Perturbative renormalization group for Hamiltonians

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We present new renormalization group equations for effective Hamiltonians in quantum field theory.

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I. INTRODUCTION

This paper presents a new, generic treatment of field theory Hamiltonians with ultraviolet divergences—reducing a Hamiltonian with divergences and appropriately chosen counterterms to an effective Hamiltonian which is finite to all orders in perturbation theory. Light-front Hamiltonians of quantum field theory are of special interest but the formalism is general. The purpose of developing a new formalism is to isolate problems of renormalization from problems of nearly degenerate states. Then perturbative renormalization can be carried through regardless of any near-degeneracy problems that ordinary perturbation theory might encounter. The reduction takes the form of a unitary transformation on the initial Hamiltonian with an ultraviolet cutoff. By "ultraviolet divergences" we mean all large energy divergences, including both small p^+ and large $p^\perp$ divergences for light-front field theory. The reduction does not handle infrared divergences associated with emission and absorption of zero-mass particles with all components of their four-momenta near zero. The paper draws on previous work by Głazek and Wilson [1]. Some original and review articles which discuss the light-front and closely related infinite momentum frame dynamics are listed in Ref. [2].

For an ultraviolet divergence to occur in nth order in perturbation theory for a Hamiltonian H with an interaction term H_I, it must be possible to reach eigenstates of the unperturbed Hamiltonian H_0 with eigenvalues of order the ultraviolet energy cutoff after at most $n/2$ transition matrix elements of H_I, starting from a state of fixed energy. If the energy cutoff is Λ, then at least one of these transition matrix elements must make a transition where the final energy is of order a factor $\Lambda^{\frac{2}{n}}$ larger than the initial energy (or vice versa). If no such matrix elements exist, then it is impossible for $n/2$ matrix elements multiplied in a sequence to reach an intermediate state energy of order the cutoff energy Λ starting from a constant energy, and this in turn makes it impossible for the Hamiltonian to exhibit cutoff-dependent ultraviolet divergences.

Our strategy for eliminating divergences is to perform a unitary transformation on the initial Hamiltonian, the result of which is an effective Hamiltonian which has no matrix elements providing energy jumps that exceed a fixed, finite factor once the initial energy is larger than a fixed bound λ_0. This means energy jumps by a factor $\Lambda^{\frac{2}{n}}$ are not possible if Λ is large enough, for any given order n, and this in turn rules out divergences as long as individual matrix elements of the effective Hamiltonian are themselves free of divergences, which has to be achieved through the presence of counterterms in the initial Hamiltonian. The required unitary transformation is introduced in Ref. [1]. In that paper, the initial Hamiltonian is algebraically transformed into a similar Hamiltonian in one step. In this paper, the strategy is to perform a series of infinitesimal unitary transformations, the end result of which is a similar effective Hamiltonian. We compare both procedures and explain the differences.

The infinitesimal unitary transformations will define a continuum of Hamiltonians H_λ which interpolate between the initial Hamiltonian ($\lambda = \Lambda$) and the effective Hamiltonian H_{λ_0}. The interpolating Hamiltonians will be defined so that no energy jumps by more than a fixed factor occur as long as the base energy is $\geq \lambda$. This means the initial Hamiltonian has bounded energy jumps only above Λ itself, whereas H_λ has bounded energy jumps above the base λ. We also forbid jumps from arbitrarily smaller energies to energies higher than the same fixed factor times λ. Finally we insist that all matrix elements of H_λ are infinitely differentiable functions of both λ and all initial and final state momenta except for explicit δ functions of momenta, the latter including both the momentum conservation δ functions and the extra p^+ δ functions of infrared counterterms and compositions thereof in light-front Hamiltonians. The infinite differentiability is needed because the compositions of infrared counterterms can lead to derivatives of δ functions of arbitrarily high orders and integrals involving these derivatives of δ functions need to be well defined.

There is one other constraint that will be imposed on the infinitesimal unitary transformations. This constraint is that energy denominators $E_i - E_f$ that occur involving the energy difference of two states $|i\rangle$ and $|f\rangle$ shall never be smaller than a constant factor times the sum $E_i + E_f + \lambda$. The presence of λ in this constraint ensures that the energy denominators are never smaller than the effective cutoff λ. This requirement will ensure

that power counting arguments can be carried through that might be voided by energy differences which are arbitrarily smaller than either energy separately. We will also try to ensure that matrix elements of H_λ which matter never act as if they were very much larger than λ, ensuring that high orders of perturbation theory cannot be more divergent than lower orders, and in particular ensuring that matrix elements of H_λ, whose size is much less than λ due to the presence of low energy external lines (in a Feynman diagramlike sense), cannot be compensated for by either small energy denominators or other matrix elements of H_λ that are much larger than λ.

The energy denominator constraint means that initial Hamiltonian cannot be fully diagonalized by the series of infinitesimal unitary transformations, except in the limit $\lambda \to 0$. Fortunately, there is no conflict between this constraint and the demand that off-diagonal matrix elements of H_λ involving sufficiently large energy jumps all vanish, leaving a form of banded matrix whose matrix elements involving smaller (but nonzero) energy changes are nonzero.

The infinitesimal unitary transformation defines the derivative of H_λ with respect to λ:

$$\frac{dH_\lambda}{d\lambda} = [H_\lambda, T_\lambda] , \tag{1.1}$$

where T_λ is anti-Hermitian, $T_\lambda^\dagger = -T_\lambda$, while H_λ is Hermitian for any λ. This equation, however, is incomplete because T_λ itself has still to be specified. To avoid small energy denominators, we will set $T_{\lambda fi} = 0$ if $E_{f\lambda}$ is close enough to $E_{i\lambda}$. However, our basic aim is to eliminate matrix elements of $H_{\lambda fi}$ whenever these matrix elements could cause large jumps in energy beyond the scale set by λ, i.e., when $E_{f\lambda}$ is sufficiently separated from $E_{i\lambda}$ and the larger of these is well above λ itself. We set this condition so that as λ is reduced more of the far-off-diagonal part of H_λ is eliminated, and so that as λ decreases it is mainly terms which jump from much lower energies to energies of order λ that are being eliminated—terms jumping to much higher energies have already been eliminated and jumps between smaller energies are yet to be handled. Equations which define T_λ are given in Sec. II.

The other issue is the boundary condition for the differential equation. The initial cutoff Hamiltonian is H_Λ which would provide the necessary boundary condition if H_Λ were known. Unfortunately, H_Λ includes unknown counterterms, and in the standard renormalization group framework, the operators in H_Λ which cause divergences are specified by conditions on H_λ at $\lambda = \lambda_0$ instead of $\lambda = \Lambda$. The remaining operators are specified at $\lambda = \Lambda$. The differential equation is solved with these mixed boundary conditions. In a perturbative treatment, we start with a bare Hamiltonian H_b cut off at Λ. This Hamiltonian is finite (no counterterms). Then the first approximation to H_λ is H_b with the cutoff λ, for all λ.

We will imagine that H_λ is dominated by its diagonal part which we will denote $H_{0\lambda}$ with eigenvalues E_λ:

$$\langle f|H_{0\lambda}|i\rangle = E_{i\lambda}\langle f|i\rangle . \tag{1.2}$$

Since $H_{0\lambda}$ is the full diagonal part of H_λ including self-energy effects, E_λ will be dependent on coupling constants.

The effective Hamiltonian H_λ will be obtained from the initial Hamiltonian $H_\Lambda \equiv H$ by a similarity transformation:

$$H_\lambda = S_\lambda^\dagger H S_\lambda . \tag{1.3}$$

Therefore, according to Eq. (1.1),

$$T_\lambda = S_\lambda^\dagger \frac{dS_\lambda}{d\lambda} . \tag{1.4}$$

Knowing S_λ and H one can find H_λ. This path is followed in Ref. [1]. One can also find H_λ knowing T_λ and integrating Eq. (1.1). This path is followed here. Note, however, that Ref. [1] introduces $H_\lambda = H_0 + H_{I\lambda}$ where H_0 is the bare free Hamiltonian which is independent of λ while here we introduce $H_{0\lambda}$ which is the diagonal part of H_λ, allowed to include self-interaction effects that make it depend on λ and the interaction.

II. DIFFERENTIAL EQUATIONS

In order to define the infinitesimal unitary transformations and the infinitesimal changes in H_λ when λ changes by an infinitesimal amount, we need to introduce various zones of the operators. Let an operator $\hat{O}$ have matrix elements $O_{fi} = \langle f|\hat{O}|i\rangle$. We introduce an auxiliary function, $x_{\lambda fi} \equiv \langle f|x_\lambda|i\rangle$, of the state labels f and i, for a given λ. For example, we choose

$$x_{\lambda fi} = \frac{E_{f\lambda} - E_{i\lambda}}{E_{f\lambda} + E_{i\lambda} + \lambda} , \tag{2.1}$$

where $E_{f\lambda}$ and $E_{i\lambda}$ are the diagonal matrix elements of H_λ. (We assume that λ dependence of $E_{f\lambda}$ and $E_{i\lambda}$ is negligible when they are of order λ, so that there are no multiple energy scales for which either $E_{f\lambda} \sim \lambda$ or $E_{i\lambda} \sim \lambda$.) The function $x_{\lambda fi}$ has the following properties. Its modulus is close to 1 when one of the energies is much larger than the other and large in comparison to the cutoff λ. On the other hand, $x_{\lambda fi}$ is close to 0 when the energies are similar or small in comparison to the cutoff. In the language of Ref. [1], moduli of the matrix elements of x_λ are close to 1 in the diagonal remotum of x_λ and close to 0 in the diagonal proximum of x_λ.

Zones of an operator $\hat{O}$ are defined by introducing smooth functions $u_{\lambda fi}$ and $r_{\lambda fi}$:

$$u_{\lambda fi} \equiv \langle f|u_\lambda|i\rangle = u(x_{\lambda fi}) \tag{2.2}$$

and

$$r_{\lambda fi} \equiv 1 - u_{\lambda fi} = r(x_{\lambda fi}) , \tag{2.3}$$

where $u(x)$ is a suitable function. We need these functions to ensure smoothness and differentiability, as befits a differential equation approach, and establish zones of operator matrix elements with smooth λ dependence in each zone and full differentiability across boundaries of the zones. The zones are introduced in order to define the Hamiltonians which cannot make large energy jumps and therefore have matrix elements mainly in the zone

close to the diagonal. The infinitesimal unitary transformations which eliminate the Hamiltonian zone far away from the diagonal have matrix elements different from 0 only in the zones away from the diagonal, so that they cannot involve small energy denominators. The full infinite differentiability is required because QCD generates Hamiltonians with derivatives of δ functions and to ensure that these δ functions have valid integrals all other momentum functions have to be differentiable.

An operator $u_\lambda[\hat{O}]$ is defined as follows. If an operator $\hat{O}$ has matrix elements $O_{fi} = \langle f|\hat{O}|i\rangle$, then the operator $u_\lambda[\hat{O}]$ has matrix elements $[u_\lambda[\hat{O}]]_{fi} = \langle f|u_\lambda|i\rangle\langle f|\hat{O}|i\rangle$. For example, according to Eq. (2.2), $[u_\lambda[\hat{O}]]_{fi} = u(x_{\lambda fi})\langle f|\hat{O}|i\rangle$. In order to simplify the notation $u_\lambda[\hat{O}]$ is often written as $u_\lambda\hat{O}$. There is also the differentiation rule $d(u\hat{O}) = (du)\hat{O} + u(d\hat{O})$. $r_\lambda[\hat{O}]$ is defined in the same way using function r in place of u.

One can choose the function $u(x)$ in various ways. For example, if one chooses $u(x) = \theta(x_0 - |x|)$, then $u_\lambda\hat{O} = \mathrm{DP}[\hat{O}]$ and $r_\lambda\hat{O} = \mathrm{DR}[\hat{O}]$, where DP and DR denote the diagonal proximum and diagonal remotum of the operator $\hat{O}$ as defined in Ref. [1]. Namely, the parameters β and E_0 introduced in Ref. [1] are $\beta = (1+x_0)(1-x_0)^{-1}$ and $E_0 = x_0(1-x_0)^{-1}\lambda/2$. In the present paper, we introduce smooth functions. The function $u(x) = u(|x|)$ stays equal 1 for $|x| \leq x_1$, drops smoothly from 1 to 0 between $|x| = x_1$ and $|x| = x_2$, and stays equal 0 for $|x| \geq x_2$. The function $r(x) = r(|x|)$ stays equal 0 for $|x| \leq x_1$, raises smoothly from 0 to 1 between $|x| = x_1$ and $|x| = x_2$, and stays equal 1 for $|x| \geq x_2$; $0 < x_1 \leq x_2 < 1$. Thus, when x_1 and x_2 approach x_0 one obtains the diagonal proximum and diagonal remotum from Ref. [1].

Zones of an operator $\hat{O}$ are defined as follows. The "band" zone of $\hat{O}$ is the part of $u_\lambda\hat{O}$ where matrix elements of u_λ equal 1, shortly expressed as $u_\lambda = 1$, $r_\lambda = 0$, and $|x| \leq x_1$. The "transition" zone is where matrix elements of u_λ vary from 1 to 0, shortly expressed as that u_λ varies from 1 to 0, while r_λ varies from 0 to 1 and $x_1 < |x| < x_2$. The "far-off-diagonal" zone is a part of $r_\lambda\hat{O}$ where u_λ equals 0 and r_λ equals 1 and $x_2 \leq |x| \leq 1$.

The infinitesimal unitary transformations and the differential equation for the Hamiltonian H_λ are now defined. We assume that

$$H_\lambda = u_\lambda G_\lambda \,. \tag{2.4}$$

Solving Eq. (1.1) means finding G_λ for a chosen u_λ, which includes determining boundary conditions (counterterms) which guarantee that H_λ is independent of the initial large cutoff Λ for $\lambda \ll \Lambda \to \infty$ and defining an iterative procedure for finding a solution which is able to interpolate between the required boundary conditions.

In terms of G_λ, Eq. (1.1) reads

$$\frac{du_\lambda}{d\lambda}G_\lambda + u_\lambda\frac{dG_\lambda}{d\lambda} = [H_{0\lambda}, T_\lambda] + [H_{I\lambda}, T_\lambda] \,. \tag{2.5}$$

Note that G_λ can be arbitrary in the far-off-diagonal region where $u_\lambda = \frac{du_\lambda}{d\lambda} = 0$ as long as it is finite and its derivative is finite. In Eq. (2.5), u_λ, $H_{0\lambda}$, $H_{I\lambda}$, and G_λ are considered to be known, and $\frac{dG_\lambda}{d\lambda}$ and T_λ to be unknown. Since we have only one equation and two unknowns, we need to introduce another equation which will define a method for finding $\frac{dG_\lambda}{d\lambda}$ and T_λ separately. We group the unknowns on the left hand side and knowns on the right hand side, putting the commutator $[H_{I\lambda}, T_\lambda]$ on the right hand side since it is assumed to be small and taken into account by successive approximations. Thus,

$$[T_\lambda, H_{0\lambda}] + u_\lambda\frac{dG_\lambda}{d\lambda} = [H_{I\lambda}, T_\lambda] - \frac{du_\lambda}{d\lambda}G_\lambda \,. \tag{2.6}$$

The right hand side of this equation is denoted by Q_λ. Then, one part of Q_λ is chosen to equal $[T_\lambda, H_{0\lambda}]$ and the remaining part gives $u_\lambda\frac{dG_\lambda}{d\lambda}$. The parts of Q_λ will be defined using the operation r_λ introduced above. Namely,

$$[T_\lambda, H_{0\lambda}] = r_\lambda[Q_\lambda] \tag{2.7}$$

and

$$u_\lambda\frac{dG_\lambda}{d\lambda} = Q_\lambda - r_\lambda[Q_\lambda] \equiv u_\lambda[Q_\lambda] \,. \tag{2.8}$$

Evaluating the matrix elements of both sides of the above equations in different zones of the operators one obtains differential equations for matrix elements of G_λ. Equation (2.4) provides matrix elements of H_λ.

In terms of the matrix elements $\langle f|H_\lambda|i\rangle = H_{\lambda fi}$ and $\langle f|T_\lambda|i\rangle = T_{\lambda fi}$ the equations are the following. In the band zone ($0 \leq |x| \leq x_1$; $r_{\lambda fi} = 0$; $u_{\lambda fi} = 1$),

$$T_{\lambda fi} = 0 \tag{2.9}$$

and

$$\frac{dH_{\lambda fi}}{d\lambda} = [H_{I\lambda}, T_\lambda]_{fi} \,. \tag{2.10}$$

In the transition zone ($x_1 < |x| \leq x_2$; both, $r_{\lambda fi}$ and $u_{\lambda fi}$ changing),

$$T_{\lambda fi} = \frac{r_{\lambda fi}}{E_{i\lambda} - E_{f\lambda}}\left\{[H_{I\lambda}, T_\lambda]_{fi} - \frac{du_{\lambda fi}}{d\lambda}\frac{H_{\lambda fi}}{u_{\lambda fi}}\right\} \tag{2.11}$$

and

$$\frac{dH_{\lambda fi}}{d\lambda} = u_{\lambda fi}[H_{I\lambda}, T_\lambda]_{fi} + r_{\lambda fi}\frac{du_{\lambda fi}}{d\lambda}\frac{H_{\lambda fi}}{u_{\lambda fi}} \,. \tag{2.12}$$

In the far-off-diagonal zone ($x_2 \leq |x| \leq 1$; $r_{\lambda fi} = 1$; $u_{\lambda fi} = 0$),

$$T_{\lambda fi} = \frac{1}{E_{i\lambda} - E_{f\lambda}}[H_{I\lambda}, T_\lambda]_{fi} \tag{2.13}$$

and

$$\frac{dH_{\lambda fi}}{d\lambda} = H_{\lambda fi} = 0 \,. \tag{2.14}$$

The beauty of these equations is that energy denominators $E_{i\lambda} - E_{f\lambda}$ only arise when $|x| \geq x_1$, which means

$$\left|\frac{1}{E_{i\lambda} - E_{f\lambda}}\right| \leq \frac{1}{x_1}\,\frac{1}{E_{i\lambda} + E_{f\lambda} + \lambda} \,, \tag{2.15}$$

and so for estimates of orders of magnitude of various

terms in perturbation theory we can replace the energy difference by a sum including λ—in particular, no reciprocal of an energy difference can be larger than a constant times λ^{-1}. But, second, when λ is small enough, $H_{fi\lambda}$ vanishes if $E_{f\lambda} \gg E_{i\lambda}$ or vice versa which means low energy eigenstates of H_λ, computed in any given order of perturbation theory, involve only a bounded and finite range of intermediate energies, independently of how large the energy cutoff Λ is. This limitation prevents any ultraviolet divergent diagrams being generated from the effective Hamiltonian if λ is held fixed while the initial cutoff Λ becomes arbitrarily large, as long as the initial Hamiltonian has all counterterms needed to ensure that the low energy matrix elements of H_λ are all independent of the cutoff Λ. Moreover, in all our equations, operators are multiplied only through commutators, which ensures that no unlinked product can be formed—all products are linked through the commutators of at least one field operator from the pair forming each link.

However, it should be noted that the operators T_λ are defined only implicitly, and it is important that $H_{I\lambda}$ be small compared to the diagonal part of H_λ in order that the T_λ equations be soluble by a convergent iteration process; otherwise T_λ may start linking states widely separated in energies, separated by much larger energy than λ itself. The iteration process is described in the next section.

A brief digression is in order concerning possible alterations of the zone structure. A situation of particular interest appears when the band and far-off-diagonal zones disappear. Namely, one may consider a smooth function $u(x)$ which equals 0 only for $|x| = 1$ and equals 1 only for $x = 0$ and makes a transition between ~ 0 and ~ 1 in the vicinity of some point $x = x_0$. In this situation there is only one zone spread over the entire Hamiltonian matrix, analogous to the transition region. The utility of the various zone structures depends on the structure of the initial bare Hamiltonian. One may also consider zones defined using functions $u(x)$ and $r(x)$ which depend on λ themselves. The idea is that the parameters x_i, $i = 1, 2$, determine the zone boundaries in H_λ and T_λ through the parameters $E_{0i} = x_i(1 - x_i)^{-1}\lambda/2$ and the slopes $\beta_i = (1 + x_i)(1 - x_i)^{-1}$ which are analogous to E_0 and β from Ref. [1]. Both the energy bounds analogous to E_0 and the slopes analogous to β are allowed to vary with λ in order to create the most convenient setup for numerical computations. Finally, operators u_λ can be defined employing also eigenvalues of operators other than $H_{0\lambda}$.

Equations (2.7) and (2.8) can be written in the operator form in different zones, using Eq. (2.4) and

$$\left[\frac{dG_\lambda}{d\lambda}\right]_{fi} = [Q_\lambda]_{fi} \tag{2.16}$$

in the region where $u_{\lambda fi} \neq 0$. In the band zone one has

$$\frac{dG_\lambda}{d\lambda} = [H_{I\lambda}, T_\lambda] , \tag{2.17}$$

$$T_\lambda = 0 , \tag{2.18}$$

in the transition zone one has

$$\frac{dG_\lambda}{d\lambda} = [H_{I\lambda}, T_\lambda] - \frac{du_\lambda}{d\lambda} G_\lambda , \tag{2.19}$$

$$[T_\lambda, H_{0\lambda}] = r_\lambda \left\{ [H_{I\lambda}, T_\lambda] - \frac{du_\lambda}{d\lambda} G_\lambda \right\} , \tag{2.20}$$

and in the far-off-diagonal zone one has

$$[T_\lambda, H_{0\lambda}] = [H_{I\lambda}, T_\lambda] , \tag{2.21}$$

$$H_\lambda = 0 . \tag{2.22}$$

[In the case of only one zone (transition) one has only three equations: (2.4), (2.19), and (2.20).]

Since $\frac{dG_\lambda}{d\lambda}$ does not vanish outside the transition zone of G_λ, one may worry that the matrix elements of the resulting operators will not necessarily be infinitely differentiable functions of the state labels f and i. We will show later that no such problem arises.

III. RENORMALIZED HAMILTONIANS

We describe the renormalization group calculation of the continuum of effective Hamiltonians H_λ.

Both H_λ and T_λ are found in an iterative procedure of successive approximations while the initial cutoff Λ is being made very large.

In the first approximation one neglects the commutators with T_λ and $H_{I\lambda}$. In this approximation the family of the effective Hamiltonians is of the form

$$H_{1\lambda} \equiv H_{b\lambda} = e_\lambda H_b , \tag{3.1}$$

where H_b is the initially given bare Hamiltonian and e_λ is similar to u_λ. According to Eq. (2.4),

$$G_{1\lambda} = \frac{e_\lambda}{u_\lambda} H_b , \tag{3.2}$$

where

$$e_\lambda = u_\lambda \exp(r_\lambda) , \tag{3.3}$$

which means $e_{\lambda fi} = u_{\lambda fi} \exp(r_{\lambda fi})$.

In the next approximations one includes corrections induced by the commutators. Once the Hamiltonian is written as

$$H_\lambda = e_\lambda X_\lambda + H_{b\lambda} , \tag{3.4}$$

one obtains

$$\frac{dX_\lambda}{d\lambda} = \exp(-r_\lambda)[H_{I\lambda}, T_\lambda] . \tag{3.5}$$

Therefore,

$$\begin{aligned} H_\lambda &= e_\lambda \left[\int_\Lambda^\lambda ds \, \exp(-r_s)[H_{Is}, T_s] + X_\Lambda \right] + H_{b\lambda} \\ &\equiv e_\lambda \left[\int_\Lambda^\lambda ds \, \exp(-r_s)[H_{Is}, T_s] + X_\Lambda + H_b \right] , \end{aligned} \tag{3.6}$$

where X_Λ is a suitable initial condition (counterterm) which guarantees Λ independence of H_λ for $\lambda = \lambda_0$ when $\Lambda \gg \lambda_0$.

The first step in analyzing and subtracting out divergences by the counterterms is to study how low energy matrix elements of H_λ vary when λ is of order the initial cutoff Λ, identifying terms that are divergent as a power of Λ or potentially divergent as $\ln\Lambda$. The latter terms arise whenever matrix elements of H_λ change by an amount that is independent of Λ but nonzero as λ drops from well above Λ to a small constant times Λ. As long as the energy scale in $H_{I\lambda}$ never exceeds the bare cutoff Λ, and in fact contains a small coupling parameter, we see that no λ-dependent changes in $H_{\lambda fi}$ can exceed the cutoff scale Λ itself, and all denominators are of order Λ too. One splits $[H_{Is},T_s]$ into the diverging and converging parts $[H_{Is},T_s]^d$ and $[H_{Is},T_s]^c$. The diverging part is the part which behaves as 1 or s^{-1}. The converging part is the remaining part which behaves as s^{-2}, s^{-3}, etc. H_s is of order s and T_s is of order s^{-1}. The counterterm X_Λ is defined by the condition that it removes the Λ-dependent parts of the integral when $\lambda=\lambda_0$:

$$X_\Lambda = \mathcal{X}_0 - \int_\Lambda^{\lambda_0} ds \exp(-r_s)[H_{Is},T_s]^d\,, \tag{3.7}$$

where $\mathcal{X}_0$ is a finite part added to the divergent counterterm given by the integral. Using this initial condition one finally obtains

$$H_\lambda = e_\lambda\left\{\int_\Lambda^{\lambda} ds \exp(-r_s)[H_{Is},T_s]^c + \int_{\lambda_0}^{\lambda} ds \exp(-r_s)[H_{Is},T_s]^d + \mathcal{X}_0 + H_b\right\}. \tag{3.8}$$

This equation is to be solved by successive approximations. The solution for $\Lambda \gg \lambda_0$ defines a continuum of renormalized Hamiltonians labeled by the parameter λ. The finite part of the counterterm, $\mathcal{X}_0$, is specified by fitting observables.

The operator H_λ is clearly infinitely differentiable because the expression in the curly brackets has no reason to diverge for any finite λ. Moreover, the factor e_λ ensures that matrix elements of H_λ vanish whenever u_λ does.

IV. CONCLUSION

We have succeeded in defining a sequence of renormalized Hamiltonians H_λ which (a) are finite if counterterms can be found, and (b) can be computed perturbatively without ever getting energy denominators with differences less than order λ. This latter result ensures that nearly degenerate states do not give large effects in the perturbative computations. This is opposed to effective Hamiltonian schemes where some basis states are kept and other basis states eliminated, where the energy difference between the highest state kept and the lowest state dropped appears and can cause difficulties.

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