

Convergence of the optimized δ expansion for the connected vacuum amplitude: Zero dimensions

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Recent proofs of the convergence of the linear δ expansion in zero and one dimension have been limited to the analogue of the vacuum generating functional in field theory. In zero dimensions it was shown that with an appropriate, N -dependent, choice of an optimizing parameter λ , which is an important feature of the method, the sequence of approximants Z_N tends to Z with an error proportional to e^{-cN} . In the present paper we establish the convergence of the linear δ expansion for the connected vacuum function $W = \ln Z$. We show that with the same choice of λ the corresponding sequence W_N tends to W with an error proportional to $e^{-c\sqrt{N}}$. The rate of convergence of the latter sequence is governed by the positions of the zeros of Z_N .

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I. INTRODUCTION

In a previous paper [1] it was proved that in zero-dimensional space-time the optimized linear δ expansion can completely cure the problems inherent in conventional perturbation theory. A conventional perturbation expansion is a formal series in powers of the coupling constant. Such a power series typically has a zero radius of convergence and sometimes is not even useful as an asymptotic series because it is not Borel summable. In contrast, the optimized linear δ expansion produces a sequence of approximants which converge rapidly to the exact answer, with an error R_N that decreases exponentially with the order N : $R_N \sim \exp(-cN)$. The proof given in Ref. [1] was subsequently extended [2] to one-dimensional space-time (the quantum anharmonic oscillator), where it was shown that for the finite-temperature partition function $R_N \sim \exp(-cN^{2/3})$. The technique can also be extended to establish a proof of convergence for cutoff $\varphi_{2,3}^4$ theories (with either sign of the squared mass) in finite volume.

However, the proofs of convergence given in Refs. [1] and [2] were limited to the partition function (vacuum-vacuum amplitude) Z , which represents the sum of all vacuum graphs, disconnected as well as connected. In quantum field theory the crucial quantity to compute is W , the logarithm of the vacuum-vacuum function, which represents the sum of the connected vacuum graphs only. It is nontrivial to show that the optimized linear δ expansion converges for W because Z_N , the N th approximation to Z , has zeros in the complex- δ plane at which $\ln Z_N$ is singular. The presence of such zeros could interfere with the convergence of the sequence W_N . It is the

purpose of this paper to show that in zero dimensions the δ expansion for W does in fact converge, in spite of the zeros of Z_N .

The specific model we consider here is the zero-dimensional analogue of the Euclidean functional integral for a φ^4 quantum field theory, which in this case amounts to the one-dimensional integral

$$Z \equiv \int_{-\infty}^{\infty} dx e^{-gx^4 - \mu^2 x^2}, \quad (1)$$

where g is the coupling constant and μ the mass. In order to perform a weak-coupling expansion of Z in powers of g it is necessary that the mass parameter μ^2 be positive; a weak-coupling expansion does not exist otherwise. However, in the linear δ expansion the value of μ^2 is immaterial. Thus, for simplicity we restrict our attention to the massless case $\mu = 0$; the analysis of the massive case (with either sign of μ^2) does not differ in any significant way.

The linear δ expansion [3] has features in common with a number of previous approaches [4–14] to improving on the convergence of ordinary perturbation theory. It involves the introduction of an artificial parameter δ which does not appear in the original problem and which interpolates linearly between the theory we hope to solve, with action S , and another soluble theory, with action S_0 . The interpolating action $S(\delta)$ is defined as

$$S(\delta) \equiv \lambda(1 - \delta)S_0 + \delta S, \quad (2)$$

so that $S(0) = S_0$ and $S(1) = S$.

Any desired quantity is evaluated as a perturbation series in powers of δ , which is then set equal to 1 at the

end of the calculation. At $\delta = 1$ the theory defined by $S(\delta)$ is independent of the value of λ . However, to any finite order N in δ there is a residual λ dependence, and the choice of λ is in fact crucial to the convergence of the δ expansion. Indeed, if λ were taken to be a constant independent of the order N , the δ expansion would have a zero radius of convergence, just as in ordinary perturbation theory. However, it was proved in Ref. [1] that if λ is chosen as $\sqrt{\alpha N}$, the sequence of approximants $Z_N(\lambda_N)$ converges to Z . The numerical value of α is given at the beginning of Sec. III. The scaling $\lambda = cN^{2/3}$ was shown in Ref. [2] to guarantee the convergence of the corresponding sequence for the finite-temperature partition function of the anharmonic oscillator. However, the proof did not extend to zero temperature because the limits $\beta \rightarrow \infty$ and $N \rightarrow \infty$ are not interchangeable.

In the context of field theory it is natural to work with the logarithm of the partition function, i.e., with connected diagrams. It is therefore desirable to extend the proof of convergence to $W = \ln Z$. To do so it is necessary to determine the location of the zeros of Z_N in the complex- δ plane in the limit of large N . This determination is performed asymptotically by a steepest-descent evaluation of the integral representing $Z(\delta)$ and an asymptotic analysis of the behavior of the remainder R_N as a function of complex δ .

In Sec. II we explain how the zeros of Z_N affect the convergence of the δ expansion for W . The asymptotic analysis of the location of these zeros is given in Sec. III. Finally, in Sec. IV we summarize our results and discuss possible extensions of the analysis presented here to higher dimensions.

II. RELEVANCE OF THE ZEROS OF Z_N

In this paper we are investigating the convergence of the δ expansion for $W = \ln Z$. This involves computing the N th partial sum W_N of the Taylor series in δ of $W(\delta) = \ln Z(\delta)$, where

$$Z(\delta) = \int_{-\infty}^{\infty} dx e^{-\lambda x^2 + \delta(\lambda x^2 - x^4)}, \quad (3)$$

in which we have taken $g = 1$ without loss of generality. Then δ is set equal to 1 and λ chosen in some appropriate fashion as a function of N .

It was shown in Ref. [1] that with the appropriate scaling of λ the sequence of approximants $Z_N(\delta)$ evaluated at $\delta = 1$ tends to the exact result. Consequently the sequence $\ln Z_N$ also tends to W . However, this does *not* constitute the systematic expansion of W in powers of δ that we seek. That is to say, W_N , the sum of the first N terms of the Taylor expansion of $\ln Z$, is not the same as $\ln Z_N$, the logarithm of the sum of the first N terms of the Taylor expansion of Z .

To examine the convergence of W_N to W we will make use of a slight generalization of some identities introduced in Ref. [2]. Given a function $F(\delta)$, the N th partial sum of its Taylor expansion evaluated at $\delta = 1$ can be represented as the contour integral

$$F_N(1) = \frac{1}{2\pi i} \oint_{C_0} \frac{dz}{z^{N+1}} \frac{1}{1-z} F(z), \quad (4)$$

where C_0 is a closed anticlockwise contour encircling the origin but not the point $z = 1$. The quantity $F(1)$ itself can be represented as

$$F(1) = \frac{1}{2\pi i} \oint_{C_1} \frac{dz}{z^{N+1}} \frac{1}{z-1} F(z), \quad (5)$$

where C_1 is a closed contour encircling the point $z = 1$ but not the origin. Subtracting Eq. (4) from (5) gives the following general integral representation for the remainder $R_N = F(1) - F_N(1)$:

$$R_N = \frac{1}{2\pi i} \oint_{C_{01}} \frac{dz}{z^{N+1}} \frac{1}{z-1} F(z), \quad (6)$$

where the contour C_{01} encircles both $z = 0$ and $z = 1$ in an anticlockwise direction, as shown in Fig. 1.

Now let us apply these identities to the function $F(\delta) = \ln Z_N(\delta)$ evaluated at $\delta = 1$ in order to obtain a bound on the remainder, which in this case we denote by

$$\mathcal{R}_N \equiv \ln Z_N - (\ln Z_N)_N. \quad (7)$$

Expressing $Z_N(z)$ in terms of its roots,

$$Z_N(z) = Z_N(0) \prod_{r=1}^N \left(1 - \frac{z}{z_r}\right), \quad (8)$$

so that

$$\ln Z_N(z) = \ln Z_N(0) + \sum_{r=1}^N \ln \left(1 - \frac{z}{z_r}\right). \quad (9)$$

Thus,

$$\mathcal{R}_N = \frac{1}{2\pi i} \sum_{r=1}^N \oint_{C_{01}} \frac{dz}{z^{N+1}} \frac{1}{z-1} \ln(z - z_r). \quad (10)$$

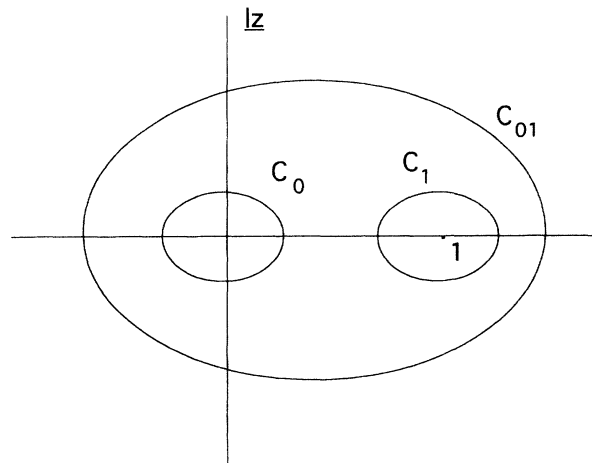


FIG. 1. Contours in the complex- z plane used for the representation of $F_N(1)$, $F(1)$, and R_N [Eqs. (4)–(6)].

Note that constant terms such as $\ln Z_N(0)$ do not contribute to the integral around the contour C_{01} .

In the derivation of Eq. (10) we are assuming that there are no singularities of the integrand inside the contour other than those at $z = 0$ and $z = 1$. We will verify this assumption in Sec. III. Let us now expand the contour C_{01} by pushing it outward in all directions. In so doing we encounter the logarithmic branch points emanating from the roots z_r . We take the branch cuts to lie along straight lines radiating directly outwards, as shown in Fig. 2. Thus, the shifted contour wraps around these branch cuts and the integral is then given by the sum of the discontinuities across each branch cut:

$$\mathcal{R}_N = \sum_{r=1}^N \int_{z_r}^{\infty} \frac{dz}{z^{N+1}} \frac{1}{z-1}. \quad (11)$$

By parametrizing the integration variable z along each branch cut by $\rho e^{i\theta_r}$ it is easy to establish the bound

$$|\mathcal{R}_N| < \sum_{r=1}^N \frac{1}{|z_r - 1|} \frac{1}{N} \frac{1}{|z_r|^N}, \quad (12)$$

where we have used the inequality $|z - 1| > |z_r - 1|$ along each radial cut. This latter inequality is valid provided that $|z_r| > 1$, which will be established in the next section. If z_{\min} denotes the root having the smallest modulus, then Eq. (12) may be replaced by the simpler inequality

$$|\mathcal{R}_N| < \frac{1}{|z_{\min} - 1|} \frac{1}{|z_{\min}|^N}. \quad (13)$$

Since $|z_{\min}| > 1$, we may conclude from Eq. (13) that the remainder \mathcal{R}_N vanishes for large N . The rate at which \mathcal{R}_N tends to 0 is crucially dependent on the behavior of z_{\min} as a function of N , which is the subject of the next section.

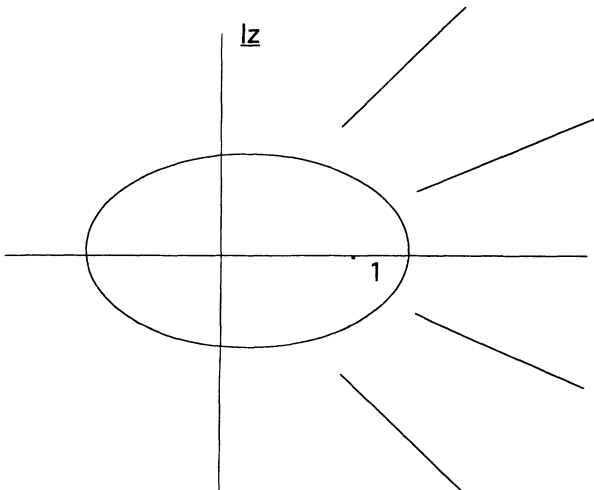


FIG. 2. Logarithmic branch cuts radiating from the zeros of $Z_N(z)$ in the representation of \mathcal{R}_N in Eq. (11).

III. SADDLE-POINT DETERMINATION OF THE SMALLEST ZERO OF Z_N

In Ref. [2] it was shown that the sequence Z_N converges to Z if the parameter λ is chosen as $\lambda = \lambda_N = \sqrt{\alpha N}$, where the numerical value $\alpha = 1.3254\dots$ is obtained from $\alpha = 2/\sinh \beta$ and β satisfies the transcendental equation $\beta = \coth \beta$. We adopt the same choice for the parameter λ here. In this case the series

$$Z_N(z) = \sum_{n=0}^N c_n z^n \quad (14)$$

looks rather simple: Apart from the last term, which is small and negative, the coefficients are all positive and monotonically decreasing, looking roughly like a geometric series. Unfortunately, it is not easy to determine the location of the roots of a polynomial from the knowledge of the coefficients c_n . As an example, consider the simple-looking case $c_n = e^{-\gamma n^2}$. For small values of γ the roots lie on a circle centered at the origin in the complex- z plane. However, as γ increases past a critical value two pairs of complex-conjugate roots break away from the circle. Additional roots eventually break away from the circle as γ increases through a whole sequence of critical values. This phenomenon is described in Ref. [15]. To our knowledge there is no simple analytic way to determine these critical values or, indeed, the positions of the roots.

Interestingly, the configuration of the roots of Z_N shares many of the characteristics of this simple model. For large odd N all but five of the roots lie almost exactly on a circle; of the remaining roots, one complex-conjugate pair lies inside the circle, another conjugate pair lies outside, and there is a single root located far away on the positive-real axis (see Fig. 3 for the case $N = 27$). When N is even the only qualitative difference is that there is no real root. As shown in Eq. (13), it is the position

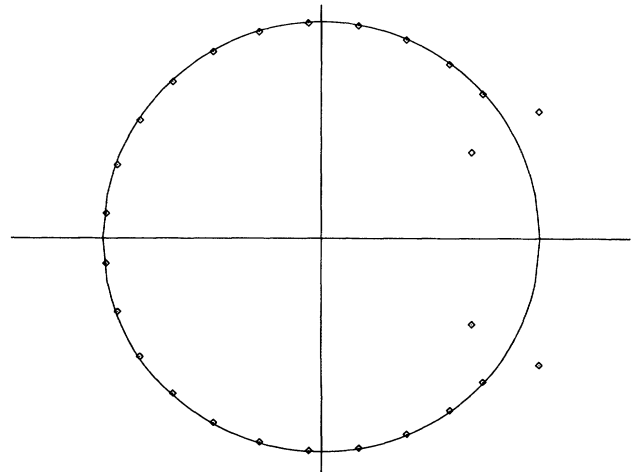


FIG. 3. Location of zeros of $Z_N(z)$ in the complex plane for $N = 27$. The circle (of radius 2.1) is just an empirical fit to the ring of zeros. The distant real root at $z = 14.124$ is not shown.

of the pair of roots nearest the origin which determines whether or not the sequence W_N converges to W .

As remarked above, it is very difficult to find the positions of the roots directly from the coefficients. Thus, we have adopted the alternative strategy of expressing $Z_N(z)$ as the difference

$$Z_N(z) = Z(z) - R_N(z) \tag{15}$$

and estimating each term on the right side of (15) by asymptotic methods. Note that while $Z(z)$ does not depend explicitly on the large parameter N , when $z \neq 1$ it does involve λ_N , which is a large parameter. It is through this dependence that we are able to estimate the integral representation for $Z(z)$ by steepest-descent analysis. The result of this analysis is that

$$|z_{\min}| = 1 + \left(\frac{3\pi}{\alpha N}\right)^{1/2} + O\left(\frac{1}{N}\right). \tag{16}$$

From this asymptotic relation we conclude that \mathcal{R}_N tends to zero like $\exp(-\sqrt{3\pi N/\alpha})$, thus proving that W_N converges to W .

The asymptotic analysis of the two terms $R_N(z)$ and $Z(z)$ on the right side of Eq. (15) follows in Secs. III A and III B, respectively.

A. Asymptotic bound on $R_N(z)$

The starting point for our analysis is the identity for $\Theta_N(y) \equiv e^{-y}\{e^y\}_N$ established in Ref. [1]:

$$\frac{d}{dy}\Theta_N = -y^N e^{-y}/N!. \tag{17}$$

This identity is to be applied under the x integration with $y = z(\lambda x^2 - x^4)$. That is,

$$R_N(z) = 2 \int_0^\infty dx e^{-\lambda x^2 + y} [1 - \Theta_N(y)]. \tag{18}$$

Let us first deal with the case $\text{Re} z > 0$, where the x integration contour can be maintained along the real axis. For complex z in this region we integrate Eq. (17) along the ray $y' = yt$, with $0 \leq t \leq 1$, to obtain

$$\Theta_N(y) = 1 - \int_0^y dy' y'^N e^{-y'}/N!, \tag{19}$$

using $\Theta_N(0) = 1$.

The x integration in Eq. (18) splits naturally into the two ranges $0 \leq x \leq \sqrt{\lambda}$ and $\sqrt{\lambda} \leq x$. In the first case $y = |y|e^{i\theta}$, where $\theta = \arg(z)$. The contribution to the remainder from this range is therefore

$$A_N(z) = \frac{2}{N!} \int_0^{\sqrt{\lambda}} dx e^{-\lambda x^2 + y} \int_0^{|y|} d\omega \omega^N e^{-\omega e^{i\theta}} e^{i(N+1)\theta}. \tag{20}$$

Thus,

$$|A_N(z)| \leq \frac{2}{N!} \int_0^{\sqrt{\lambda}} dx e^{-\lambda x^2 + |y| \cos \theta} \int_0^{|y|} d\omega \omega^N e^{-\omega \cos \theta}. \tag{21}$$

The maximum of the ω integrand occurs at $\omega = N/\cos \theta \geq N$, whereas the upper limit is less than or equal to $N\alpha|z|/4$. Thus, provided that $|z| < 4/\alpha$, we may bound $|A_N(z)|$ by

$$\begin{aligned} |A_N(z)| &\leq \frac{2}{N!} \int_0^{\sqrt{\lambda}} dx e^{-\lambda x^2 + |y| \cos \theta} |y|^{N+1} e^{-y \cos \theta} \\ &= \frac{2}{N!} |z|^{N+1} \int_0^{\sqrt{\lambda}} dx e^{-\lambda x^2} (\lambda x^2 - x^4)^{N+1}. \end{aligned} \tag{22}$$

Apart from the factor of $|z|^{N+1}$, the right side of Eq. (22) is precisely $A_N(1)$, which was shown in Ref. [1] to be bounded by $CN^{3/4}e^{-N\alpha/2}$. Thus,

$$|A_N(z)| \leq CN^{3/4}|z|^{N+1}e^{-N\alpha/2}. \tag{23}$$

In the second of the two regions of the x integration, the quantity $\lambda x^2 - x^4$ is negative, so that $y = \omega e^{i(\theta+\pi)}$. The corresponding contribution to the remainder is then

$$\begin{aligned} B_N(z) &= \frac{2}{N!} \int_{\sqrt{\lambda}}^\infty dx e^{-\lambda x^2 + y} \\ &\quad \times \int_0^{|y|} d\omega \omega^N e^{\omega e^{i\theta}} e^{i(N+1)(\theta+\pi)}, \end{aligned} \tag{24}$$

so that

$$|B_N(z)| \leq \frac{2}{N!} \int_{\sqrt{\lambda}}^\infty dx e^{-\lambda x^2 - |y| \cos \theta} \int_0^{|y|} d\omega \omega^N e^{\omega \cos \theta}. \tag{25}$$

In this case the ω integrand is a monotone function whose maximum occurs at the upper limit. Thus,

$$\begin{aligned} |B_N(z)| &\leq \frac{2}{N!} \int_{\sqrt{\lambda}}^\infty dx e^{-\lambda x^2 - |y| \cos \theta} |y|^{N+1} e^{|y| \cos \theta} \\ &= \frac{2}{N!} |z|^{N+1} \int_{\sqrt{\lambda}}^\infty dx e^{-\lambda x^2} (x^4 - \lambda x^2)^{N+1}. \end{aligned} \tag{26}$$

Again, the right side is $B_N(1)$ multiplied by a factor of $|z|^{N+1}$. In Ref. [1], $B_N(1)$ was shown to be bounded for large N in precisely the same way as $A_N(1)$.

Thus, altogether,

$$|R_N(z)| \leq CN^{3/4}|z|^{N+1}e^{-N\alpha/2}, \tag{27}$$

provided that

$$|z| \leq \frac{4}{\alpha}. \tag{28}$$

Now let us consider $\text{Re} z < 0$. The only difference is that since $\cos \theta < 0$ the roles of A_N and B_N are reversed. That is, $|A_N(z)|$ is bounded as in Eq. (23) independent of $|z|$, while $|B_N(z)|$ is bounded in the same way, provided that Eq. (28) holds. Altogether, the conclusion for $|R_N(z)|$ remains the same.

B. Steepest-descent evaluation of $Z(z)$

For $\text{Re} z > 0$ the integral in Eq. (3) is an adequate definition of the function $Z(z)$ in the complex- z plane.

Later on in this subsection we will examine the analytic continuation of $Z(z)$ to the left-half z plane. Because $\lambda = \lambda_N$ is large, we evaluate the integral by looking for the saddle points of the exponent φ in the integrand:

$$\varphi(x) \equiv -\lambda x^2 + z(\lambda x^2 - x^4). \quad (29)$$

The three saddle points satisfying $\varphi'(x) = 0$ are $x_0 = 0$ and $x_{\pm} = \pm\sqrt{\lambda(z-1)/(2z)}$. At these saddle points $\varphi(x_0) = 0$ and $\varphi(x_{\pm}) = -\lambda^2(z-1)^2/(4z)$.

The method of steepest descents requires that for each value of complex z we deform the integration path in Eq. (3) to a stationary-phase contour in the complex- x plane connecting the original end points at $x = \pm\infty$. We find that there are two cases to consider. When $|z| < 1$ (region \mathcal{A}) the stationary-phase contour passes through the saddle point x_0 only and not through the others. In contrast, when $|z| > 1$ the stationary-phase contour passes through all three saddle points. The appropriate integration contours for these two cases are illustrated in Figs. 4 and 5, respectively. In the second case there are two subregions in the complex- z plane to consider, region \mathcal{B} , in which $\exp[\varphi(x_{\pm})]$ is subdominant with respect to $\exp[\varphi(x_0)] = 1$, and region \mathcal{C} , in which the reverse is true. The boundary curve Γ between subregions \mathcal{B} and \mathcal{C} is given by the polar equation

$$\Gamma: \quad \cos \theta = \frac{2r}{r^2 + 1}, \quad (30)$$

where $z = re^{i\theta}$. The above regions and the boundary curves are illustrated in Fig. 6.

In regions \mathcal{A} and \mathcal{B} the dominant contribution to $Z_N(z)$ comes from the saddle point at the origin, whose contribution is a slowly varying (nonexponential) function of N : $Z(z) \sim N^{-1/4}$. Referring to Eq. (15), we see that a zero of $Z_N(z)$ must arise from a cancellation between this contribution and $R_N(z)$, which as we have

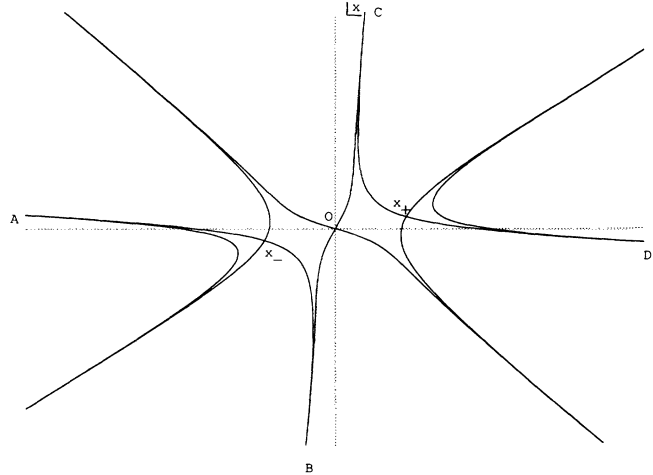


FIG. 5. Stationary-phase contours of the function $\varphi(x)$ in Eq. (29) for a typical complex value of z with $|z| > 1$. The original integration contour lying along the real axis must be distorted into the contour marked $Ax-Bx_0Cx_+D$, passing through the three saddle points x_0 and x_{\pm} .

seen is bounded by $\text{const} \times \exp[N(\ln|z| - \alpha/2)]$. Thus, no zero is possible for $|z| < e^{\alpha/2} = 1.94\dots$. Since this value is strictly greater than one, the zeros in regions \mathcal{A} and \mathcal{B} will not affect the convergence of \mathcal{R}_N . From the numerical plot in Fig. 3 it appears that the bound on $|z|$ is saturated, giving rise to a ring of zeros.

In region \mathcal{C} the dominant contribution to $Z_N(z)$ comes from the saddle points x_{\pm} . These saddle points give a nontrivial exponential dependence to $Z_N(z)$ of the form $\exp[N\alpha(z-1)^2/(4z)]$. Since this exponent is positive,

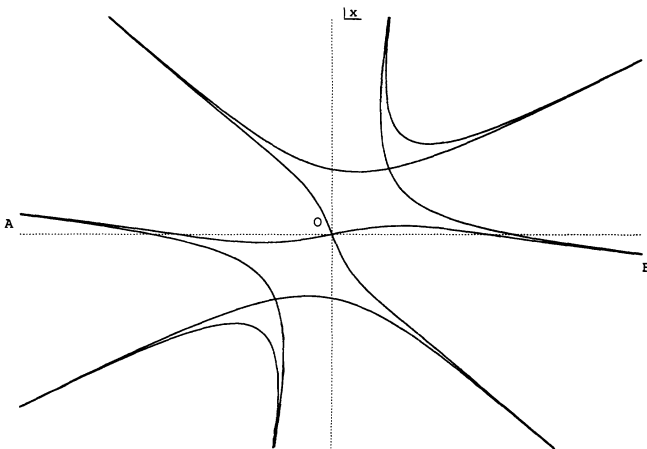


FIG. 4. Stationary-phase contours of the function $\varphi(x)$ in Eq. (29) for a typical complex value of z with $|z| < 1$. The original integration contour lying along the real axis must be distorted into the contour marked Ax_0B , passing through the saddle point x_0 . The other saddle points play no role in the asymptotic evaluation of the integral.

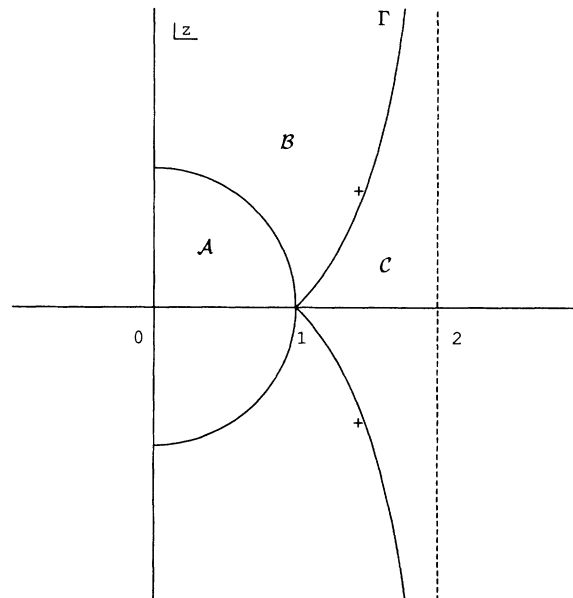


FIG. 6. Right-half z plane showing the three regions \mathcal{A} , \mathcal{B} , \mathcal{C} discussed in Sec. III B. The boundary between regions \mathcal{B} and \mathcal{C} , labeled Γ , is given in Eq. (30). Also shown are the two smallest-modulus roots of $Z_N(z)$ for $N = 27$.

the necessary cancellation required for a zero can only occur at still larger values of $|z|$. This accounts for the conjugate pair of complex zeros lying outside the ring in Fig. 3 and the large real root. None of these zeros affects the convergence of \mathcal{R}_N .

Finally, we examine the boundary curve Γ separating regions \mathcal{B} and \mathcal{C} . On this curve the exponent of the saddle points at x_{\pm} is the same as that of x_0 . There thus arises the possibility of a cancellation between their contributions, leaving a remainder which can be compensated by $R_N(z)$ at a smaller value of $|z|$ inside the ring of zeros in Fig. 3. Note that the smallest zero marked in Fig. 6 lies almost exactly on Γ , indicating that this is indeed the mechanism that produces the zero of smallest absolute value. In order for this cancellation to occur the two saddle-point contributions must be completely out of phase. The phase of each contribution is of course uniquely determined by the direction of the stationary-phase path going through the saddle point. As can be seen from Fig. 5, the stationary-phase path directions differ by $\pi/2$. Thus, the required condition for cancellation is

$$\text{Im } \varphi(x_{\pm}) = \pi/2 + (2n + 1)\pi, \quad (31)$$

where n is any integer.

This equation is to be solved in conjunction with Eq. (30). The condition $|z| > 1$ excludes negative values of n and in fact the smallest value of the modulus of the root z is obtained for $n = 0$. With $n = 0$ the simultaneous solution of Eqs. (30) and (31) is

$$z = a + \sqrt{a^2 - 1}, \quad (32)$$

where $a = 1 + 3\pi i/(N\alpha)$. The accuracy of our asymptotic analysis improves with increasing N . For $N = 27$ this saddle-point analysis predicts the positions of the smallest roots quite accurately: $z = 1.481\dots \pm (0.811\dots)i$, to be compared with the actual numerical values $z = 1.443\dots \pm (0.835\dots)i$, as shown in Fig. 6. For larger values of N the roots approach their asymptotic values with an error that behaves like $1/\sqrt{N}$.

From Eq. (32) we can determine the behavior for large N of the modulus of the smallest root z_{\min} . This is given in Eq. (16), which we have checked numerically by performing a fit to a series in inverse powers of \sqrt{N} up to $N = 59$. The crucial feature of Eq. (16) is that $|z_{\min}|$ approaches 1 from above sufficiently slowly that \mathcal{R}_N , the difference between W_N and W , tends to zero. While the convergence is not as rapid as that of the sequence Z_N , which converges like $\exp(-\alpha N/2)$, it still converges like an exponential: $\exp(-\sqrt{3\pi N/\alpha})$.

Up until now we have not considered the case $\text{Re } z < 0$. For this case the integral representation for $Z(z)$ is no longer valid as it stands because it is divergent. As z is rotated into the left-half complex plane the end points of the integration contour must be rotated in the opposite direction (and at one-quarter of the rate) in order for the integral to continue to exist. A description of this analytic continuation procedure may be found in Ref. [16] for the case of boundary conditions on differential equation

eigenvalue problems. Once the integral representation for $Z(z)$ has been continued to negative values of $\text{Re } z$, it may be subjected to the same saddle-point analysis as above. For this case, there are just two regions, $|z| < 1$ in which the stationary-phase contour passes only through the saddle point x_0 and $|z| > 1$ in which the contour passes through all three saddle points, but the contribution from x_0 dominates. These two cases correspond to what happens in regions \mathcal{A} and \mathcal{B} for $\text{Re } z > 0$. Thus, the resulting zeros complete the ring of zeros on Fig. 3 but do not affect our conclusions regarding the convergence of W_N .

IV. DISCUSSION AND CONCLUSIONS

The arguments presented above establish the convergence and bound the remainder in an optimized expansion of $W \equiv \ln Z$ for the non-Gaussian integral (3), provided we choose to vary the optimizing parameter λ with N at large N so as to optimize the convergence of the partials Z_N (i.e., we take $\lambda \sim \sqrt{N}$):

$$\ln Z - \ln Z_N \simeq e^{-N\alpha/2}$$

(ignoring power prefactors). With this choice of N dependence for λ , we have shown that

$$\ln Z_N - (\ln Z_N)_N \simeq e^{-\sqrt{3\pi N/\alpha}}.$$

The remainders in the partials $W_N \equiv (\ln Z_N)_N$ for the connected function W are thus asymptotically

$$W - W_N \simeq e^{-\sqrt{3\pi N/\alpha}}.$$

Although the numerical illustrations have been for odd N , we should emphasize that with λ chosen as $\sqrt{\alpha N}$ the

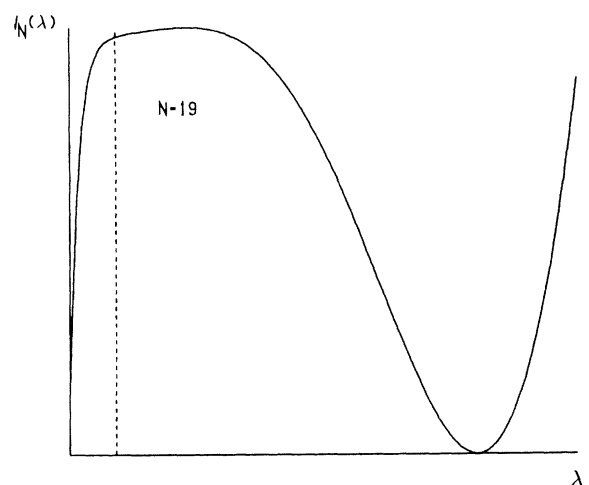


FIG. 7. $W_N(\lambda)$ versus λ for $N = 19$, showing the lower maximum and the minimum. The dotted line is at $\lambda = \lambda_N = \sqrt{\alpha N}$. Note that the vertical scale is highly magnified, with a range of about 7×10^{-5} .

results for even N interpolate smoothly between those for odd N . The distinction originally arose [2] in the context of the principle of minimal sensitivity (PMS [7]) as applied to $Z_N(\lambda)$. There it turns out that for odd N there is a single stationary point of Z_N as a function of λ , whereas for even N there is a point of inflection but no stationary point. The present paper marks a step away from reliance on the PMS philosophy. Instead we are simply choosing λ in such a way that the convergence of the sequence W_N is guaranteed. This methodology eliminates the arbitrariness that often occurs in applying the PMS criterion. That is, there may well be several stationary points [17], and one then has to choose between them by some further criterion which itself needs to be justified. In the case of W_N there are indeed several stationary points, and one no longer has a strict inequality such as $Z_N < Z$ to help one distinguish between them. For example, for $N = 19$ there are two maxima and one minimum in λ . The first maximum and minimum, illustrated in Fig. 7, are reasonably close to the exact value of W , while the second maximum exceeds it by some 0.4%. The value of λ given by $\lambda = \sqrt{\alpha N}$ is slightly lower than

the position of the first maximum, and gives a better estimate of W : 0.5948757 compared with the exact value of 0.5948753

The convergence of the optimized expansion for W is slower than that for Z with this choice of $\lambda(N)$. It is of course possible that a more rapid convergence [with $\ln(W - W_N) \simeq -N^\nu$, $\frac{1}{2} < \nu < 1$] might be obtained with a different choice of $\lambda(N)$. Our main object here has been to provide an existence proof for a convergent procedure for the connected generating function. An understanding of the convergence at the level of connected quantities is crucial in higher dimensions, where the δ expansion for the full partition function converges at any finite space-time volume, but at a rate which deteriorates as the volume is increased. As W is linear in the volume (for large volume) a convergent procedure at any finite volume will be uniformly convergent (for connected quantities) as the volume cutoff is removed. Finally, we note that the techniques used above involve only saddle-point estimates which should generalize readily to functional integrals defining the partition functions for quantum mechanics or field theories.

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- [1] I. R. C. Buckley, A. Duncan, and H. F. Jones, *Phys. Rev. D* **47**, 2554 (1993).
 [2] A. Duncan and H. F. Jones, *Phys. Rev. D* **47**, 2560 (1993).
 [3] A. Duncan and M. Moshe, *Phys. Lett. B* **215**, 352 (1988).
 [4] W. E. Caswell, *Ann. Phys. (N.Y.)* **123**, 153 (1979).
 [5] J. Killingbeck, *J. Phys. A* **14**, 1005 (1981).
 [6] I. G. Halliday and P. Suranyi, *Phys. Lett.* **85B**, 421 (1979); *Phys. Rev. D* **21**, 1529 (1981).
 [7] P. M. Stevenson, *Phys. Rev.* **23**, 2916 (1981).
 [8] B. S. Shaverdyan and A. G. Ushveridze, *Phys. Lett.* **123B**, 316 (1983).
 [9] A. Okopińska, *Phys. Rev. D* **35**, 1835 (1987).
 [10] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., *Phys. Rev. D* **37**, 1472 (1988).
 [11] I. Stancu and P. M. Stevenson, *Phys. Rev. D* **42**, 2710 (1990).
 [12] V. I. Yukalov, *J. Math. Phys.* **32**, 1235 (1991).
 [13] A. Neveu, *Festschrift for A. Martin* (Springer, New York, 1991).
 [14] A. N. Sissakian and I. L. Solovtsov, *Z. Phys. C* **54**, 263 (1992).
 [15] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1980), pp. 225–229.
 [16] C. M. Bender and A. Turbiner, *Phys. Lett. A* **173**, 442 (1993).
 [17] F. Cooper, H. F. Jones, and L. M. Simmons, Jr., *Phys. Rev. D* **43**, 3396 (1991).