Angular distribution functions in the decays of ψ', ψ'' and the singlet D charmonium state directly produced in $\bar{p}p$ collisions

F. L. Ridener, Jr.
Pennsylvania State University at New Kensington, New Kensington, Pennsylvania 15068

K. J. Sebastian
University of Massachusetts at Lowell, Lowell, Massachusetts 01854
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We calculate the angular distribution of the two γ photons produced in the processes $\bar{p}p \to \psi', \psi'' \to \chi_J + \gamma_1 \to \psi + \gamma_1 + \gamma_2$ ($J=0,1,2$), and $\bar{p}p \to 1^1D_2 \to 1\,^1P_1 + \gamma_1 \to 1\,^1S_0 + \gamma_1 + \gamma_2$ in terms of the helicity amplitudes of the individual processes. There is enough information in the angular distributions, even when $\bar{p}$ and p are unpolarized, to extract the magnitudes of all the helicity amplitudes as well as the cosine of the relative phases of the helicity amplitudes or equivalently the multipole amplitudes in the processes ψ' or $\psi'' \to \chi_J + \gamma$ ($J=0,1,2$) and $1\,^1D_2 \to 1\,^1P_1 + \gamma$. Finally we also derive the combined angular distribution of the electron and the photon in the cascade process $\bar{p}p \to 1\,^1D_2 \to 1\,^3S_1 + \gamma \to e^+ + e^- + \gamma$ in terms of the helicity amplitudes of the individual processes. Here again, even when $\bar{p}$ and p are unpolarized there is enough information in the angular distribution to determine the magnitudes of all the helicity amplitudes as well as the cosine of the relative phases of the helicity amplitudes or equivalently radiation multipole amplitudes in the parity-conserving one-photon radiative transitions $1\,^1D_2(2^{-+}) \to 1\,^3S_1(1^{--}) + \gamma$.

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I. INTRODUCTION

Recently, the singlet and triplet P states of charmonium have been directly produced in $\bar{p}p$ collisions and their decays have been studied [1]. It is quite possible that in the near future the $\psi'(2\,^3S_1)$, $\psi''(1\,^3D_1)$, and $1\,^1D_2$ states of charmonium will also be produced in $\bar{p}p$ collision experiments at Fermilab. In potential models [2] the mass of the $^1D_2$ state is predicted to be around 3822 MeV. Even though this state is above the charm threshold, its strong decay into $\bar{D}_0 + D_0$ is forbidden by parity conservation and its decay into $\bar{D} + D^*$ or $D + \bar{D}^*$ is forbidden by energy conservation as long as the mass of the $1\,^1D_2$ state does not exceed 3870 MeV. Estimates of the radiative decay widths of the $1\,^1D_2$ state in potential models [2] give the results $\Gamma(1\,^1D_2 \to 1\,^1P_1 + \gamma) \simeq 600\text{–}700$ keV and $\Gamma(1\,^1D_2 \to 1\,^3S_1 + \gamma) \simeq 60$ keV. If we assume that the strong decay widths of the $1\,^1D_2$ state in the Okubo-Zweig-Iizuka- (OZI-) violating channels are of the same order of magnitude as those of the 1S, 2S, and 1P states, the radiative decays of the $1\,^1D_2$ state will have significant branching ratios and it may be possible to study such decays in $\bar{p}p$ collisions.

The angular distribution functions of the decay products of the triplet P states [χ_J ($J=0,1,2$)] directly produced in unpolarized as well as polarized $\bar{p}p$ collisions have been discussed before in the literature [3–5]. The model-independent angular distribution functions in terms of the helicity amplitudes are important not only to extract the helicity amplitudes from the experimentally measured angular distributions of the decay products, but also to establish the J^{PC} quantum numbers of the postulated resonance in the $\bar{p}p$ channel. In this paper we derive and discuss the angular distributions of the two γ photons in the two cascade processes

$$\bar{p}p \to \psi'$$

or

$$\psi''(1^{--}) \to \chi_J(J^{++}) + \gamma_1 \to \psi(1^{--}) + \gamma_1 + \gamma_2$$

and

$$p\bar{p} \to 1\,^1D_2(2^{-+}) \to 1\,^1P_1(1^{+-}) + \gamma_1$$
$$\to 1\,^1S_0(0^{-+})\gamma + \gamma_1 + \gamma_2 \,.$$

We also derive the combined angular distribution of the photon and electron in the cascade process

$$\bar{p}p \to 1\,^1D_2(2^{-+}) \to 1\,^3S_1(1^{--}) + \gamma \to e^+ + e^- + \gamma \,.$$

In each case we assume that the incident proton (p) and antiproton ($\bar{p}$) are unpolarized. Even in this case there is enough information in the angular distribution to calculate the magnitudes of all the helicity amplitudes in the individual processes. There is also enough information to calculate the cosine of the relative phase of the helicity or equivalently multipole amplitudes in the middle processes:

$$\psi' \text{ or } \psi'' \to \chi_J + \gamma \,, \quad 1\,^1D_2 \to 1\,^1P_1 + \gamma$$

and

$$1\,^1D_2 \to 1\,^3S_1 + \gamma \,.$$

From angular momentum and parity conservation, the first two radiative processes are expected to have only the $E1$, $M2$, and $E3$ multimode amplitudes whereas in the last radiative process the $M1$, $E2$, and $M3$ amplitudes are expected to be present.

The format of the rest of the paper is as follows. In Sec. II we discuss and derive the angular distribution function of the two photons in the cascade process

$$\bar{p}p\to\psi' \text{ or } \psi''\to\chi_J+\gamma_1\to\psi+\gamma_1+\gamma_2 \; (J=0,1,2) .$$

In Sec. III we do the same thing for the process

$$\bar{p}p\to 1\,{}^1D_2\to 1\,{}^1P_1+\gamma_1\to 1\,{}^1S_0+\gamma_1+\gamma_2 ,$$

and in Sec. IV we discuss the combined angular distribution of the photon and electron in the process

$$\bar{p}p\to 1\,{}^1D_2\to 1\,{}^3S_1+\gamma\to e^++e^-+\gamma .$$

Finally, in Sec. V we make some concluding remarks.

II. $\bar{p}p\to\psi'$ OR $\psi''\to\chi_J+\gamma_1\to\psi+\gamma_1+\gamma_2$ $(J=0,1,2)$

The probability amplitude for the process

$$\bar{p}(\lambda_1)+p(\lambda_2)\to X(\delta)\to\chi_J(\nu)+\gamma_1(\mu)$$
$$\to\psi(\sigma)+\gamma_2(\kappa)+\gamma_1(\mu) ,$$

where X is ψ' or ψ'' $(J^{PC}=1^{--})$ and λ_1, λ_2, δ, ν, μ, σ, and κ are the particle helicities, can be written as the product of the amplitudes of three sequential events:

$$\bar{p}(\lambda_1)+p(\lambda_2)\to X(\delta) , \quad X(\delta)\to\chi_J(\nu)+\gamma_1(\mu) ,$$

and

$$\chi_J(\nu)\to\psi(\sigma)+\gamma_2(\kappa) .$$

We will work in the X rest frame or the c.m. frame of $\bar{p}p$ with the positive z axis taken to be along the momentum of χ_J and γ_1 along the negative z axis. We assume the momentum of $\bar{p}$, namely, $\mathbf{p}$ is in the xz plane making an angle θ with the z axis. The y and x axes are, respectively, defined by the unit vectors

$$\hat{\mathbf{j}}=\frac{\hat{\mathbf{k}}\times\mathbf{p}}{|\hat{\mathbf{k}}\times\mathbf{p}|} \quad\text{and}\quad \hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{k}} . \tag{1}$$

If $|p,\theta,\phi;\lambda_1\lambda_2\rangle$ represents a two-particle helicity state in the zero momentum (c.m.) frame where p is the magnitude of either particle's momentum and the angles (θ,ϕ) represent the direction of the first particle's momentum and λ_1,λ_2 the helicities of the two particles, then using the expansion of the two-particle helicity states in terms of the angular momentum states we can write [4] the matrix element for the first process $\bar{p}(\lambda_1)+p(\lambda_2)\to X(\delta)$ as

$$\langle 1\delta|B|p,\theta,0;\lambda_1\lambda_2\rangle=\left[\frac{3}{4\pi}\right]^{1/2}B_{\lambda_1\lambda_2}d^1_{\delta\lambda}(\theta) , \tag{2a}$$

where

$$\lambda=\lambda_1-\lambda_2 , \tag{2b}$$

$$d^1_{\delta\lambda}(\theta)=D^1_{\delta\lambda}(0,\theta,0) , \tag{2c}$$

and B is a transition operator. By charge-conjugation invariance [6],

$$B_{\lambda_1\lambda_2}=B_{\lambda_2\lambda_1} . \tag{3}$$

By parity invariance [5], we get

$$B_{\lambda_1\lambda_2}=B_{-\lambda_1-\lambda_2} . \tag{4}$$

Next, we consider the decay $X(\delta)\to\chi_J(\nu)+\gamma_1(\mu)$. Since χ_J and γ_1 move in the $+z$ and $-z$ directions, respectively, we can write the matrix element for this process as

$$\langle 00;\nu\mu|A|1\delta\rangle=\left[\frac{3}{4\pi}\right]^{1/2}D^1_{\delta,\nu-\mu}(0,0,0)A^J_{\nu\mu} , \tag{5a}$$

where

$$\mu=\pm1 , \quad \nu=-J,-J+1,\ldots,0,\ldots,+J . \tag{5b}$$

In Eq. (5a),

$$D^1_{\delta,\nu-\mu}(0,0,0)=\delta_{\delta,\nu-\mu} . \tag{5c}$$

Because of parity invariance [6], the helicity amplitude $A^J_{\nu\mu}$ satisfies the condition

$$A^J_{\nu\mu}=A^J_{-\nu-\mu}(-1)^J . \tag{6}$$

The charge-conjugation invariance [6] is trivially satisfied in this process.

Assuming the direction of the final γ_2 in the rest frame of χ_J is given by (θ',ϕ') in our coordinate system (defined earlier), the matrix element for the process $\chi_J(\nu)\to\psi(\sigma)+\gamma_2(\kappa)$ can be written as

$$\langle\kappa\sigma,\theta'\phi'|C|J\nu\rangle=\left[\frac{2J+1}{4\pi}\right]^{1/2}D^{J*}_{\nu,\kappa-\sigma}(\phi',\theta',-\phi')C^J_{\kappa\sigma} , \tag{7a}$$

where

$$\kappa=\pm1 \quad\text{and}\quad \sigma=-1,0,+1 . \tag{7b}$$

By parity invariance the helicity amplitude $C^J_{\sigma\kappa}$ satisfies the constraint

$$C^J_{\kappa\sigma}=(-1)^JC^J_{-\sigma-\kappa} . \tag{8}$$

C invariance is trivially satisfied in this process.

The amplitude $T^{\sigma\kappa\mu}_{\lambda_1\lambda_2}$ for the cascade process to go from the initial state $\bar{p}(\lambda_1)p(\lambda_2)$ to the final state $\psi(\sigma)+\gamma_1(\mu)+\gamma_2(\kappa)$ through all possible helicity states δ of X and ν of χ_J is given by

$$\begin{aligned}T^{\sigma\kappa\mu}_{\lambda_1\lambda_2}&=\sum_{\delta=-1}^{+1}\sum_{\nu=-J}^{+J}\left[\frac{3}{4\pi}\right]\left[\frac{2J+1}{4\pi}\right]^{1/2}D^{J*}_{\nu,\kappa-\sigma}(\phi',\theta',-\phi')d^1_{\delta,\lambda_1-\lambda_2}(\theta)C^J_{\kappa\sigma}A^J{\nu\mu}B_{\lambda_1\lambda_2}\delta_{\delta,\nu-\mu}\\&=\left[\frac{3}{4\pi}\right]\left[\frac{2J+1}{4\pi}\right]^{1/2}C^J_{\kappa\sigma}B_{\lambda_1\lambda_2}\sum_{\nu(\mu)}D^{J*}_{\nu,\kappa-\sigma}(\phi',\theta',-\phi')d^1_{\nu-\mu,\lambda_1-\lambda_2}(\theta)A^J_{\nu\mu} ,\end{aligned} \tag{9a}$$

where

$$\nu(\mu)=-J,-J+1,\ldots,0\ (\mu=-1)$$
$$=0,1,\ldots,J\ (\mu=+1)\ . \tag{9b}$$

When $\bar{p}$ and p are unpolarized, the normalized angular distribution function for the two photons is given by

$$W(\theta;\theta',\phi')=\frac{\tilde{N}_J}{4}\sum_{\lambda_1\lambda_2}^{\pm}\sum_{\sigma}^{-1,0+1}\sum_{\kappa,\mu}^{\pm 1}T_{\lambda_1\lambda_2}^{\sigma\kappa\mu}T_{\lambda_1\lambda_2}^{\sigma\kappa\mu *}\ , \tag{10}$$

where $\tilde{N}_J$ is a normalization constant, which will make

$$\int W(\theta;\theta',\psi')d\Omega\, d\Omega'=1\ . \tag{11}$$

Using Eqs. (9), Eq. (10) can be rewritten as

$$W(\theta;\theta',\phi')=N_J\sum_{\lambda_1\lambda_2}^{\pm}|B_{\lambda_1\lambda_2}|^2\sum_{\kappa\sigma}|C_{\kappa\sigma}^J|^2\sum_{\mu}^{\pm 1}\sum_{\nu(\mu)\nu'(\mu)}[D_{\nu(\mu),\kappa-\sigma}^{J*}(\phi',\theta',-\theta')d_{\nu-\mu,\lambda_1-\lambda_2}^{1}(\theta)A_{\nu\mu}^J]$$
$$\times[D_{\nu',\kappa-\sigma}^{J}(\phi',\theta',-\phi')d_{\nu'-\mu,\lambda_1-\lambda_2}^{1}(\theta)A_{\nu'\mu}^J]^*\ , \tag{12}$$

where N_J is another normalization constant. Using the Clebsch-Gordan series for the D^J functions [7] and the fact that

$$D_{M0}^{L}(\phi,\theta,-\phi)=\left[\frac{4\pi}{2L+1}\right]^{1/2}Y_{LM}(\theta,\phi)\ , \tag{13}$$

we can express the right-hand side of Eq. (12) entirely in terms of the spherical harmonics. The techniques used are the same as we employed in a recent paper [5]. In terms of spherical harmonics, we finally get the normalized angular distribution function as

$$W(\theta;\theta',\phi')=\frac{1}{8\pi}\sum_{L'}^{0,2,\ldots,2J}\sum_{\delta=0,1}^{\mathrm{Min}(J,L')}\sum_{\kappa(\delta)}C_{L'\kappa\delta}^{J}[Y_{L'\delta}(\theta',\phi')+Y_{L'\delta}^{*}(\theta',\phi')]Y_{\kappa N(\delta)}(\theta,0)\ , \tag{14a}$$

where

$$\kappa(\delta)=\begin{cases}0,2 & \text{when } \delta \text{ is even}\ ,\\ 2 & \text{when } \delta \text{ is odd}\ ,\end{cases} \tag{14b}$$

and

$$N(\delta)=\begin{cases}0 & \text{when } \delta \text{ is even}\ ,\\ 1 & \text{when } \delta \text{ is odd}\ .\end{cases} \tag{14c}$$

The coefficients $C_{L'\kappa\delta}^J$ are expressed in terms of the helicity amplitudes $A_{\nu\mu}^J$, B_ν, and C_ν^J, which are defined in terms of the previously introduced helicity amplitudes $B_{\lambda_1\lambda_2}$, $A_{\nu\mu}^J$, and $C_{\sigma\kappa}^J$ as

$$B_1=\sqrt{2}B_{+-}=\sqrt{2}B_{-+}=B_{-1}\ ,$$
$$B_0=\sqrt{2}B_{++}=\sqrt{2}B_{--}\ , \tag{15}$$
$$A_\nu^J=A_{\nu 1}^J=A_{-\nu-1}^J\ ,$$

where ν is the helicity of χ_J and can vary from zero to J:

$$C_\zeta^J=\sqrt{2}C_{\sigma\kappa}\ ,$$

where

$$\zeta=\sigma-\kappa \text{ varies from 0 to } J\ .$$

These newly defined helicity amplitudes satisfy the normalization conditions

$$B_0^2+B_1^2=1\ ,$$
$$\sum_{\nu=0}^{J}|A_\nu^J|^2=1\ , \tag{16}$$
$$\sum_{\zeta=0}^{J}|C_\zeta^J|^2=1\ .$$

In terms of the helicity amplitudes defined by Eqs. (15) and (16), the nonvanishing C coefficients of Eq. (14a) are given below.

$J=0$:

$$C_{000}=1\ ,$$
$$C_{020}=\frac{(\frac{3}{2}|B_1|^2-1)}{\sqrt{5}}\ . \tag{17}$$

$J=1$:

$$C_{000}=1\ ,$$
$$C_{020}=\frac{(\frac{3}{2}|B_1|^2-1)}{\sqrt{5}}(|A_0|^2-2|A_1|^2)\ ,$$
$$C_{200}=\frac{(1-\frac{3}{2}|C_1|^2)}{\sqrt{5}}(2|A_0|^2-|A_1|^2)\ , \tag{18}$$
$$C_{220}=-\tfrac{2}{5}(1-\tfrac{3}{2}|C_1|^2)(1-\tfrac{3}{2}|B_1|^2)(|A_0|^2+|A_1|^2)\ ,$$
$$C_{221}=-\tfrac{6}{5}(1-\tfrac{3}{2}|C_1|^2)\mathrm{Re}(A_1A_0^*)(1-\tfrac{3}{2}|B_1|^2)\ .$$

$J=2$:

$$C_{000}=1\ ,$$

$$C_{020}=\frac{(\frac{3}{2}|B_1|^2-1)}{\sqrt{5}}[|A_0|^2-2|A_1|^2+|A_2|^2]\ ,$$

$$C_{200}=\frac{\sqrt{5}}{7}(1-\tfrac{1}{2}|C_1|^2-2|C_2|^2)\times(2|A_0|^2+|A_1|^2-2|A_2|^2)\ ,$$

$$C_{220}=\tfrac{2}{7}(\tfrac{3}{2}|B_1|^2-1)(1-\tfrac{1}{2}|C_1|^2-2|C_2|^2)\times(|A_0|^2-|A_1|^2-|A_2|^2)\ ,$$

$$C_{221}=\frac{2\sqrt{3}}{7}(\tfrac{3}{2}|B_1|^2-1)(1-\tfrac{1}{2}|C_1|^2-2|C_2|^2)\times[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\,\mathrm{Re}(A_2A_1^*)]\ ,$$

$$C_{202}=-\frac{4\sqrt{5}}{7}(\tfrac{3}{2}|B_1|^2-1)(1-\tfrac{1}{2}|C_1|^2-2|C_2|^2)\times\mathrm{Re}(A_2A_0^*)\ ,$$

$$C_{222}=+\tfrac{4}{7}(\tfrac{3}{2}|B_1|^2-1)(1-\tfrac{1}{2}|C_1|^2-2|C_2|^2)\times\mathrm{Re}(A_2A_0^*)\ , \quad (19)$$

$$C_{400}=\tfrac{1}{21}(3-5|C_1|^2-\tfrac{5}{2}|C_2|^2)\times(6|A_0|^2-4|A_1|^2+|A_2|^2)\ ,$$

$$C_{420}=\frac{1}{21\sqrt{5}}(\tfrac{3}{2}|B_1|^2-1)(3-5|C_1|^2-\tfrac{5}{2}|C_2|^2)\times(6|A_0|^2+8|A_1|^2+|A_2|^2)\ ,$$

$$C_{421}=\frac{2}{7\sqrt{3}}(\tfrac{3}{2}|B_1|^2-1)(3-5|C_1|^2-\tfrac{5}{2}|C_2|^2)\times[\sqrt{6}\,\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)]\ ,$$

$$C_{402}=+\frac{10}{7\sqrt{15}}(\tfrac{3}{2}|B_1|^2-1)(3-5|C_1|^2-\tfrac{5}{2}|C_2|^2)\times\mathrm{Re}(A_2A_0^*)\ ,$$

$$C_{422}=-\frac{2}{7\sqrt{3}}(\tfrac{3}{2}|B_1|^2-1)(3-5|C_1|^2-\tfrac{5}{2}|C_2|^2)\times\mathrm{Re}(A_2A_0^*)\ .$$

From Eqs. (14)–(19), it is clear that a measurement of the angular distribution of the two photons will enable us to calculate the magnitudes of all the helicity amplitudes A, B, and C as well as the cosine of the relative phase among the A helicity amplitudes of the decay ψ' or $\psi''\rightarrow\chi_J+\gamma$ $(J=0,1,2)$. From the A and C helicity amplitudes, one can construct the radiative multipole amplitudes as appropriate linear combinations [3,4,8].

III. $\bar{p}p\rightarrow 1\,{}^1D_2\rightarrow 1\,{}^1P_1+\gamma_1\rightarrow 1\,{}^1S_0+\gamma_1+\gamma_2$

Here, by parity and C invariance [6], the B helicity amplitudes in the process

$$\bar{p}(\lambda_1)p(\lambda_2)\rightarrow 1\,{}^1D_2(\nu)$$

satisfy the conditions

$$B_{\lambda_1\lambda_2}=B_{\lambda_2\lambda_1}\quad (C\ \text{invariance})\ , \quad (20)$$

$$B_{\lambda_1\lambda_2}=-B_{-\lambda_1-\lambda_2}\quad (P\ \text{invariance})\ , \quad (21)$$

and so

$$B_1=B_{+-}=B_{-+}=-B_{+-}=0 \quad (22)$$

and we define

$$B_0=B_{++}=-B_{--}\ . \quad (23)$$

Because of parity invariance [6], the A helicity amplitudes in the process $1\,{}^1D_2(\nu)\rightarrow 1\,{}^1P_1(\sigma)+\gamma_1(\mu)$ satisfy the relations

$$A_{\sigma\mu}=A_{-\sigma-\mu}\ . \quad (24)$$

We define

$$A_\nu=A_{\nu-1,-1}=A_{-\nu+1,1}\ . \quad (25)$$

There are three independent helicity amplitudes A_0, A_1, and A_2.

The C helicity amplitude in the process $1\,{}^1P_1(\sigma)\rightarrow 1\,{}^1S_0+\gamma_2(\kappa)$ is defined by the matrix element

$$\langle\theta',\phi',\kappa 0|C|1\sigma\rangle=\left[\frac{3}{4\pi}\right]^{1/2}D^{1*}_{\sigma\kappa}(\theta',\phi')C_\kappa\quad(\kappa=\pm 1)\ , \quad (26)$$

where θ',ϕ' represent the direction of γ_2 in the ${}^1P_1$ rest frame. The ${}^1P_1$ moves along the positive z axis, and the momentum vector of $\bar{p}$ is in the xz plane, making an angle θ with the positive z axis. By parity invariance the helicity amplitudes C_κ $(\kappa=\pm1)$ satisfy

$$C_\kappa=C_{-\kappa}\ . \quad (27)$$

We define

$$C=C_1=C_{-1}\ . \quad (28)$$

The normalized angular distribution function of the two photons is now given by the expression

$$W(\theta;\theta',\phi')=\frac{1}{8\pi}\sum_{L'}^{0,2}\sum_{\delta=0,1}^{\mathrm{Min}(2,L')}\sum_{\kappa(\delta)}C_{L'\kappa\delta}[Y^*_{L'\delta}(\theta',\phi')+Y_{L'\delta}(\theta',\phi')]Y^{(\theta)}_{\kappa N(\delta)}\ , \quad (29a)$$

where

$$\kappa(\delta)=[1-(-1)^\delta],[1-(-1)^\delta]+2,\ldots,4\ ,$$

$$N(\delta)=\begin{cases}0 & \text{when } \delta \text{ is even}\ ,\\ 1 & \text{when } \delta \text{ is odd}\ .\end{cases} \quad (29b)$$

We use the normalizations

$$|C|^2=|B_0|^2=1 \quad (30a)$$

and

$$\sum_{\nu=0}^{2} |A_\nu|^2=1 \ . \tag{30b}$$

The expressions for the nonzero coefficients $C_{L'\kappa\delta}$ in terms of the helicity amplitudes A_0, A_1, and A_2 are

$$\begin{aligned}
&C_{000}=1 \ ,\\
&C_{020}=\frac{\sqrt{5}}{7}[2|A_0|^2+|A_1|^2-2|A_2|^2] \ ,\\
&C_{040}=\tfrac{1}{7}[6|A_0|^2-4|A_1|^2+|A_2|^2] \ ,\\
&C_{200}=\frac{1}{2\sqrt{5}}[|A_0|^2-2|A_1|^2+|A_2|^2] \ ,\\
&C_{220}=\tfrac{1}{7}[|A_0|^2-|A_1|^2-|A_2|^2] \ ,\\
&C_{240}=\frac{1}{14\sqrt{5}}[6|A_0|^2+8|A_1|^2+|A_2|^2] \ ,\\
&C_{221}=\frac{\sqrt{3}}{7}[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\,\mathrm{Re}(A_2A_1^*)] \ ,\\
&C_{241}=\frac{\sqrt{3}}{7}[\sqrt{6}\,\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)] \ ,\\
&C_{202}=\frac{1}{\sqrt{5}}\mathrm{Re}(A_2A_0^*) \ ,\\
&C_{222}=\tfrac{5}{7}\mathrm{Re}(A_2A_0^*) \ ,\\
&C_{242}=-\frac{6}{7\sqrt{5}}\mathrm{Re}(A_2A_0^*) \ .
\end{aligned} \tag{31}$$

From Eqs. (29)–(31), it is clear that we can determine the magnitudes of all the helicity amplitudes A_0, A_1, and A_2, as well as the cosine of their relative phases, once the angular distributions of the two photons are measured. From A_0, A_1, and A_2, we can construct the $E1$, $M2$, and $E3$ multiple amplitudes in ${}^1D_2\to{}^1P_1+\gamma$.

IV. $\bar{p}p\to 1\,{}^1D_2\to 1\,{}^3S_1+\gamma\to e^++e^-+\gamma$

As in the previous case, the B_1 helicity amplitude in the production process $\bar{p}p\to 1\,{}^1D_2$ vanishes by parity and charge-conjugation invariance. Only B_0 is nonzero. The A helicity amplitudes of the $1\,{}^1D_2^{(\nu)}\to 1\,{}^3S_1^{(\sigma)}+\gamma(\mu)$ or $2^{-+}\to 1^{--}+\gamma$ are defined by the matrix element

$$\langle p00;\sigma\mu|A|2\nu\rangle=\left[\frac{5}{4\pi}\right]^{1/2}\delta\nu,\sigma-\mu A_{\sigma\mu} \ . \tag{32}$$

By parity invariance of the A transition operator,

$$A_{\sigma\mu}=-A_{-\sigma-\mu} \ . \tag{33}$$

By angular momentum conservation along the z axis,

$$\nu=\sigma-\mu \ . \tag{34}$$

We can define the three independent helicity amplitudes to be

$$\begin{aligned}
&A_0=A_{-1-1}=-A_{11} \ ,\\
&A_1=A_{0-1}=-A_{01} \ ,\\
&A_2=A_{1-1}=-A_{-11} \ .
\end{aligned} \tag{35}$$

The C helicity amplitudes in the process $\psi(\sigma)\to e^-(\kappa_1)+e^+(\kappa_2)$, with e^- momentum in the direction (θ',ϕ') in the rest frame of ψ, are defined by the equation

$$\langle\theta',\phi';\kappa_1\kappa_2|C|1\sigma\rangle=\left[\frac{3}{4\pi}\right]^{1/2}D^{1*}_{\sigma,\kappa_1-\kappa_2}(\theta',\phi')C_{\kappa_1\kappa_2} \ . \tag{36}$$

By charge-conjugation invariance [6],

$$C_{\kappa_1\kappa_2}=C_{\kappa_2\kappa_1} \ . \tag{37}$$

By parity invariance [5],

$$C_{\kappa_1\kappa_2}=C_{-\kappa_1-\kappa_2} \ . \tag{38}$$

We can define two independent helicity amplitudes C_0 and C_1 to be

$$C_0=\sqrt{2}C_{++}=\sqrt{2}C_{--}$$

and (39)

$$C_1=C_{+-}=C_{-+} \ .$$

We also have the normalizations

$$|B_0|^2=1 \ , \quad \sum_{\nu=0}^{2}|A_\nu|^2=1 \ , \quad \text{and } C_0^2+C_1^2=1 \ . \tag{40}$$

The normalized angular distribution function of the final photon and electron in the cascade process is now given by

$$W(\theta;\theta',\phi')=\frac{1}{8\pi}\sum_{L'}^{0,2}\sum_{\delta=0,1}^{\mathrm{Min}(L',2)}\sum_{\kappa(\delta)}C_{L'\kappa\delta}[Y^*_{L'\delta}(\theta',\phi')+Y_{L'\delta}(\theta',\phi')]Y_{\kappa N(\delta)}(\theta) \ , \tag{41}$$

where $\kappa(\delta)$ and $N(\delta)$ are given by Eqs. (29b). The expressions for the nonzero coefficients $C_{L'\kappa\delta}$ in terms of the helicity amplitudes are

$$\begin{aligned}
&C_{000}=1 \ ,\\
&C_{020}=\frac{\sqrt{5}}{7}(2|A_0|^2+|A_1|^2-2|A_2|^2) \ ,\\
&C_{040}=\tfrac{1}{7}(6|A_0|^2-4|A_1|^2+|A_2|^2) \ ,\\
&C_{200}=\frac{1}{2\sqrt{5}}(3|C_1|^2-2)(|A_0|^2-2|A_1|^2+|A_2|^2) \ ,\\
&C_{220}=\tfrac{1}{7}(3|C_1|^2-2)(|A_0|^2-|A_1|^2-|A_2|^2) \ ,\\
&C_{240}=\frac{1}{14\sqrt{5}}(3|C_1|^2-2)(6|A_0|^2+8|A_1|^2+|A_2|^2) \ ,\\
&C_{221}=\frac{\sqrt{3}}{7}(3|C_1|^2-2)[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\,\mathrm{Re}(A_2A_1^*)] \ ,
\end{aligned} \tag{42}$$

$$C_{241}=\frac{\sqrt{3}}{7}(3|C_1|^2-2)[\sqrt{6}\,\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)]\ ,$$

$$C_{202}=-\frac{1}{\sqrt{5}}(3|C_1|^2-2)\mathrm{Re}(A_2A_0^*)\ ,$$

$$C_{222}=\tfrac{2}{5}(3|C_1|^2-2)\mathrm{Re}(A_2A_0^*)\ ,$$

$$C_{242}=-\frac{6}{7\sqrt{5}}(3|C_1|^2-2)\mathrm{Re}(A_2A_0^*)\ .$$

From Eqs. (41) and (42), it is clear that we can determine the magnitudes of all the A helicity amplitudes as well as the cosine of their relative phases once the angular distributions of the photon and electron are measured. We can also determine the C_0 helicity amplitude in the process $\psi\to e^-+e^+$. If the e^+e^- system is produced by the process $q\bar{q}\to\gamma\to e^-e^+$, C_0 is of the order of $m/E\simeq 3.3\times 10^{-4}$, and with our normalization of Eq. (40), C_0 should turn out to be extremely small compared to 1.

We should also mention that the angles (θ',ϕ') are defined in the ψ rest frame where as θ is measured in the $^1D_2$ rest frame.

V. CONCLUDING REMARKS

We have derived the angular distribution functions of the decay products of the charmonium states produced in unpolarized $\bar{p}p$ collisions. In particular, we considered the three processes

(i) $\bar{p}p\to\psi'$ or $\psi''\to\chi_J+\gamma_1\to\psi+\gamma_1+\gamma_2$,

(ii) $\bar{p}p\to 1\,{}^1D_2\to 1\,{}^1P_1+\gamma_1\to 1\,{}^1S_0+\gamma_1+\gamma_2$,

and

(iii) $\bar{p}p\to 1\,{}^1D_2\to 1\,{}^3S_1+\gamma\to e^++e^-+\gamma$.

The expressions for the coefficients $C_{L'\kappa\delta}$ in terms of the relevant helicity amplitudes are identical in the angular distribution functions in cases (ii) and (iii) except for an additional factor of $(1-3|C_0|^2)$ for $L'=2$ in case (iii).

Our expressions for the angular distributions should prove to be useful to the experimentalists in extracting the helicity or multipole amplitudes in the radiative processes discussed above. In comparing Eqs. (14), (29), and (41) with experiment, we should point out an important point. The angles (θ',ϕ') are defined in the rest frame of the final decaying particle where as θ is measured in the $\bar{p}p$ c.m. frame. These two frames are never identical. In the $\bar{p}p$ c.m. frame, the final decaying particle has a significant velocity ($v/c\simeq 0.15-0.2$). Since the helicity amplitudes are Lorentz invariant, all we have to do is to reexpress the angles θ',ϕ' in Eqs. (14), (29), and (41) in terms of the angles θ'',ϕ'' defined in the $\bar{p}p$ c.m. frame. The relation between the two sets of angles is given by (to first order in $\beta=v/c$)

$$\cos\theta'\simeq\cos\theta''-\beta\sin^2\theta''\ ,$$
$$\sin\theta'\simeq\sin\theta''+\beta\sin\theta''\cos\theta''\ , \qquad (43)$$
$$\phi'=\phi''\ ,$$

where β is the velocity (in units of c) of the final decaying particle in the $\bar{p}p$ c.m. frame.

Finally, the relationship between the helicity amplitudes of the radiative transitions where $J=1$ state goes into a $J=2$ state or vice versa is given by [3,4,8]

$$A_\nu=\sum_{\kappa=1}^{3}a_k\left[\frac{2k+1}{5}\right]^{1/2}\langle k1;1,\nu-1|2\nu\rangle\ , \qquad (44)$$

where

$$\nu=0,1,2\ .$$

Equation (44) represents a real orthogonal transformation between (A_0,A_1,A_2) and (a_1,a_2,a_3). So if the A's are normalized, the a's will be too. That is,

$$\sum_{\nu=0}^{2}|A_\nu|^2=\sum_{k=1}^{3}|a_k|^2=1\ . \qquad (45)$$

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