Massive Schwinger model with $SU(2)_f$ on the light cone

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The massive Schwinger model with two flavors is studied in the strong coupling region by using the light-front Tamm-Dancoff approximation. The mass spectrum of the lightest particles is obtained numerically. We find that the mass of the lightest isotriplet ("pion") behaves as $m^{0.50}$ for strong coupling, where m is the fermion mass. We also find that the lightest isosinglet is not in the valence state ("η") which is much heavier in the strong coupling region, but can be interpreted as a bound state of two pions. It is 1.762 times heavier than the pion at $m=1.0\times10^{-3}(e/\sqrt{\pi})$, while Coleman predicted that the ratio is $\sqrt{3}$ in the strong coupling limit. The "pion decay constant" is calculated to be 0.3945.

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I. INTRODUCTION

The light-front Tamm-Dancoff (LFTD) approximation has attracted much attention recently as an alternative nonperturbative method to lattice theory [1]. It is the Tamm-Dancoff approximation [2] applied to field theory quantized on the light cone [3,4]. The light-cone quantization provides a cure for one of the most serious problems of the Tamm-Dancoff approximation; in the application of the Tamm-Dancoff approximation, one must first specify the ground state, while in the light-cone quantization the ground state is relatively simple because the Fock vacuum is an eigenstate of the light-front Hamiltonian. The LFTD field theory is a very attractive and efficient numerical method for relativistic bound state problems and is intuitively appealing because it is based on diagonalization of Hamiltonians with the eigenstates being wave functions for bound states.

There are, however, several problems in LFTD field theory: (1) (nonperturbative) renormalization, (2) spontaneous symmetry breaking (or the "zero-mode problem"), and (3) recovery of rotational symmetry. These problems are very important in the development of LFTD field theory. We think, however, that it is useful to see how far the LFTD field theory goes by studying simple models for which we can circumvent these problems.

In this paper we study the massive Schwinger model with two flavors in the strong coupling region nonperturbatively in the LFTD approximation. Because it is a two-dimensional model, there is no renormalization problem. Because the fermions are massive, spontaneous symmetry breaking does not come into trouble [5,6]; the global $SU(2)_A$ symmetry is explicitly (and softly) broken. Because there are no transverse directions, the rotational symmetry is not broken from the outset. As we will see, in the strong coupling region, the structure of the mass spectrum is relatively simple.

The massive Schwinger model [7,8] is a generalization of the massless Schwinger model [9–11] with the massive fermion. Both of them have been playing a unique role as simple toy models for QCD [12]. Although the massless Schwinger model is exactly solvable, the massive model is no longer exactly solvable. One has to employ some nonperturbative methods, such as bosonization and Monte Carlo simulations [13], to *solve* it. In his beautiful paper [8], Coleman studied the massive Schwinger model and its extension with $SU_f(2)$ flavor (isospin) symmetry by using the bosonization technique. Among important results, he found the following for the model with $SU(2)_f$ symmetry in the strong coupling limit. (i) The model is equivalent to the sine-Gordon theory with $\beta=\sqrt{2\pi}$. (ii) The lightest particle is an isotriplet, and the next lightest is an isosinglet. (iii) The isosinglet/isotriplet mass ratio is $\sqrt{3}$. (iv) The isotriplet is $I^{PG}=1^{-+}$, while the isosinglet is 0^{++}, not 0^{--}. He confessed that he did not understand why it is so. In this paper we examine these results numerically in the light-front Tamm-Dancoff approximation including up to four-body states. We not only confirm his results, but also obtain several new results. Our main results are the following. (i) We also find that the lightest particle is an isotriplet and the next is an isosinglet in the strong coupling limit. The isotriplet may be called a "pion," because the valence component is dominant. (ii) We calculate the "pion" mass as a function of the fermion mass numerically. It behaves like $m^{0.5007(2)}$ in the strong coupling limit. (iii) We calculate the isosinglet/isotriplet mass ratio for various values of the fermion mass and find that it is 1.762 for $m=1.0\times10^{-3}(e/\sqrt{\pi})$. (iv) We argue that the lightest isosinglet is a bound state of two "pions." This is our answer to Coleman's question. The valence state ("η") is much heavier due to the annihilation force. (v) We find no η-η, π-η, or π-π in the isotriplet or isoquintet bound states in the strong coupling region. (vi) We calculate the

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"pion decay constant" to be 0.3945. (Of course, this is merely a two-dimensional analogue and has no physical importance; the "pions" do not decay. What we would really like to do is to demonstrate that we can calculate such phenomenological quantities such as this from the fundamental field theory.)

The massive Schwinger model in the LFTD approximation has been studied by Bergknoff [14] and Mo and Perry [15]. Our work is based on these works, especially on that of Mo and Perry. We refer the readers to them. There are also some papers on the massive Schwinger model in discretized light-cone quantization (DLCQ) [16–19], which is closely related to the LFTD approximation.

The paper is organized as follows. In Sec. II we will describe the light-cone quantized massive Schwinger model with SU(2)$_f$ and its Tamm-Dancoff approximation. The states are classified by the exact flavor (isospin) symmetry. We truncate the states, keeping only two- and four-body components. The inclusion of four-body states is essential for our work. As we will see, the lowest isosinglet state is not in its valence state and therefore cannot be found in the lowest approximation (up to four-body states). In Sec. III we will describe our numerical method. The method is a generalization of that of Mo and Perry. We expand the wave functions in terms of basis functions, which satisfy assumed symmetries under exchanges of momenta and boundary conditions, and diagonalize the matrices for the "norm" and the light-cone Hamiltonian. The "norm" is necessary because the basis functions are not orthonormal. We then give results. The hope for the LFTD approximation is that the components with a large number of constituents are suppressed at least for the low-lying states. We will see that it is the case (except for the lightest isosinglet) by comparing the calculations with two- and four-body states with those with only two-body states. We will also show that a small number of basis functions is sufficient to produce quite accurate values. We will then identify states by examining the wave functions. It is a peculiar feature of the LFTD approximation that we can utilize detailed information of the wave functions. It is also exploited in calculating the "pion decay constant." Section IV is devoted to discussions. The appendixes are collections of conventions, lengthy expressions, and some formulas.

II. LIGHT-FRONT TAMM-DANCOFF APPROXIMATION

A. Massive Schwinger model with two flavors

In this section we will establish the conventions and provide the readers with some basic formulas.

The massive Schwinger model is two-dimensional QED with massive fermions. The model discussed in this paper involves a flavor SU(2)$_f$ (isospin) symmetry:

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\sum_{i=1}^{2}\bar{\psi}_i[\gamma^\mu(i\partial_\mu-eA_\mu)-m]\psi_i\ , \tag{2.1}$$

where i ($i=1,2$) is the label for flavors. Because it is regarded as a toy model for QCD, we sometimes use words such as "quark," "pion," and "η." They should not be confused with *real* ones, of course. The equations of motion in the $A^+=0$ gauge,

$$\begin{aligned}
&-\partial_-^2 A^-=\sqrt{2}e\sum_{i=1}^{2}\psi_{iR}^\dagger\psi_{iR}\ ,\\
&\partial_+\partial_- A^-=\sqrt{2}e\sum_{i=1}^{2}\psi_{iL}^\dagger\psi_{iL}\ ,\\
&i\sqrt{2}\partial_-\psi_{iL}=m\psi_{iR}\ ,\\
&i\sqrt{2}\partial_+\psi_{iR}=m\psi_{iL}+\sqrt{2}eA^-\psi_{iR}\ ,
\end{aligned}\tag{2.2}$$

show that A^- and $\psi_{iL}\equiv\frac{1}{2}(1-\gamma^5)\psi_i$ are dependent variables and that only $\psi_{iR}\equiv\frac{1}{2}(1+\gamma^5)\psi_i$ is independent. (See Appendix A for notation.) These dependent variables can be eliminated:

$$\begin{aligned}
&\psi_{iL}(x^-)=-i\frac{m}{2\sqrt{2}}\int dy^-\epsilon(x^--y^-)\psi_{iR}(y^-)\ ,\\
&A^-(x^-)=-\frac{e}{\sqrt{2}}\int dy^-|x^--y^-|\sum_{i=1}^{2}\psi_{iR}^\dagger\psi_{iR}(y^-)\ .
\end{aligned}\tag{2.3}$$

(There can be an x^- independent background electric field, which is related to the vacuum angle, as Coleman discussed [8]. We will not consider this parameter at all in this paper and concentrate only on the case $\theta=0$.)

Canonical quantization is performed by assuming the following anticommutation relations for only independent variables:

$$\begin{aligned}
&\{\psi_{iR}(x),\psi_{jR}^\dagger(y)\}_{x^+=y^+}=\frac{\delta_{ij}}{\sqrt{2}}\delta(x^--y^-)\ ,\\
&\{\psi_{iR}(x),\psi_{jR}(y)\}_{x^+=y^+}=\{\psi_{iR}^\dagger(x),\psi_{jR}^\dagger(y)\}_{x^+=y^+}=0\ .
\end{aligned}\tag{2.4}$$

In terms of independent variables, the Lagrangian can be written as

$$\begin{aligned}
L=\int dx^-\mathcal{L}=&\,i\sqrt{2}\int dx^-\sum_{i=1}^{2}:\psi_{iR}^\dagger\partial_+\psi_{iR}:\\
&+\frac{im^2}{2\sqrt{2}}\int dx^-dy^-\sum_{i=1}^{2}\psi_{iR}^\dagger(x^-)\epsilon(x^--y^-)\psi_{iR}(y^-)\\
&+\frac{e^2}{4}\int dx^-dy^-\sum_{i=1}^{2}j^+(x^-)|x^--y^-|j^+(y^-)\ .
\end{aligned}\tag{2.5}$$

In order to construct a well-defined quantum theory, we have to restrict ourselves to the $Q=\int dx^-j^+(x^-)=0$ subspace, where the conserved U(1)$_V$ current is defined by

$$j^\mu=\sum_{i=1}^{2}:\bar{\psi}_i\gamma^\mu\psi_i:\ ; \tag{2.6}$$

otherwise, we would have infinite energy.

There is also the (anomalous) U(1)$_A$ current

$$\begin{aligned}
&j_5^\mu=\sum_{i=1}^{2}:\bar{\psi}_i\gamma^\mu\gamma_5\psi_i:\ ,\\
&\partial_\mu j_5^\mu=2im\sum_{i=1}^{2}:\bar{\psi}_i\gamma_5\psi_i:+\frac{2e}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu\ ,
\end{aligned}\tag{2.7}$$

as well as $SU(2)_V$ and $SU(2)_A$ currents

$$\begin{aligned} j^{a\mu} &= \sum_{i,j}^{2} :\bar{\psi}_i \gamma^\mu (T^a)_{ij} \psi_j: \,, \\ j_5^{a\mu} &= \sum_{i,j}^{2} :\bar{\psi}_i \gamma^\mu \gamma_5 (T^a)_{ij} \psi_j: \,, \\ \partial_\mu j^{a\mu} &= 0 \,, \\ \partial_\mu j_5^{a\mu} &= 2im \sum_{i,j}^{2} :\bar{\psi}_i \gamma_5 (T^a)_{ij} \psi_j: \,, \end{aligned} \tag{2.8}$$

where $T^a = \sigma^a/2$. In the above the normal ordering is defined with respect to the free field expansion:

$$\psi_{iR}(x^-) = \frac{1}{2^{1/4}} \int_0^\infty \frac{dk^+}{2\pi\sqrt{k^+}} [b_i(k^+) e^{-ik^+x^-} + d_i^\dagger(k^+) e^{ik^+x^-}] \,, \tag{2.9}$$

where, from (2.4),

$$\{b_i(k^+), b_j^\dagger(l^+)\} = \{d_i(k^+), d_j^\dagger(l^+)\} = 2\pi k^+ \delta_{ij} \delta(k^+ - l^+) \,. \tag{2.10}$$

By substituting (2.9), the light-cone Hamiltonian P^- is written, in terms of these creation and annihilation operators, as

$$\begin{aligned} P^- &= P^-_{\text{free}} + P^-_{\text{self}} + P^-_0 + P^-_2 \,, \\ P^-_{\text{free}} &= \frac{m^2}{4\pi} \sum_{i=1}^{2} \int_0^\infty \frac{dk}{k^2} [b_i^\dagger(k) b_i(k) + d_i^\dagger(k) d_i(k)] \,, \\ P^-_{\text{self}} &= \frac{e^2}{8\pi^2} \sum_{i=1}^{2} \int_0^\infty \frac{dk_1}{k_1} [b_i^\dagger(k_1) b_i(k_1) + d_i^\dagger(k_1) d_i(k_1)] \int_0^\infty dk_2 \left[\frac{1}{(k_1-k_2)^2} - \frac{1}{(k_1+k_2)^2} \right] \,, \\ P^-_0 &= \frac{e^2}{8\pi^3} \sum_{i,j}^{2} \int_0^\infty \frac{\Pi_i^4 dk_i}{\sqrt{k_1 k_2 k_3 k_4}} \Bigg\{ [b_i^\dagger(k_1) b_j^\dagger(k_2) b_j(k_3) b_i(k_4) + d_i^\dagger(k_1) d_j^\dagger(k_2) d_j(k_3) d_i(k_4)] \frac{\delta(k_1+k_2-k_3-k_4)}{2(k_1-k_4)^2} \\ &\qquad - b_i^\dagger(k_1) d_j^\dagger(k_2) d_j(k_3) b_i(k_4) \frac{\delta(k_1+k_2-k_3-k_4)}{(k_1-k_4)^2} \\ &\qquad + b_i^\dagger(k_1) d_i^\dagger(k_2) d_j(k_3) b_j(k_4) \frac{\delta(k_1+k_2-k_3-k_4)}{(k_1+k_2)^2} \Bigg\} \,, \\ P^-_2 &= \frac{e^2}{8\pi^3} \sum_{i,j}^{2} \int_0^\infty \frac{\Pi_i^4 dk_i}{\sqrt{k_1 k_2 k_3 k_4}} [b_i^\dagger(k_1) b_j^\dagger(k_2) d_j^\dagger(k_3) b_i(k_4) + b_i^\dagger(k_4) d_j(k_3) b_j(k_2) b_i(k_1) \\ &\qquad + d_i^\dagger(k_1) d_j^\dagger(k_2) b_j^\dagger(k_3) d_i(k_4) + d_i^\dagger(k_4) b_j(k_3) d_j(k_2) d_i(k_1)] \\ &\qquad \times \frac{\delta(k_1+k_2+k_3-k_4)}{(k_1-k_4)^2} \,. \end{aligned} \tag{2.11}$$

Because of light-cone kinematics, the momentum integrations are restricted to $[0,\infty)$. It explains why the Hamiltonian does not contain the terms which consist of only creation operators or only annihilation operators; they would break momentum conservation. The only exception is the "zero modes" $k^+=0$. They are in general supposed to be responsible for the nontrivial structure of vacua, such as spontaneous symmetry breaking. In the present case, however, we have tentatively dropped them because the presence of the mass term forces the wave functions to vanish at $k^+=0$.

B. Isospin multiplets

States are classified by the irreducible representations of the isospin $[SU(2)_V]$ symmetry with the conserved isospin charge:

$$I^a = \int dx^- j^{a+}(x^-) \,. \tag{2.12}$$

Let us first count how many independent wave functions there are. It is trivial to see that there are a triplet and a singlet for two-body states, $2^2 = 3+1$. For four-body states, we have $2^4 = 16$ states. An elementary consideration tells that there are one quintet, three triplets, and two singlets, $2^4 = 5 + 3\times 3 + 2\times 1$. Each multiplet has one independent wave function. Explicitly,

$$|2,2\rangle=\tfrac{1}{2}\int_0^{\mathcal{P}}\frac{\prod_i^4 dk_i}{(2\pi)^2\sqrt{k_1k_2k_3k_4}}\delta\left[\sum_{i=1}^4 k_i-\mathcal{P}\right]\psi_4(k_1,k_2,k_3,k_4)b_1^\dagger(k_1)b_1^\dagger(k_2)d_2^\dagger(k_3)d_2^\dagger(k_4)|0\rangle\ , \tag{2.13}$$

$$\begin{aligned}|1,1\rangle=&\int_0^{\mathcal{P}}\frac{dk_1dk_2}{2\pi\sqrt{k_1k_2}}\delta(k_1+k_2-\mathcal{P})\psi_2(k_1,k_2)b_1^\dagger(k_1)d_2^\dagger(k_2)|0\rangle\\&+\frac{1}{2}\int_0^{\mathcal{P}}\frac{\prod_i^4 dk_i}{(2\pi)^2\sqrt{k_1k_2k_3k_4}}\delta\left[\sum_{i=1}^4 k_i-\mathcal{P}\right]\\&\quad\times\{\psi^A(k_1,k_2,k_3,k_4)[b_1^\dagger(k_1)b_2^\dagger(k_2)d_2^\dagger(k_3)d_2^\dagger(k_4)+b_1^\dagger(k_1)b_1^\dagger(k_2)d_1^\dagger(k_3)d_2^\dagger(k_4)]\\&\quad+\sqrt{2}\psi^{1S}(k_1,k_2,k_3,k_4)b_1^\dagger(k_1)b_2^\dagger(k_2)d_2^\dagger(k_3)d_2^\dagger(k_4)\\&\quad+\sqrt{2}\psi^{2S}(k_1,k_2,k_3,k_4)b_1^\dagger(k_1)b_1^\dagger(k_2)d_1^\dagger(k_3)d_2^\dagger(k_4)\}|0\rangle\ ,\end{aligned} \tag{2.14}$$

$$\begin{aligned}|0,0\rangle=&\int_0^{\mathcal{P}}\frac{dk_1dk_2}{2\pi\sqrt{k_1k_2}}\delta(k_1+k_2-\mathcal{P})\psi_2'(k_1,k_2)\frac{1}{\sqrt{2}}[b_1^\dagger(k_1)d_1^\dagger(k_2)+b_2^\dagger(k_1)d_2^\dagger(k_2)]|0\rangle\\&+\frac{1}{2\sqrt{3}}\int_0^{\mathcal{P}}\frac{\prod_i^4 dk_i}{(2\pi)^2\sqrt{k_1k_2k_3k_4}}\delta\left[\sum_{i=1}^4 k_i-\mathcal{P}\right]\\&\quad\times\{\psi_3(k_1,k_2,k_3,k_4)[b_1^\dagger(k_1)b_1^\dagger(k_2)d_1^\dagger(k_3)d_1^\dagger(k_4)\\&\qquad+2b_1^\dagger(k_1)b_2^\dagger(k_2)d_1^\dagger(k_3)d_2^\dagger(k_4)+b_2^\dagger(k_1)b_2^\dagger(k_2)d_2^\dagger(k_3)d_2^\dagger(k_4)]\\&\quad+2\sqrt{3}\psi_0(k_1,k_2,k_3,k_4)b_1^\dagger(k_1)b_2^\dagger(k_2)d_1^\dagger(k_3)d_2^\dagger(k_4)\}|0\rangle\ ,\end{aligned} \tag{2.15}$$

where $\mathcal{P}$ is the total light-cone momentum. Other states are obtained by the application of I^-. Note that the $I=2$ states have no two-body components. The wave functions are assumed to have the following symmetries under exchanges of momenta:

$$\begin{aligned}&\psi_4(1,2,3,4)=-\psi_4(2,1,3,4)=-\psi_4(1,2,4,3)\ ,\\&\psi_4(1,2,3,4)=-\psi^A(2,1,3,4)=-\psi^A(1,2,4,3)\ ,\\&\psi^{1S}(1,2,3,4)=\psi^{1S}(2,1,3,4)=-\psi^{1S}(1,2,4,3)\ ,\\&\psi^{2S}(1,2,3,4)=-\psi^{2S}(2,1,3,4)=\psi^{2S}(1,2,4,3)\ ,\\&\psi_3(1,2,3,4)=-\psi_3(2,1,3,4)=-\psi_3(1,2,4,3)\ ,\\&\psi_0(1,2,3,4)=\psi_0(2,1,3,4)=\psi_0(1,2,4,3)\ .\end{aligned} \tag{2.16}$$

Charge conjugation invariance leads to further restrictions:

$$\begin{aligned}&\psi_2(1,2)=\pm\psi_2(2,1),\quad \psi_2'(1,2)=\mp\psi_2'(2,1)\ ,\\&\psi_4(1,2,3,4)=\pm\psi_4(3,4,1,2)\ ,\\&\psi^A(1,2,3,4)=\mp\psi^A(3,4,1,2)\ ,\\&\psi^{1S}(1,2,3,4)=\pm\psi^{2S}(3,4,1,2)\ ,\\&\psi_0(1,2,3,4)=\pm\psi_0(3,4,1,2)\ ,\\&\psi_3(1,2,3,4)=\pm\psi_3(3,4,1,2)\ ,\end{aligned} \tag{2.17}$$

where the upper (lower) sign corresponds to charge conjugation even (odd). In the following we will not exploit these restrictions, but rather use them as an important check for the results.

C. Einstein-Schrödinger equations

The LFTD approximation is to diagonalize the light-front Einstein-Schrödinger equation

$$2P^-P^+|\psi\rangle=M^2|\psi\rangle\ , \tag{2.18}$$

in the truncated Fock space. M^2 is the invariant mass. A constant of motion P^+ may be replaced by its eigenvalue $\mathcal{P}$ for our states (2.13)–(2.15). This simple equation leads to complicated coupled integral eigenvalue equations for wave functions. They are collected in Appendix B. In the following, we discuss the crudest Tamm-Dancoff truncation (keeping only two-body states) for the purpose of illustration. For the isotriplet the Einstein-Schrödinger equation (2.18) becomes

$$\frac{M^2}{2}\psi_2(x,1-x)=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\left[\frac{1}{x}+\frac{1}{1-x}\right]\psi_2(x,1-x)-\frac{e^2}{2\pi}\int_0^1 dy\frac{\psi_2(y,1-y)}{(x-y)^2}\ . \tag{2.19}$$

This is the same as that obtained by 't Hooft [20,21] in his study of two-dimensional QCD (QCD_2) in the $1/N$ expansion. It represents a pion consisting of a quark and an antiquark interacting through a linear potential. By integrating over x, one gets

$$M^2\int_0^1 dx\ \psi_2(x,1-x)=m^2\int_0^1 dx\left[\frac{1}{x}+\frac{1}{1-x}\right]\psi_2(x,1-x)\ . \tag{2.20}$$

One may easily see that, in the massless limit, it has an eigenvalue $M^2=0$. On the other hand, from the equation for the isosinglet,

$$\frac{M^2}{2}\psi_2'(x,1-x) = \left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\left[\frac{1}{x}+\frac{1}{1-x}\right]\psi_2'(x,1-x) + \frac{e^2}{2\pi}\int_0^1 dy\, \psi_2'(y,1-y)\left[2-\frac{1}{(x-y)^2}\right], \quad (2.21)$$

one sees that $M^2=\sqrt{2}e^2/\pi$ is an eigenvalue in the massless limit. The η is heavier than the pion. The difference comes from the "2" in Eq. (2.21), which is due to the annihilation force (the last term of P_0^-). In the next section, we will show that if we include four-body states there appears a lighter isosinglet state than the η.

III. NUMERICAL METHOD AND RESULTS

A. Basis functions

There are several ways to discretize the coupled integral eigenvalue equations [16,22]. Among them, we think that the method of basis functions [15] is most appropriate for our purpose. For the strong coupling, the behavior of the wave functions near the edges of momentum region is very important. The DLCQ is unable to express this behavior very well, though it is good for moderate values of the coupling.

The choice of basis functions is essential for efficient numerical methods. Mo and Perry studied the massive Schwinger model by using several basis functions. They showed that exact integrability of the matrix elements is very important because otherwise it would take much CPU time for numerical integrations for matrix elements. Their conclusion is that Jacobi polynomials are the most appropriate basis functions. When we include four-body states, however, the orthogonality of Jacobi polynomials does not help us much. In fact, their basis functions for four-body states are no longer orthogonal. We therefore use simpler basis functions which are equivalent to their basis functions.

One can easily see by inspection that the two-body wave functions must vanish at $x=0$ and 1 because of the mass term. (Here x is a fraction of momentum. See Appendix B.) Bergknoff [14] showed that they behave as x^β, where β is the solution of the equation

$$m^2-1+\pi\beta\cot(\pi\beta)=0 . \quad (3.1)$$

[Here and hereafter, we work in units of $e/\sqrt{\pi}=1$. The strong coupling limit is thus equivalent to the massless limit ($m\to 0$).] We expand the two-body wave functions in terms of $f_k(x)$ ($k=0,1,2,\ldots$):

$$\psi(x,1-x)=\sum_{k=0}^{N_2} a_k f_k(x) , \quad (3.2)$$

where

$$f_k(x)=\begin{cases}[x(1-x)]^{\beta+k} , \\ [x(1-x)]^{\beta+k}(2x-1) .\end{cases} \quad (3.3)$$

The four-body wave functions (except for ψ_4) must also vanish as x_i^β at $x_i=0$ ($i=1,2,3,4$). We expand the four-body wave functions in terms of $G_{\mathbf{k}}(x_1,x_2,x_3,x_4)$:

$$\psi(x_1,x_2,x_3,x_4)=\sum_{\mathbf{k}}^{N_4} b_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4) , \quad \sum_{i=1}^{4} x_i=1 , \quad (3.4)$$

where

$$G_{\mathbf{k}}(x_1,x_2,x_3,x_4)=\begin{cases}(x_1x_2x_3x_4)^\beta(x_1-x_2)^{k_1}(x_{12}x_{34})^{k_2}(x_3-x_4)^{k_3} , \\ (x_1x_2x_3x_4)^\beta(x_1-x_2)^{k_1}(x_{12}x_{34})^{k_2}(x_3-x_4)^{k_3}x_{12} ,\end{cases} \quad (3.5)$$

where $x_{12}=x_1+x_2$ and $x_{34}=x_3+x_4$. One can easily show that $(x_1x_2x_3x_4)^\beta x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}$ can be expressed in terms of $G_{\mathbf{k}}(x_1,x_2,x_3,x_4)$ of (3.5) uniquely, because of $x_{12}+x_{34}=1$. According to the symmetries under exchanges, k_1 and k_3 may take only odd or even integers. For example, in the expansion of ψ^A, k_1 and k_3 take only odd values.

B. Eigenvalue equations

By inserting the expansions of wave functions in terms of basis functions and projecting the equations to two- and four-body basis functions, we obtain eigenvalue equations of matrix form. For example, for isosinglet states, we have

$$\begin{aligned}&\psi_2'(x,1-x)=\sum_{k=0} a_k f_k(x) ,\\ &\psi_3(x_1,x_2,x_3,x_4)=\sum_{\mathbf{k}} b_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4) ,\\ &k_1,k_3 \text{ odd} ,\\ &\psi_0(x_1,x_2,x_3,x_4)=\sum_{\mathbf{k}} c_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4) ,\\ &k_1,k_3 \text{ even} ,\end{aligned} \quad (3.6)$$

and

$$M^2\begin{bmatrix}A&0&0\\0&B&0\\0&0&B\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}(m^2-1)C+D&\sqrt{6}(\tilde{E}-E)&-\sqrt{2}(\tilde{E}-E)\\\sqrt{6}(\tilde{E}-E)&(m^2-1)Q+R+S-4T+6U&-2\sqrt{3}U\\-\sqrt{2}(\tilde{E}-E)&-2\sqrt{3}U&(m^2-1)Q+R+S-4T+2U\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}, \quad (3.7)$$

where

$$A_{kl}=\int_0^1 dx f_k(x)f_l(x)\ ,$$

$$B_{kl}=\int_{(4)}G_k(x_1,x_2,x_3,x_4)G_l(x_1,x_2,x_3,x_4)\ ,$$

$$C_{kl}=\int_0^1 dx\frac{f_k(x)f_l(x)}{x(1-x)}\ ,$$

$$D_{kl}=\int_0^1 dx\,dy f_k(x)\left[2-\frac{1}{(x-y)^2}\right]f_l(y)\ ,$$

$$E_{kl}=\int_{(4)}f_k(x_1)\frac{1}{(x_2+x_3)^2}G_l(x_1,x_2,x_3,x_4)\ ,$$

$$\tilde{E}_{kl}=\int_{(4)}f_k(1-x_4)\frac{1}{(x_2+x_3)^2}G_l(x_1,x_2,x_3,x_4)\ , \quad (3.8)$$

$$Q_{kl}=\int_{(4)}G_k(x_1,x_2,x_3,x_4)\sum_{i=1}^{4}\frac{1}{x_i}G_l(x_1,x_2,x_3,x_4)\ ,$$

$$R_{kl}=\int_{(6)}G_k(x_1,x_2,x_3,x_4)\frac{1}{(x_1-y_1)^2}G_l(y_1,y_2,x_3,x_4)\ ,$$

$$S_{kl}=\int_{(6)'}G_k(x_1,x_2,x_3,x_4)\frac{1}{(x_3-y_3)^2}G_l(x_1,x_2,y_3,y_4)\ ,$$

$$T_{kl}=\int_{(6)''}G_k(x_1,x_2,x_3,x_4)\frac{1}{(x_1-y_1)^2}G_l(y_1,x_2,x_3,y_4)\ ,$$

$$U_{kl}=\int_{(6)''}G_k(x_1,x_2,x_3,x_4)\frac{1}{(x_1+x_4)^2}G_l(y_1,x_2,x_3,y_4)\ ,$$

with

$$\int_{(4)}\equiv\int_0^1\prod_{i=1}^4 dx_i\delta\left[\sum_{i=1}^4 x_i-1\right]\ ,$$

$$\int_{(6)}\equiv\int_0^1\prod_{i=1}^4 dx_i dy_1 dy_2\delta\left[\sum_{i=1}^4 x_i-1\right]\times\delta(x_1+x_2-y_1-y_2)\ ,$$

$$\int_{(6)'}\equiv\int_0^1\prod_{i=1}^4 dx_i dy_3 dy_4\delta\left[\sum_{i=1}^4 x_i-1\right]\times\delta(x_3+x_4-y_3-y_4)\ , \quad (3.9)$$

$$\int_{(6)''}\equiv\int_0^1\prod_{i=1}^4 dx_i dy_1 dy_4\delta\left[\sum_{i=1}^4 x_i-1\right]\times\delta(x_1+x_4-y_1-y_4)\ .$$

These are calculated by using the formulas collected in Appendix C. The matrix eigenvalue equations for the isotriplet and isoquintet are presented in Appendix D.

The matrix eigenvalue equation has the following form containing a "norm" A:

$$H\mathbf{x}=EA\mathbf{x}\ , \quad (3.10)$$

where H and A are $N\times N$ matrices. By first solving the eigenvalue problem for the norm A,

$$A\mathbf{v}_i=\lambda_i\mathbf{v}_i,\quad (\mathbf{v}_i,\mathbf{v}_j)=\delta_{ij}\ , \quad (3.11)$$

and then by using the rescaled eigenvectors $\mathbf{w}_i=\mathbf{v}_i/\sqrt{\lambda_i}$, we can transform (3.10) to the usual form

$$W^THW\mathbf{y}=E\mathbf{y}\ , \quad (3.12)$$

where

$$W=(\mathbf{w}_1,\ldots,\mathbf{w}_N)\ , \quad (3.13)$$

and $\mathbf{x}=W\mathbf{y}$. We diagonalize this rotated H numerically. Note that the eigenvectors $\mathbf{x}_i$ satisfy the relation $\mathbf{x}_i^TA\mathbf{x}_j=\delta_{ij}$ if $\mathbf{y}_i$ are orthonormalized.

In the following we will be concerned with only the strong coupling region because the structure of the mass spectrum is relatively simple there.

C. Two-body Tamm-Dancoff approximation

Let us briefly summarize the results for the crudest Tamm-Dancoff approximation, keeping only two-body states.

We have three parameters: the fermion mass (in units of $e/\sqrt{\pi}$) m, the largest value for k in the expansion (3.2), N_2, and the largest value for k_i in the expansion (3.4), N_4, which is neglected in this subsection [23]. First of all, we have to know how many basis functions are necessary in order to produce sufficiently accurate values. Figure 1 is the result for the lightest isotriplets. Remarkably, even only one basis function is enough to get sufficiently good values for the mass eigenvalues. The lowest mass at $m=1.0\times10^{-3}$ is $6.025\,93\times10^{-2}$ for one basis function and $6.025\,93\times10^{-2}$ for eight basis functions. This should be called a "pion," whose mass vanishes in the massless limit. Figure 2 shows the same result for the lightest isosinglets. The lowest mass at $m=1.0\times10^{-3}$ is 1.415 50 for one basis function and 1.415 50 for eight basis functions. As we have seen, it should become $\sqrt{2}$ in the massless limit. We call this state "η," the lightest isosinglet in the valence state. For both cases the convergence is very fast even for higher states. The differences between the values for one basis function and for eight basis functions becomes appreciable for larger fermion mass, but they are the same in the first five digits even for $m=1.0$.

In the next subsection, we will include four-body states keeping $N_2=3$ which seems large enough.

D. Two- and four-body Tamm-Dancoff approximation

The convergence of the mass spectrum is not so drastic when the number of four-body basis functions changes.

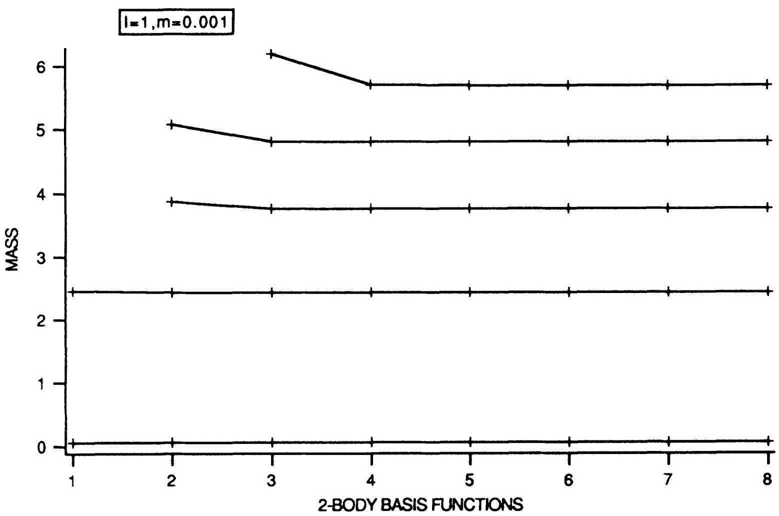

FIG. 1. Two-body Tamm-Dancoff approximation for isotriplets. The lightest five states are shown with the total number of basis functions. The convergence is faster for smaller masses. The lowest state has a very small mass. We call this the "pion."

This is presumably because of the complicated shapes of the four-body wave functions. Figure 3 shows for isotriplets how the masses converge when N_4 increases. The total number of four-body basis functions is

$$N_A+N_{1S}+N_{2S} =2\left[\frac{N_4+1}{2}\right]^2\times(N_4+1) +2\left[2\times\left[\frac{N_4+1}{2}\right]\times\left[\frac{N_4+2}{2}\right]\times(N_4+1)\right],$$

where [ ] is Gauss' symbol. For $N_4=3$, it is 96. Figure 4 is for isosinglets. The isotriplet and isosinglet mass spectra are shown in Fig. 5.

From these, we see that $N_4=3$ is large enough. In the following, most calculations are done with $N_4=3$.

The lightest state is pion. Compare with Fig. 1. The pion mass at $m=1.0\times10^{-3}$ is 5.52473×10^{-2} for $N_4=4$, and it is 5.53050×10^{-2} for $N_4=1$. See Fig. 6. The fact that the value does not change much (only 9%) by the inclusion of four-body states implies that this state is in its valence state. In fact, one can see by examining the wave functions that the probability of being in the two-body (symmetric under $x_1\leftrightarrow x_2$) state is 98.36% at $m=1.0\times10^{-3}$. This state is G-parity even. If we could confirm that it is parity odd, our identification would be complete, i.e., 1^{-+}. But in the light-cone quantization, parity is very difficult to implement. In more than two spatial dimensions, one may define a kind of parity operation which leaves the quantization plane intact by using a spatial rotation. But in one spatial dimension, there are no rotations. In conclusion, we fail to implement parity in our scheme. It turns out, however, that charge conjugation and the probabilities of being in specific states are, in most cases, very powerful in identifying states.

It is interesting to see how the pion mass varies with the fermion mass. Figure 7 shows that the pion mass is roughly proportional to the square root of the fermion mass:

$$\ln m_\pi=0.564(2)+0.5007(2)\ln m \quad (3.14)$$

Note that it seems consistent with the usual notion of current quark masses. Grady [13] got 0.58±0.10 instead

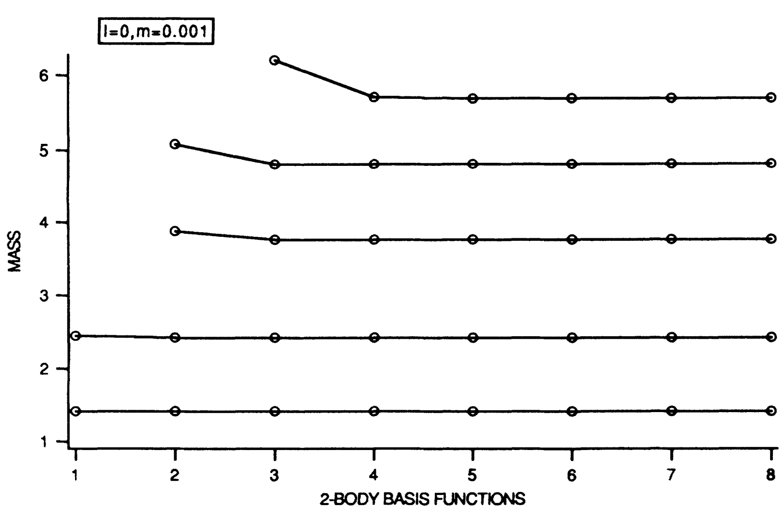

FIG. 2. Two-body Tamm-Dancoff approximation for isosinglets. The lightest five states are shown. Note that the spectrum is very similar to that of the one-flavor model. We call the lowest state the "η."

of 0.5007(2) and claimed that it is consistent with $\frac{2}{3}$. We will discuss this in the next section.

It is also easy to find η in the spectrum. Its mass at $m=1.0\times10^{-3}$ is 1.4154873 for $N_4=4$ and 1.4154960 for $N_4=1$. See Fig. 8. The two-body result is extremely close to this value. [The difference is less than $(1\times10^{-3})\%$.] The probability of being in the two-body (symmetric under $x_1 \leftrightarrow x_2$) is more than 99.999% at $m=1.0\times10^{-3}$. The dependence of the mass on the fermion mass is shown in Fig. 9:

$$\ln(m_\eta-\sqrt{2})=0.19(1)+0.993(2)\ln m \ . \tag{3.15}$$

Note that $m_\eta-\sqrt{2}$ is roughly proportional to m, in contrast with the pion mass, $m_\pi \sim m^{1/2}$. This state is G-parity odd.

It is very interesting to see that η is no longer the lightest isosinglet once four-body states are included. Figure 4 shows that the lightest isosinglet is not η, but it tends to be massless in the $m \to 0$ limit. Its mass is 9.73543×10^{-2} at $m=1.0\times10^{-3}$ for $N_4=4$. The convergence is relatively slow. The isosinglet/isotriplet mass ratio is 1.762. This should be compared with $\sqrt{3}$ in the strong coupling limit, obtained by Coleman. Note, however, that the pion/η mass ratio is $(5.5247\times10^{-2})/1.41549=3.65\times10^{-2}$; the coupling is not quite strong. Actually, $1.762=\sqrt{3}+O(m_\pi/m_\eta)$. Thus we think that the above value is consistent with Coleman's result. We will shortly show that this state is G-parity even.

In order to identify states, it is useful to introduce the *meson* operators

$$\begin{aligned}
A^\dagger_{11}(k_1,k_2)&=b^\dagger_1(k_1)d^\dagger_2(k_2)\ ,\\
A^\dagger_{10}(k_1,k_2)&=-\frac{1}{\sqrt{2}}[b^\dagger_1(k_1)d^\dagger_1(k_2)-b^\dagger_2(k_1)d^\dagger_2(k_2)]\ ,\\
A^\dagger_{1-1}(k_1,k_2)&=-b^\dagger_2(k_1)d^\dagger_1(k_2)\ ,\\
A^\dagger_{00}(k_1,k_2)&=\frac{1}{\sqrt{2}}[b^\dagger_1(k_1)d^\dagger_1(k_2)+b^\dagger_2(k_1)d^\dagger_2(k_2)]\ .
\end{aligned} \tag{3.16}$$

The isosinglet state $|0,0\rangle$ is rewritten in terms of the meson operators as

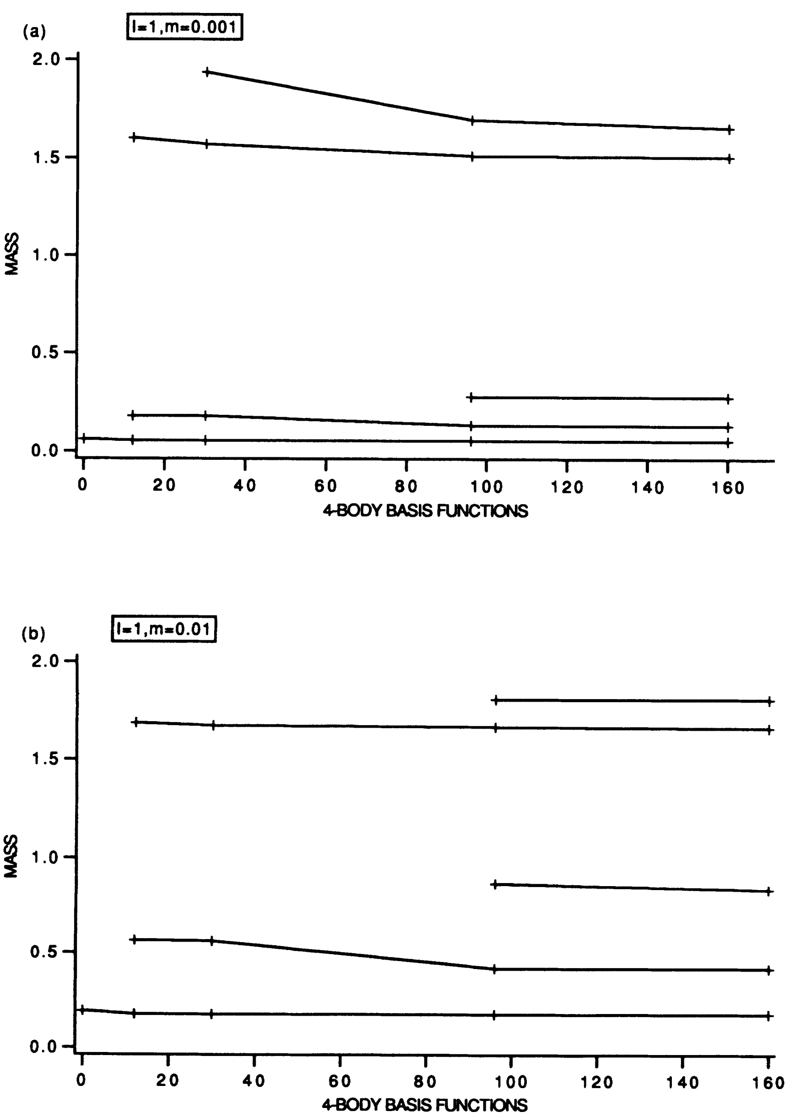

FIG. 3. Two- and four-body Tamm-Dancoff approximation for isotriplets. The lightest five states are shown with the total number of four-body basis functions. (a) is for $m=1.0\times10^{-3}$ and (b) for $m=1.0\times10^{-2}$. The increase of the number of basis functions leads to finer resolution. Some ("spurious") states appear when the number of basis functions is large enough. We think that they are scattering states. They do not appear to reach the asymptotic values within our calculations.

$$|0,0\rangle = \int_0^{\mathcal{P}} \frac{dk_1 dk_2}{2\pi\sqrt{k_1 k_2}} \delta(k_1+k_2-\mathcal{P})\psi_2'(k_1,k_2) A_{00}^{\dagger}(k_1,k_2)|0\rangle$$
$$+\frac{1}{2\sqrt{3}}\int_0^{\mathcal{P}} \frac{\prod_i^4 dk_i}{(2\pi)^2\sqrt{k_1k_2k_3k_4}}\delta\left[\sum_{i=1}^{4} k_i - \mathcal{P}\right]$$
$$\times\{\psi_3(k_1,k_3,k_4,k_2)[A_{00}^{\dagger}(k_1,k_2)A_{00}^{\dagger}(k_3,k_4) - A_{00}^{\dagger}(k_1,k_4)A_{00}^{\dagger}(k_3,k_2)]$$
$$-2\sqrt{3}\psi_0(k_1,k_3,k_4,k_2)[A_{11}^{\dagger}(k_1,k_2)A_{1-1}^{\dagger}(k_3,k_4)$$
$$-A_{10}^{\dagger}(k_1,k_2)A_{10}^{\dagger}(k_3,k_4)+A_{1-1}^{\dagger}(k_1,k_2)A_{11}^{\dagger}(k_3,k_4)]\}|0\rangle \ . \qquad (3.17)$$

From this expression it is now clear that ψ_0 is the wave function for the π-π system. For the lightest isosinglet, the probability of being in the four-body π-π state (ψ_0) is calculated to be 54.07%, while it is 41.84% for the two-body (antisymmetric under $x_1 \leftrightarrow x_2$) state. (All the rest is for η-η component.) This state is G-parity even. [See (2.17).]

Now we have obtained the answer to Coleman's question. (1) The lightest isosinglet 0^{++} is a pion-pion bound state. (2) In the weak coupling region, its mass will become almost twice the pion mass, while the pion and η masses will become almost degenerate. This is why the 0^{++} state is far away up from these states in the weak coupling region. (3) Why is it so light? (a) Because it is a bound state of two pions. Its existence would be found by nonrelativistic reasoning as Coleman demonstrated for

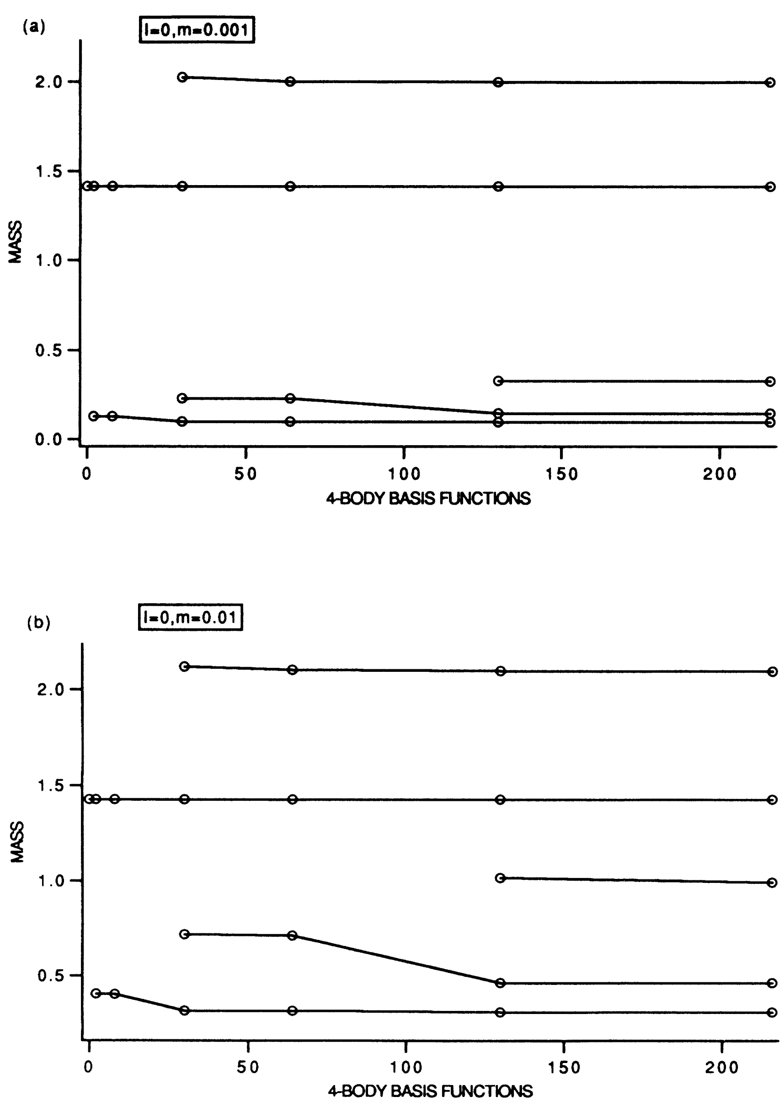

FIG. 4. Two- and four-body Tamm-Dancoff approximation for isosinglets. The lightest five states are shown with the total number of four-body basis functions. (a) is for $m=1.0\times10^{-3}$ and (b) for $m=1.0\times10^{-2}$. Compare with Fig. 2. The lightest state is not η, but tends to massless in the $m\rightarrow0$ limit. There are also "spurious" states.

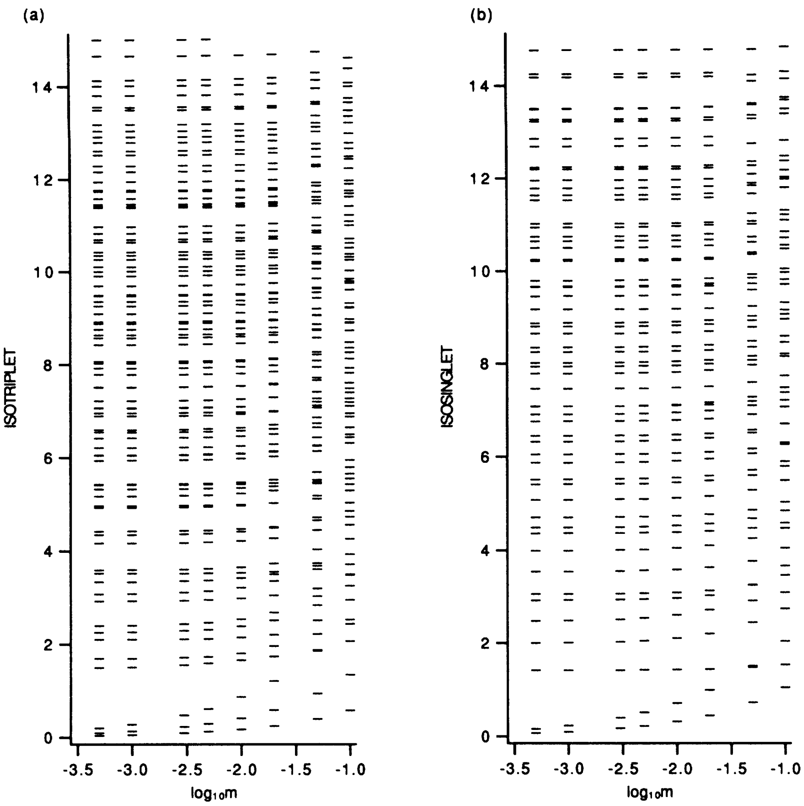

FIG. 5. Isotriplet mass spectrum below $M=15$ with $(N_2,N_4)=(3,3)$ is shown in (a). (b) is the same result for the isosinglet.

the one-flavor model. Its mass should be smaller than $2m_\pi$. (b) The annihilation force makes significant effects only on the two-body (symmetric under $x_1 \leftrightarrow x_2$) [24] component of isosinglets. As Coleman noted, it shifts symmetric (parity odd) states up. In fact, by examining the wave functions of isosinglets lighter than η, one can see that they do not have the two-body component which is symmetric under $x_1 \leftrightarrow x_2$ (parity odd). What prevented Coleman from saying that this is the reason is that the story seems different for the one-flavor model. The point is that in the one-flavor model there are no light valence states; the meson has mass $e/\sqrt{\pi}$ in the strong coupling

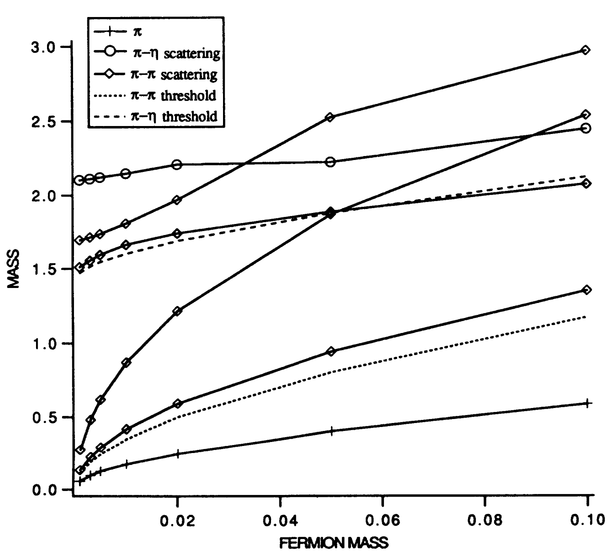

FIG. 6. Low-lying isotriplet states. The pion-pion and pion-η thresholds are shown. We do not find any candidates for pion-pion and pion-η bound states but only pion-pion scattering states.

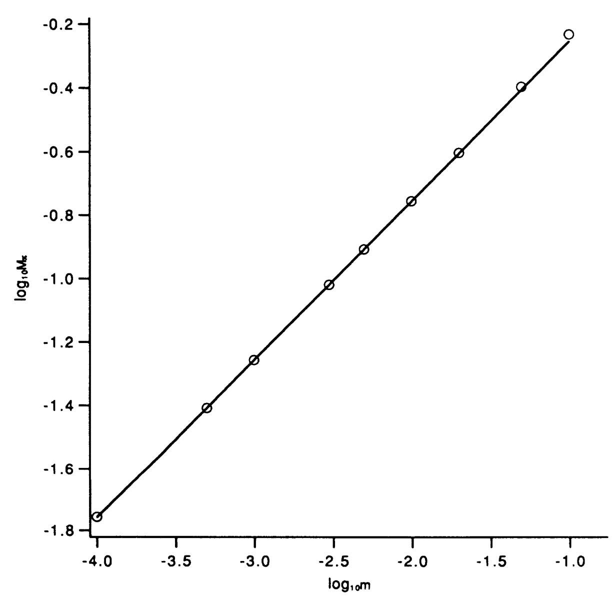

FIG. 7. Relation between the fermion and pion masses. The straight line is obtained by fitting the first six data by the method of least squares.

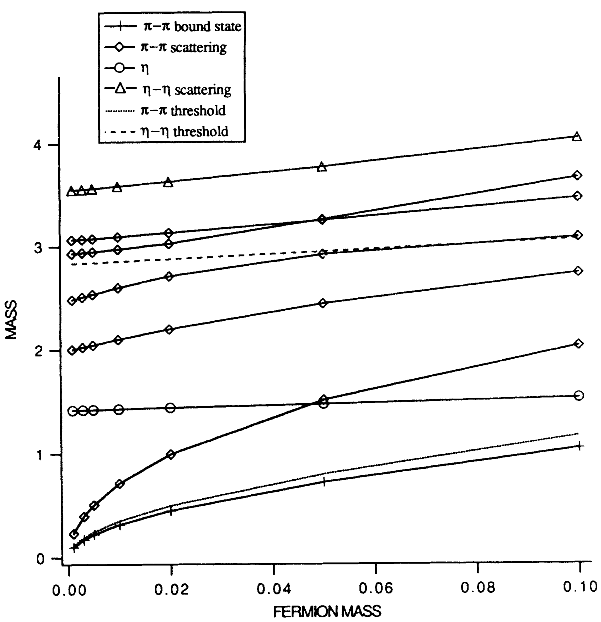

FIG. 8. Low-lying isosinglet states. The pion state and the pion-pion and η-η thresholds are shown for convenience. It is clear that the lowest state is below the threshold for a wide range of the coupling constant.

limit, while in the two-flavor model there is a light valence state (pion). The isosinglet sector knows the existence of the light isotriplet. Equation (3.17) shows why the one-flavor model does not have light states. The information of the light state is encoded through ψ_0, while the four-body wave function in the one-flavor model is similar to ψ_3. If we had put $\psi_0=0$ by hand, we would have obtained a spectrum very similar to that of the one-flavor model.

Let us look for other bound states for the strong couplings in Figs. 6 and 8. We look for η-η bound states near the threshold, but such states cannot be found below the threshold. Above the threshold we find a state which

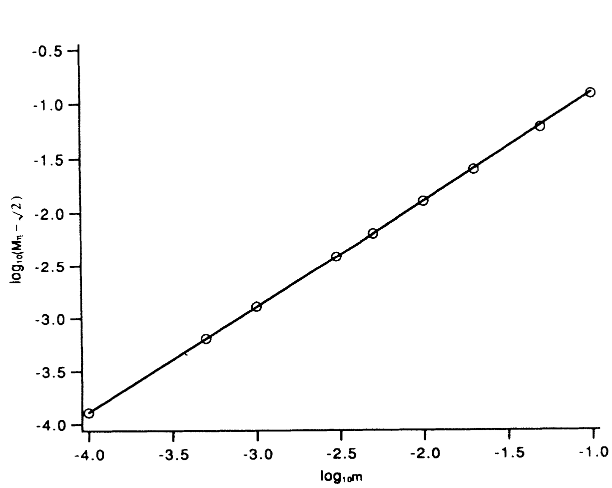

FIG. 9. Relation between the fermion and η masses. The straight line is obtained by fitting the first six data by the method of least squares.

may be regarded as a scattering state. (This is supported by the fact that the ψ_3 component is dominant for this state and that for other states below the threshold the ψ_0 and/or $4'_2$ components are dominant. The following analysis proceeds analogously.) We look for π-η bound states in the isotriplet spectrum. We cannot find any candidate, but above the threshold, we find a state which may be a scattering state. The π-π bound states in the isotriplet are not found, either.

Finally, let us discuss isoquintets. Because there are no two-body wave functions, we could not determine analytically how the four-body wave function behaves near the edge of momenta. We have to vary the β to find the best trial functions. Interestingly, however, for the small fermion mass the β for the best trial functions is just what we obtain by using (3.1), though the difference becomes appreciable for large values. The spectrum with such β is shown with the total number of four-body basis functions in Fig. 10. Because the masses for low-lying states are the same for $N_4=3$ and 4, the following calculations are done with $N_4=3$.

Figure 11 shows the mass spectrum for various values of the fermion mass.

The low-lying states are shown in Fig. 12. The lightest state is above the pion-pion threshold, indicating that there are no isoquintet bound states.

E. Pion decay constant

Let us first normalize the pion state in the Lorentz-invariant way:

$$\langle \pi^a(p)|\pi^b(q)\rangle=\delta^{ab}2p^+(2\pi)\delta(p^+-q^+) \ . \tag{3.18}$$

The pion decay constant may be defined by analogy as

$$\langle 0|\partial_\mu j_5^{a\mu}(0)|\pi^b(p)\rangle=m_\pi^2 f_\pi \delta^{ab} \ . \tag{3.19}$$

Note that in two dimensions f_π is a dimensionless constant.

We define

$$|\pi^1(\mathcal{P})\rangle=i\sqrt{2\pi}(|1,1\rangle-|1,-1\rangle) \ ,$$

which satisfies the normalization condition (3.18), assuming

$$\int_0^1 dx_1 dx_2 \delta(x_1+x_2-1)|\psi_2|^2 + \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta\left[\sum_{i=1}^4 x_i - 1\right] \times(|\psi^A|^2+|\psi^{1S}|^2+|\psi^{2S}|^2)=1 \ . \tag{3.20}$$

The (phase) factor i has been introduced for later convenience. Now, from (2.8) and (2.14), one can easily obtain

$$\langle 0|\partial_\mu j_5^{a\mu}(0)|\pi^b(p)\rangle=\delta^{ab}\frac{m^2}{\sqrt{2\pi}}\int_0^1 dx \frac{\psi_2(x,1-x)}{x(1-x)} \ ; \tag{3.21}$$

thus,

$$f_\pi=\frac{m^2}{\sqrt{2\pi}m_\pi^2}\int_0^1 dx\frac{\psi_2(x,1-x)}{x(1-x)} \ . \tag{3.22}$$

The right-hand side can be calculated numerically, as shown in Fig. 13. In the strong coupling region, it is almost independent of the fermion mass, indicating that PCAC (partial conservation of axial vector current) is a valid concept. In the limit, it is 0.3945,

$$f_\pi = 0.394\,518(4) + 2.26(8)\times 10^{-2} m \ . \tag{3.23}$$

IV. DISCUSSION

We have obtained the mass spectrum of the massive Schwinger model numerically in the LFTD approximation and have seen that the LFTD approach is very powerful in the study of bound states. In particular, because we obtain the wave functions simultaneously, the identification of the states is easy. We also examine PCAC by seeing if the "pion decay constant" is really a constant in the small mass region and how the pion mass changes with the fermion mass. Remarkably, PCAC is very good even for this two-dimensional toy model.

In the following we list several unsolved problems.

(1) The relation between the pion mass and the fermion mass is consistent with the usual notion of current quark masses and the pion decay constant depends only weakly on the fermion mass in the strong coupling region. On the other hand, however, if we use the current algebra relation,

$$f_\pi^2 m_\pi^2 = m \left\langle 0 \left| \sum_{i=1}^{2} \bar{\psi}_i \psi_i \right| 0 \right\rangle , \tag{4.1}$$

it follows that the condensation $\langle 0|\sum_{i=1}^{2}\bar{\psi}_i\psi_i|0\rangle$ is independent of m. (Note that unfortunately we are not able to calculate it directly in our scheme because we normal order the fermion bilinear and discard the zero modes completely.) On the other hand, since it is naively [25] expected that pion mass is proportional to $m^{2/3}$ in the strong coupling limit from the work by Dashen, Hasslacher, and Neveu [26] and Coleman [8]. It seems, however, to us that the value obtained by Grady is also

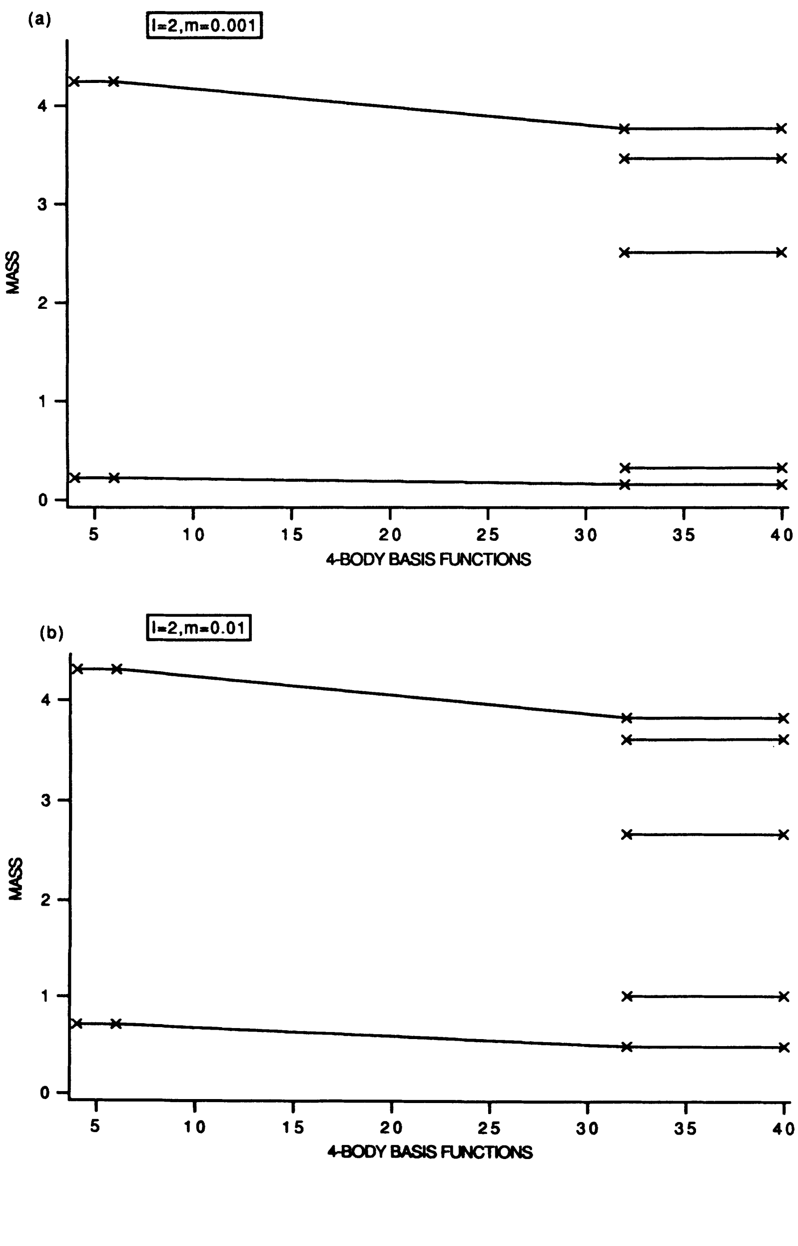

FIG. 10. Four-body Tamm-Dancoff approximation for isoquintet states. The first five states are shown with the total number of four-body basis functions. (a) is for $m = 1.0\times 10^{-3}$ and (b) for $m = 1.0\times 10^{-2}$.

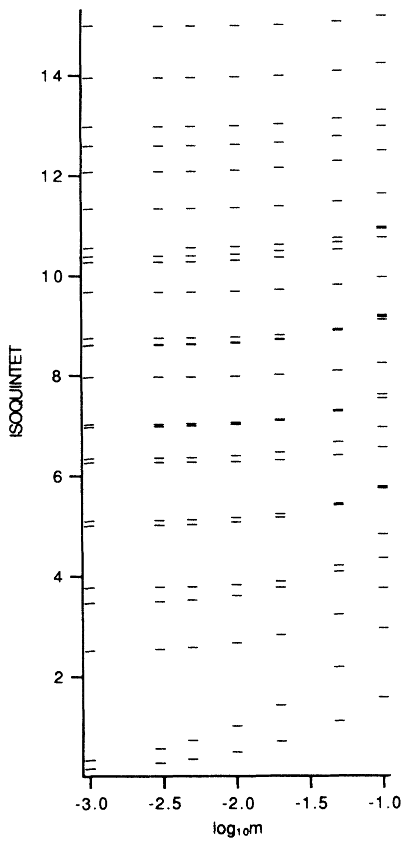

FIG. 11. Isoquintet mass spectrum with $N_4=3$.

consistent with $m^{1/2}$. On the other hand, Grady [13] also showed that the condensation behaves like $m^{0.32\pm0.02} \simeq m^{1/3}$, consistent with the massless case, where it should be zero because spontaneous symmetry breaking cannot occur. At present, we are not able to point out the reason for this discrepancy.

(2) It is difficult to implement parity in our formulation. At the present, we do not know how to do it.

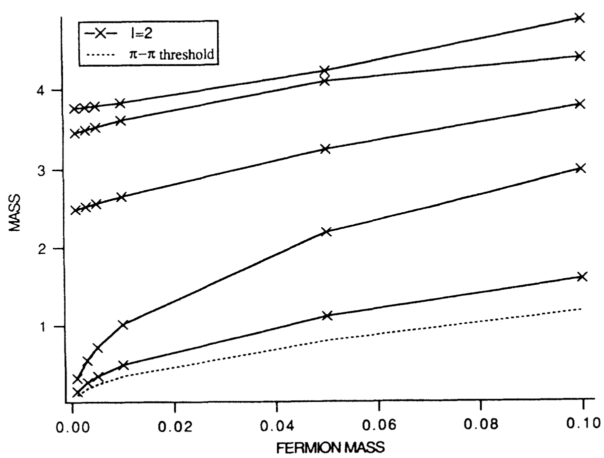

FIG. 12. Low-lying isoquintet states. The pion state and pion-pion threshold are shown for convenience. We do not find any candidates for bound states.

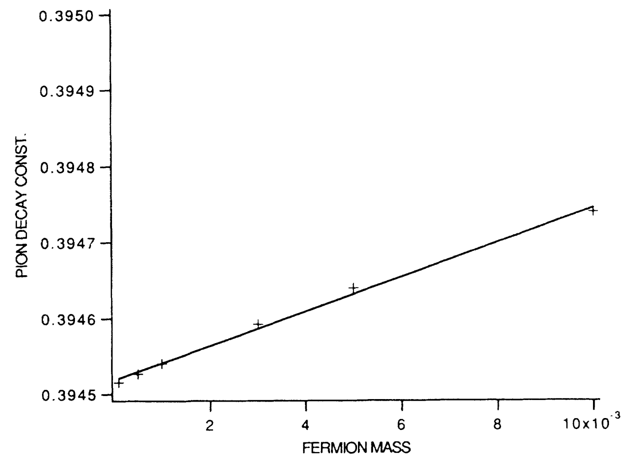

FIG. 13. Pion decay constant f_π as a function of the fermion mass. It is almost independent of the fermion mass. The straight line is the least-squares fitting.

(3) It is interesting to investigate effects of the nonzero vacuum angle. One could calculate numerically how θ affects the spectrum. The case of $\theta=\pi$ is of particular interest because Coleman [8] showed that in this case there appear half-asymptotic particles. There is a lattice study [27] which finds evidence for a phase transition.

(4) The inclusion of six-body states will reveal the existence (or nonexistence) of a bound state of three pions. One may infer its existence from the analysis of the one-flavor model when $\theta=0$. It might be simpler to use DLCQ for this purpose.

(5) Because we know the (sufficiently accurate) wave functions of bound states, we may formulate scattering of a bound state off a bound state. Although the scattering in one spatial dimension appears trivial, we may study the off-shell physics of the model.

ACKNOWLEDGMENTS

K.H. is grateful to T. Fujita for discussions, especially for the suggestion that the lightest isosinglet might be a bound state of two pions. This work was supported by a Grant-in-Aid for Encouragement of Young Scientists (No. 05740181) from the Ministry of Education, Science and Culture.

APPENDIX A: NOTATION AND CONVENTIONS

In this appendix we summarize the notation and conventions used in this paper. They are essentially the same as those of Perry and Harindranath [28]. The metric is given by

$$g^{\mu\nu}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad (\mu,\nu=0,1)\ ,$$
$$g^{\mu\nu}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad (\mu,\nu=+,-)\ , \tag{A1}$$

where

$$x^{\pm}=(x^0\pm x^1)/\sqrt{2}\ . \tag{A2}$$

We treat x^+ as our "time."

Accordingly, the γ matrices are defined as

$$\gamma^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma^1=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$
$$\gamma^5=\gamma^0\gamma^1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$
$$\gamma^+=\begin{bmatrix}0 & 0\\ \sqrt{2} & 0\end{bmatrix},\quad \gamma^-=\begin{bmatrix}0 & \sqrt{2}\\ 0 & 0\end{bmatrix},\tag{A3}$$
$$\gamma^+\gamma^-=\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix},\quad \gamma^-\gamma^+=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix};$$

thus, $\psi=(\psi_R,\psi_L)^T$. The totally antisymmetric tensor $\epsilon^{\mu\nu}$ is defined by

$$\epsilon^{01}=\epsilon^{-+}=+1\ .\tag{A4}$$

APPENDIX B: EINSTEIN-SCHRÖDINGER EQUATIONS

We give a complete set of coupled integral eigenvalue equations obtained by applying the Hamiltonian (2.11) to the states (2.13)–(2.15). We have rescaled the total momentum $\mathcal{P}$ out by changing variables to momentum fractions $x_i=k_i/\mathcal{P}$. The two-body wave function $\psi_2(k_1,k_2)$ is replaced by $\psi_2(x_1,x_2)$, but the four-body wave function $\psi^A(k_1,k_2,k_3,k_4)$ by $\psi^A(x_1,x_2,x_3,x_4)/\mathcal{P}$ and other wave functions analogously. For the four-body wave functions, $\sum_{i=1}^4 x_i=1$. We introduce Ψ and Φ for notational convenience:

$$\Psi=\psi^{1S}+\psi^{2S}+\sqrt{2}\psi^A\ ,\qquad \Phi=\frac{1}{2}\psi_0-\frac{\sqrt{3}}{2}\psi_3\ .\tag{B1}$$

For isospin=2,

$$\begin{aligned}\frac{M^2}{2}\psi_4(x_1,x_2,x_3,x_4)=&\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\sum_{i=1}^{4}\frac{1}{x_i}\psi_4(x_1,x_2,x_3,x_4)\\&+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_2-y_1-y_2)\psi_4(y_1,y_2,x_3,x_4)\left[\frac{1}{2(x_1-y_1)^2}-\frac{1}{2(x_2-y_1)^2}\right]\right.\\&+\delta(x_3+x_4-y_1-y_2)\psi_4(x_1,x_2,y_1,y_2)\left[\frac{1}{2(x_3-y_1)^2}-\frac{1}{2(x_4-y_1)^2}\right]\\&+\delta(x_1+x_4-y_1-y_2)\psi_4(y_1,x_2,y_2,x_3)\frac{1}{(x_1-y_1)^2}\\&-\delta(x_2+x_4-y_1-y_2)\psi_4(y_1,x_1,y_2,x_3)\frac{1}{(x_2-y_1)^2}\\&-\delta(x_1+x_3-y_1-y_2)\psi_4(y_1,x_2,y_2,x_4)\frac{1}{(x_1-y_1)^2}\\&\left.+\delta(x_2+x_3-y_1-y_2)\psi_4(y_1,x_1,y_2,x_4)\frac{1}{(x_2-y_1)^2}\right\}.\end{aligned}\tag{B2}$$

For isospin=1,

$$\begin{aligned}\frac{M^2}{2}\psi_2(x,1-x)=&\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\left[\frac{1}{x}+\frac{1}{1-x}\right]\psi_2(x,1-x)-\frac{e^2}{2\pi}\int_0^1 dy\frac{\psi_2(y,1-y)}{(x-y)^2}\\&+\frac{e^2}{\pi}\int_0^1 dy_1dy_2dy_3\frac{\delta(y_1+y_2+y_3-x)}{(x-y_1)^2}\\&\times\left[\psi^A(y_1,y_2,y_3,1-x)+\frac{1}{\sqrt{2}}[\psi^{1S}(y_1,y_2,y_3,1-x)+\psi^{2S}(y_1,y_2,y_3,1-x)]\right]\\&-\frac{e^2}{\pi}\int_0^1 dy_2dy_3dy_4\frac{\delta(y_2+y_3+y_4-(1-x))}{[(1-x)-y_4]^2}\\&\times\left[\psi^A(x,y_2,y_3,y_4)+\frac{1}{\sqrt{2}}[\psi^{1S}(x,y_2,y_3,y_4)+\psi^{2S}(x,y_2,y_3,y_4)]\right],\end{aligned}\tag{B3}$$

$$
\begin{aligned}
\frac{M^2}{2}\psi^A(x_1,x_2,x_3,x_4)
&=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\sum_{i=1}^{4}\frac{1}{x_i}\psi^A(x_1,x_2,x_3,x_4)\\
&\quad+\frac{e^2}{4\pi}\left\{\psi_2(x_1,1-x_1)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_2+x_3)^2}\right]-\psi_2(x_2,1-x_2)\left[\frac{1}{(x_1+x_4)^2}-\frac{1}{(x_1+x_3)^2}\right]\right.\\
&\qquad\left.-\psi_2(1-x_3,x_3)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_1+x_4)^2}\right]+\psi_2(1-x_4,x_4)\left[\frac{1}{(x_2+x_3)^2}-\frac{1}{(x_1+x_3)^2}\right]\right\}\\
&\quad+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_2-y_1-y_2)\psi^A(y_1,y_2,x_3,x_4)\left[\frac{1}{2(x_1-y_1)^2}-\frac{1}{2(x_2-y_1)^2}\right]\right.\\
&\qquad+\delta(x_3+x_4-y_1-y_2)\psi^A(x_1,x_2,y_1,y_2)\left[\frac{1}{2(x_3-y_1)^2}-\frac{1}{2(x_4-y_1)^2}\right]\\
&\qquad-\delta(x_1+x_4-y_1-y_2)\psi^A(x_2,y_1,y_2,x_3)\frac{1}{(x_1-y_1)^2}\\
&\qquad+\delta(x_2+x_4-y_1-y_2)\psi^A(x_1,y_1,y_2,x_3)\frac{1}{(x_2-y_1)^2}\\
&\qquad+\delta(x_1+x_3-y_1-y_2)\psi^A(x_2,y_1,y_2,x_4)\frac{1}{(x_1-y_1)^2}\\
&\qquad\left.-\delta(x_2+x_3-y_1-y_2)\psi^A(x_1,y_1,y_2,x_4)\frac{1}{(x_2-y_1)^2}\right\}\\
&\quad+\frac{e^2}{2\sqrt{2}\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_4-y_1-y_2)\Psi(x_2,y_1,y_2,x_3)\frac{1}{(x_1+x_4)^2}\right.\\
&\qquad-\delta(x_2+x_4-y_1-y_2)\Psi(x_1,y_1,y_2,x_3)\frac{1}{(x_2+x_4)^2}\\
&\qquad-\delta(x_1+x_3-y_1-y_2)\Psi(x_2,y_1,y_2,x_4)\frac{1}{(x_1+x_3)^2}\\
&\qquad\left.+\delta(x_2+x_3-y_1-y_2)\Psi(x_1,y_1,y_2,x_4)\frac{1}{(x_2+x_3)^2}\right\},
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
\frac{M^2}{2}\psi^{1S}(x_1,x_2,x_3,x_4)
&=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\sum_{i=1}^{4}\frac{1}{x_i}\psi^{1S}(x_1,x_2,x_3,x_4)+\frac{e^2}{4\sqrt{2}\pi}\left\{\psi_2(x_1,1-x_1)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_2+x_3)^2}\right]\right.\\
&\qquad+\psi_2(x_2,1-x_2)\left[\frac{1}{(x_1+x_4)^2}-\frac{1}{(x_1+x_3)^2}\right]\\
&\qquad-\psi_2(1-x_3,x_3)\left[\frac{1}{(x_2+x_4)^2}+\frac{1}{(x_1+x_4)^2}\right]\\
&\qquad\left.+\psi_2(1-x_4,x_4)\left[\frac{1}{(x_2+x_3)^2}+\frac{1}{(x_1+x_3)^2}\right]\right\}\\
&\quad+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_2-y_1-y_2)\psi^{1S}(y_1,y_2,x_3,x_4)\left[\frac{1}{2(x_1-y_1)^2}+\frac{1}{2(x_2-y_1)^2}\right]\right.\\
&\qquad+\delta(x_3+x_4-y_1-y_2)\psi^{1S}(x_1,x_2,y_1,y_2)\left[\frac{1}{2(x_3-y_1)^2}-\frac{1}{2(x_4-y_1)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\delta(x_1+x_4-y_1-y_2)\psi^{1S}(x_2,y_1,y_2,x_3)\frac{1}{(x_1-y_1)^2}\\
&+\delta(x_2+x_4-y_1-y_2)\psi^{1S}(x_1,y_1,y_2,x_3)\frac{1}{(x_2-y_1)^2}\\
&-\delta(x_1+x_3-y_1-y_2)\psi^{1S}(x_2,y_1,y_2,x_4)\frac{1}{(x_1-y_1)^2}\\
&\left.-\delta(x_2+x_3-y_1-y_2)\psi^{1S}(x_1,y_1,y_2,x_4)\frac{1}{(x_2-y_1)^2}\right\}\\
&+\frac{e^2}{4\pi}\int_0^1 dy_1 dy_2\left\{-\delta(x_1+x_4-y_1-y_2)\Psi(x_2,y_1,y_2,x_3)\frac{1}{(x_1+x_4)^2}\right.\\
&-\delta(x_2+x_4-y_1-y_2)\Psi(x_1,y_1,y_2,x_3)\frac{1}{(x_2+x_4)^2}\\
&+\delta(x_1+x_3-y_1-y_2)\Psi(x_2,y_1,y_2,x_4)\frac{1}{(x_1+x_3)^2}\\
&\left.+\delta(x_2+x_3-y_1-y_2)\Psi(x_1,y_1,y_2,x_4)\frac{1}{(x_2+x_3)^2}\right\},
\end{aligned}
\tag{B5}
$$

$$
\begin{aligned}
&\frac{M^2}{2}\psi^{2S}(x_1,x_2,x_3,x_4)\\
&=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\sum_{i=1}^{4}\frac{1}{x_i}\psi^{2S}(x_1,x_2,x_3,x_4)\\
&+\frac{e^2}{4\sqrt{2}\pi}\left\{-\psi_2(x_1,1-x_1)\left[\frac{1}{(x_2+x_4)^2}+\frac{1}{(x_2+x_3)^2}\right]+\psi_2(x_2,1-x_2)\left[\frac{1}{(x_1+x_4)^2}+\frac{1}{(x_1+x_3)^2}\right]\right.\\
&\left.+\psi_2(1-x_3,x_3)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_1+x_4)^2}\right]+\psi_2(1-x_4,x_4)\left[\frac{1}{(x_2+x_3)^2}-\frac{1}{(x_1+x_3)^2}\right]\right\}\\
&+\frac{e^2}{2\pi}\int_0^1 dy_1 dy_2\left\{\delta(x_1+x_2-y_1-y_2)\psi^{2S}(y_1,y_2,x_3,x_4)\left[\frac{1}{2(x_1-y_1)^2}-\frac{1}{2(x_2-y_1)^2}\right]\right.\\
&+\delta(x_3+x_4-y_1-y_2)\psi^{2S}(x_1,x_2,y_1,y_2)\left[\frac{1}{2(x_3-y_1)^2}+\frac{1}{2(x_4-y_1)^2}\right]\\
&+\delta(x_1+x_4-y_1-y_2)\psi^{2S}(x_2,y_1,y_2,x_3)\frac{1}{(x_1-y_1)^2}\\
&-\delta(x_2+x_4-y_1-y_2)\psi^{2S}(x_1,y_1,y_2,x_3)\frac{1}{(x_2-y_1)^2}\\
&+\delta(x_1+x_3-y_1-y_2)\psi^{2S}(x_2,y_1,y_2,x_4)\frac{1}{(x_1-y_1)^2}\\
&\left.-\delta(x_2+x_3-y_1-y_2)\psi^{2S}(x_1,y_1,y_2,x_4)\frac{1}{(x_2-y_1)^2}\right\}\\
&+\frac{e^2}{4\pi}\int_0^1 dy_1 dy_2\left\{-\delta(x_1+x_4-y_1-y_2)\Psi(x_2,y_1,y_2,x_3)\frac{1}{(x_1+x_4)^2}\right.\\
&+\delta(x_2+x_4-y_1-y_2)\Psi(x_1,y_1,y_2,x_3)\frac{1}{(x_2+x_4)^2}\\
&-\delta(x_1+x_3-y_1-y_2)\Psi(x_2,y_1,y_2,x_4)\frac{1}{(x_1+x_3)^2}\\
&\left.+\delta(x_2+x_3-y_1-y_2)\Psi(x_1,y_1,y_2,x_4)\frac{1}{(x_2+x_3)^2}\right\}.
\end{aligned}
\tag{B6}
$$

For isospin=0,

$$\frac{M^2}{2}\psi_2'(x,1-x)=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\left[\frac{1}{x}+\frac{1}{1-x}\right]\psi_2'(x,1-x)+\frac{e^2}{2\pi}\int_0^1 dy\,\psi_2'(y,1-y)\left[2-\frac{1}{(x-y)^2}\right]$$
$$+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2dy_3\frac{\delta(y_1+y_2+y_3-x)}{(x-y_1)^2}[\sqrt{6}\psi_3(y_1,y_2,y_3,1-x)-\sqrt{2}\psi_0(y_1,y_2,y_3,1-x)]$$
$$-\frac{e^2}{2\pi}\int_0^1 dy_2dy_3dy_4\frac{\delta(y_2+y_3+y_4-(1-x))}{[(1-x)-y_4]^2}[\sqrt{6}\psi_3(x,y_2,y_3,y_4)-\sqrt{2}\psi_0(x,y_2,y_3,y_4)]\,, \tag{B7}$$

$$\frac{M^2}{2}\psi_3(x_1,x_2,x_3,x_4)$$
$$=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\sum_{i=1}^{4}\frac{1}{x_i}\psi_3(x_1,x_2,x_3,x_4)$$
$$+\frac{\sqrt{3}e^2}{4\sqrt{2}\pi}\left\{\psi_2'(x_1,1-x_1)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_2+x_3)^2}\right]-\psi_2'(x_2,1-x_2)\left[\frac{1}{(x_1+x_4)^2}-\frac{1}{(x_1+x_3)^2}\right]\right.$$
$$\left.-\psi_2'(1-x_3,x_3)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_1+x_4)^2}\right]+\psi_2'(1-x_4,x_4)\left[\frac{1}{(x_2+x_3)^2}-\frac{1}{(x_1+x_3)^2}\right]\right\}$$
$$+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_2-y_1-y_2)\psi_3(y_1,y_2,x_3,x_4)\left[\frac{1}{2(x_1-y_1)^2}-\frac{1}{2(x_2-y_1)^2}\right]\right.$$
$$+\delta(x_3+x_4-y_1-y_2)\psi_3(x_1,x_2,y_1,y_2)\left[\frac{1}{2(x_3-y_1)^2}-\frac{1}{2(x_4-y_1)^2}\right]$$
$$-\delta(x_1+x_4-y_1-y_2)\psi_3(x_2,y_1,y_2,x_3)\frac{1}{(x_1-y_1)^2}$$
$$+\delta(x_2+x_4-y_1-y_2)\psi_3(x_1,y_1,y_2,x_3)\frac{1}{(x_2-y_1)^2}$$
$$+\delta(x_1+x_3-y_1-y_2)\psi_3(x_2,y_1,y_2,x_4)\frac{1}{(x_1-y_1)^2}$$
$$\left.-\delta(x_2+x_3-y_1-y_2)\psi_3(x_1,y_1,y_2,x_4)\frac{1}{(x_2-y_1)^2}\right\}$$
$$+\frac{\sqrt{3}e^2}{2\pi}\int_0^1 dy_1dy_2\left\{-\delta(x_1+x_4-y_1-y_2)\Phi(x_2,y_1,y_2,x_3)\frac{1}{(x_1+x_4)^2}\right.$$
$$+\delta(x_2+x_4-y_1-y_2)\Phi(x_1,y_1,y_2,x_3)\frac{1}{(x_2+x_4)^2}$$
$$+\delta(x_1+x_3-y_1-y_2)\Phi(x_2,y_1,y_2,x_4)\frac{1}{(x_1+x_3)^2}$$
$$\left.-\delta(x_2+x_3-y_1-y_2)\Phi(x_1,y_1,y_2,x_4)\frac{1}{(x_2+x_3)^2}\right\}, \tag{B8}$$

$$\frac{M^2}{2}\psi_0(x_1,x_2,x_3,x_4)$$
$$=\left[\frac{m^2}{2}-\frac{e^2}{2\pi}\right]\sum_{i=1}^{4}\frac{1}{x_i}\psi_0(x_1,x_2,x_3,x_4)$$
$$+\frac{e^2}{4\sqrt{2}\pi}\left\{\psi_2'(x_1,1-x_1)\left[\frac{1}{(x_2+x_4)^2}+\frac{1}{(x_2+x_3)^2}\right]+\psi_2'(x_2,1-x_2)\left[\frac{1}{(x_1+x_4)^2}+\frac{1}{(x_1+x_3)^2}\right]\right.$$
$$\left.-\psi_2'(1-x_3,x_3)\left[\frac{1}{(x_2+x_4)^2}+\frac{1}{(x_1+x_4)^2}\right]-\psi_2'(1-x_4,x_4)\left[\frac{1}{(x_2+x_3)^2}+\frac{1}{(x_1+x_3)^2}\right]\right\}$$

$$+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_2-y_1-y_2)\psi_0(y_1,y_2,x_3,x_4)\left[\frac{1}{2(x_1-y_1)^2}+\frac{1}{2(x_2-y_1)^2}\right]\right.$$
$$+\delta(x_3+x_4-y_1-y_2)\psi_0(x_1,x_2,y_1,y_2)\left[\frac{1}{2(x_3-y_1)^2}+\frac{1}{2(x_4-y_1)^2}\right]$$
$$-\delta(x_1+x_4-y_1-y_2)\psi_0(x_2,y_1,y_2,x_3)\frac{1}{(x_1-y_1)^2}$$
$$-\delta(x_2+x_4-y_1-y_2)\psi_0(x_1,y_1,y_2,x_3)\frac{1}{(x_2-y_1)^2}$$
$$-\delta(x_1+x_3-y_1-y_2)\psi_0(x_2,y_1,y_2,x_4)\frac{1}{(x_1-y_1)^2}$$
$$\left.-\delta(x_2+x_3-y_1-y_2)\psi_0(x_1,y_1,y_2,x_4)\frac{1}{(x_2-y_1)^2}\right\}$$
$$+\frac{e^2}{2\pi}\int_0^1 dy_1dy_2\left\{\delta(x_1+x_4-y_1-y_2)\Phi(x_2,y_1,y_2,x_3)\frac{1}{(x_1+x_4)^2}\right.$$
$$+\delta(x_2+x_4-y_1-y_2)\Phi(x_1,y_1,y_2,x_3)\frac{1}{(x_2+x_4)^2}$$
$$+\delta(x_1+x_3-y_1-y_2)\Phi(x_2,y_1,y_2,x_4)\frac{1}{(x_1+x_3)^2}$$
$$\left.+\delta(x_2+x_3-y_1-y_2)\Phi(x_1,y_1,y_2,x_4)\frac{1}{(x_2+x_3)^2}\right\}. \tag{B9}$$

APPENDIX C: SOME FORMULAS

This is a list of useful integral formulas for calculating matrix elements by using the basis functions (3.3) and (3.5) (principal-value integrals are understood):

$$I^S(\alpha,\beta)=\int_0^1 dx\,dy\frac{[x(1-x)]^\alpha[y(1-y)]^\beta}{(x-y)^2}=-\frac{\alpha\beta}{2(\alpha+\beta)}B(\alpha,\alpha)B(\beta,\beta)\,, \tag{C1}$$

$$I^A(\alpha,\beta)=\int_0^1 dx\,dy\frac{[x(1-x)]^\alpha[y(1-y)]^\beta}{(x-y)^2}(2x-1)(2y-1)$$
$$=-\frac{\alpha\beta}{2(\alpha+\beta)(\alpha+\beta+1)}B(\alpha,\alpha)B(\beta,\beta)\,,$$

$$\int\prod_{i=1}^4 dx_i\delta\left[\sum_{i=1}^4 x_i-1\right](x_2+x_4)^{-2}[x_1(1-x_1)]^{\beta+k}(x_1x_2x_3x_4)^\beta x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}$$
$$=B(4\beta+k+n_2+n_3+n_4+1,2\beta+k+n_1+1)B(\beta+n_4+1,\beta+n_2+1)B(2\beta+n_2+n_4,\beta+n_3+1)\,, \tag{C2}$$

$$\int\prod_{i=1}^4 dx_i\delta\left[\sum_{i=1}^4 x_i-1\right](x_1x_2x_3x_4)^\beta x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}=\frac{\Gamma(2\beta+n_1+1)\Gamma(2\beta+n_2+1)\Gamma(2\beta+n_3+1)\Gamma(2\beta+n_4+1)}{\Gamma(8\beta+n_1+n_2+n_3+n_4+4)}\,, \tag{C3}$$

$$\int\prod_{i=1}^4 dx_idy_1dy_2\delta\left[\sum_{i=1}^4 x_i-1\right]\delta(x_1+x_2-y_1-y_2)(x_1+x_2)^{-2}(x_1x_2x_3x_4)^\beta(y_1y_2)^\beta(x_3x_4)^\beta x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}y_1^{m_1}y_2^{m_2}$$
$$=B(\beta+m_1+1,\beta+m_2+1)B(\beta+n_1+1,\beta+n_2+1)$$
$$\times\frac{\Gamma(4\beta+n_1+n_2+m_1+m_2+1)\Gamma(2\beta+n_3+1)\Gamma(2\beta+n_4+1)}{\Gamma(8\beta+n_1+n_2+n_3+n_4+m_1+m_2+3)}\,, \tag{C4}$$

$$\int \prod_{i=1}^{4} dx_i dy_1 dy_2 \delta \left[\sum_{i=1}^{4} x_i - 1 \right] \delta(x_1+x_2-y_1-y_2)(x_1-y_1)^{-2}(x_1x_2x_3x_4)^{\beta}(y_1y_2)^{\beta}(x_3x_4)^{\beta} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} y_1^{m_1} y_2^{m_2}$$
$$= \frac{\Gamma(4\beta+n_1+n_2+m_1+m_2+1)\Gamma(2\beta+n_3+1)\Gamma(2\beta+n_4+1)}{\Gamma(8\beta+n_1+n_2+n_3+n_4+m_1+m_2+3)} D(n_1,n_2,m_1,m_2) , \quad \text{(C5)}$$

where

$$D(n_1,n_2,m_1,m_2)$$
$$= \int_0^1 dx\, dy \frac{x^{\beta+n_1}(1-x)^{\beta+n_2} y^{\beta+m_1}(1-y)^{\beta+m_2}}{(x-y)^2}$$
$$= 2^{-|n_1-n_2|-|m_1-m_2|} \sum_{\substack{k=0\\ \text{even}}}^{|n_1-n_2|} \sum_{r=0}^{k/2} \begin{bmatrix} |n_1-n_2| \\ k \end{bmatrix} \begin{bmatrix} k/2 \\ r \end{bmatrix} \sum_{\substack{l=0\\ \text{even}}}^{|m_1-m_2|} \sum_{s=0}^{l/2} \begin{bmatrix} |m_1-m_2| \\ l \end{bmatrix} \begin{bmatrix} l/2 \\ s \end{bmatrix} (-4)^{r+s} I^S(\beta+\underline{n}+r, \beta+\underline{m}+s)$$
$$+ 2^{-|n_1-n_2|-|m_1-m_2|} \epsilon(n_1-n_2)\epsilon(m_1-m_2)$$
$$\times \sum_{\substack{k=0\\ \text{odd}}}^{|n_1-n_2|} \sum_{r=0}^{k/2} \begin{bmatrix} |n_1-n_2| \\ k \end{bmatrix} \begin{bmatrix} k/2 \\ r \end{bmatrix} \sum_{\substack{l=0\\ \text{odd}}}^{|m_1-m_2|} \sum_{s=0}^{l/2} \begin{bmatrix} |m_1-m_2| \\ l \end{bmatrix} \begin{bmatrix} l/2 \\ s \end{bmatrix} (-4)^{r+s} I^A(\beta+\underline{n}+r, \beta+\underline{m}+s) , \quad \text{(C6)}$$

with $\underline{n} = \min(n_1, n_2)$ and $\underline{m} = \min(m_1, m_2)$.

APPENDIX D: MATRIX EIGENVALUE EQUATIONS

The following are matrix eigenvalue equations for the isotriplet and for the isoquintet.

For the isotriplet,

$$\begin{aligned} \psi_2(x,1-x) &= \sum_{k=0} a_k f_k(x) , \\ \psi^A(x_1,x_2,x_3,x_4) &+ \sum_{\mathbf{k}} b_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4), \quad k_1,k_3 \text{ odd} , \\ \psi^{1S}(x_1,x_2,x_3,x_4) &= \sum_{\mathbf{k}} c_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4), \quad k_1 \text{ even}, k_3 \text{ odd} , \\ \psi^{2S}(x_1,x_2,x_3,x_4) &= \sum_{\mathbf{k}} d_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4), \quad k_1 \text{ odd}, k_3 \text{ even} . \end{aligned} \quad \text{(D1)}$$

$$M^2 \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$
$$= \begin{bmatrix} (m^2-1)C+\bar{D} & 2(\tilde{E}-E) & \sqrt{2}(\tilde{E}-E) & \sqrt{2}(\tilde{E}-E) \\ 2(\tilde{E}-E) & (m^2-1)Q+R+S-4T+4U & -2\sqrt{2}U & -2\sqrt{2}U \\ \sqrt{2}(\tilde{E}-E) & -2\sqrt{2}U & (m^2-1)Q+R+S-4T+2U & 2U \\ \sqrt{2}(\tilde{E}-E) & -2\sqrt{2}U & 2U & (m^2-1)Q+R+S-4T+2U \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} , \quad \text{(D2)}$$

where

$$\bar{D}_{kl} = -\int_0^1 dx\, dy \frac{f_k(x) f_l(y)}{(x-y)^2} . \quad \text{(D3)}$$

For the isoquintet,

$$\psi_4(x_1,x_2,x_3,x_4) = \sum_{\mathbf{k}} b_{\mathbf{k}} G_{\mathbf{k}}(x_1,x_2,x_3,x_4), \quad k_1,k_3 \text{ odd} , \quad \text{(D4)}$$

$$M^2 B b = [(m^2-1)Q+R+S-4T] b . \quad \text{(D5)}$$

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