $1/m_b$ corrections. We estimate that using this method one could extract $V_{ub}$ with a theoretical uncertainty of about 20%, which is an order of magnitude better than the present theoretical uncertainty in this parameter.

Since the $b$-quark structure function is a rather fundamental object, one should try to calculate it using nonperturbative techniques such as lattice gauge theory or QCD sum rules. We believe that the relations derived in Sec. IV will be helpful for such calculations. In particular, it may be of advantage to calculate the Fourier transform of the structure function, which is given by the forward matrix element of a bilocal operator. Any theoretical insight into the behavior of the structure function and its moments will have a direct impact on the analysis of inclusive decay spectra, from which one hopes to extract accurate values for some standard model parameters or, in the case of rare decays, even indications for new physics beyond the standard model. On the other hand, we would like to emphasize that a measurement of some of the moments of the shape function in semileptonic decays [28], or of the photon spectrum in rare decays, would provide us with some fundamental QCD matrix elements and is thus interesting in its own right.

Finally, we mention that before the formalism developed here can be applied to the analysis of data, it is necessary to include QCD radiative corrections. This can be done by calculating virtual and real gluon corrections in the free quark decay model, and convoluting the result with the structure function $f(k_+)$. This procedure has its subtleties, however, due to the presence of end point singularities in the perturbative expansion. We have indicated this for the important case of the ratio of integrated decay rates in (56). A more complete discussion will be given elsewhere [44].

## ACKNOWLEDGMENTS

It is a pleasure to thank Ahmed Ali, Christoph Greub, Andrey Grozin, Matthias Jamin, Thomas Mannel, and Daniel Wyler for inspiring discussions. I am particularly indebted to Andrey Grozin for pointing out relation (43).

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