# Zero momentum limits of two loop finite temperature self-energies in $\phi^4$ and $\phi^3$ coupling theories

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Calculations of two loop finite temperature self-energies of $\phi^4$ and $\phi^3$ coupling theories, in the imaginary time formalism, are reported, and their zero external momentum limits are examined. In $\phi^4$ theory the two loop self-energy is analytic in the zero momentum limit, but in $\phi^3$ theory it is nonanalytic and diverges.

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## I. INTRODUCTION

Recently much attention has been paid to the zero external momentum properties of self-energies in finite temperature quantum field theory. The point is that, when the external momentum tends to zero, the limiting value of the self-energy does or does not depend on the direction of the limiting procedures: the spacelike limit ($p_0 \to 0$ then $p \to 0$) and timelike limit ($p \to 0$ then $p_0 \to 0$) [1]. More recently, Weldon [2] has shown that the two limiting procedures lead to different values based on the detailed analysis of calculations, and Bedaque and Das [3] confirmed those results.

However, these arguments are limited to one loop contributions to the self-energy. So we wish to present here two loop calculations and discuss the zero momentum limits. We take up two theories: one is scalar fields with $\phi^4$ coupling and the other is $\phi^3$ coupling. We use the imaginary time formalism of finite temperature quantum field theory.

## II. TWO LOOP CALCULATIONS IN $\phi^4$ COUPLING THEORY

In Fig. 1 we show two loop diagrams of the self-energies in $\phi^4$ coupling theory. The contribution from Fig. 1(a) is easily calculated and is independent of the external momentum.

The contribution from Fig. 1(b), denoted as $\Pi(p)$, is given as

$$\Pi(p) \sim \lambda^2 T^2 \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \times \frac{1}{k^2-m^2} \frac{1}{q^2-m^2} \frac{1}{(p-k+q)^2-m^2}, \quad (1)$$

where $\lambda$ is the coupling constant, $T$ is temperature, and $p_0 = 2\pi i n T$, $k_0 = 2\pi i j T$, $q_0 = 2\pi i l T$ ($n, j, l = 0, \pm 1, \pm 2, \ldots$). The double summation of $j$ and $l$ can be performed by splitting each propagator into partial fractions; for example,

$$\frac{1}{k^2-m^2} = \frac{1}{(2\pi i j T)^2 - k^2 - m^2} = \frac{i}{4\pi E_k T} \left[ \frac{1}{j - iE_k/2\pi T} - \frac{1}{j + iE_k/2\pi T} \right],$$

$$E_k = \sqrt{k^2+m^2}.$$

An example of a double summation is given in the Appendix. After the summation and analytic continuation of $p_0$ to a real continuous value we obtain

$$\Pi(p) \sim \frac{\lambda^2}{16} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{E_k E_q E_{p-k+q}} \left[ \frac{E_k+E_q-E_{p-k+q}}{(E_k+E_q-E_{p-k+q})^2 - p_0{}^2}(U-V-W+1) - \frac{E_k+E_q+E_{p-k+q}}{(E_k+E_q+E_{p-k+q})^2 - p_0{}^2}(U+V+W+1) + \frac{E_k-E_q-E_{p-k+q}}{(E_k-E_q-E_{p-k+q})^2 - p_0{}^2}(U+V-W-1) - \frac{E_k-E_q+E_{p-k+q}}{(E_k-E_q+E_{p-k+q})^2 - p_0{}^2}(U-V+W-1) \right], \quad (2)$$

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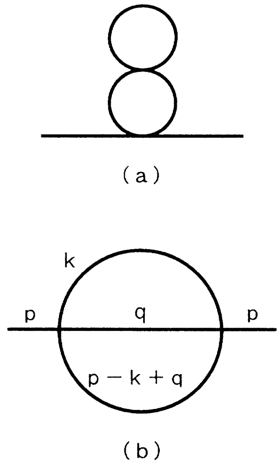


FIG. 1. Two loop diagrams of the self-energy in $\phi^4$ coupling theory.

where

$$U=\coth(E_k/2T)\coth(E_q/2T)\ ,$$

$$V=\coth(E_k/2T)\coth(E_{p-k+q}/2T)\ ,$$

$$W=\coth(E_q/2T)\coth(E_{p-k+q}/2T)\ .$$

As can be seen from this result, $\Pi(p)$ approaches to one and the same limit when $p\to 0$ in a spacelike and timelike way.

In $\phi^4$ theory, the one loop contribution is independent of the external momentum. Thus there is no nonanalytic behavior in the limit $p\to 0$, up to two loop order.

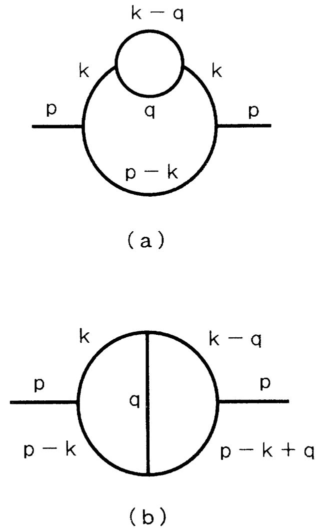


FIG. 2. Two loop diagrams of the self-energy in $\phi^3$ coupling theory.

## III. TWO LOOP CALCULATIONS IN $\phi^3$ COUPLING THEORY

In $\phi^3$ coupling theory, the nonanalytic behavior of the finite temperature self-energy in the limit $p\to 0$ seems to be established, in one loop calculations. In Fig. 2 we show two loop diagrams. We first calculate the contribution of Fig. 2(a), which we denote $\Pi_a(p)$:

$$\Pi_a(p)\sim\lambda^4T^2\sum_j\int\sum_l\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{k^2-m^2}\frac{1}{(k-q)^2-m^2}\frac{1}{q^2-m^2}\frac{1}{k^2-m^2}\frac{1}{(p-k)^2-m^2}\ , \tag{3}$$

where $p_0,k_0,q_0$ are the same as in Eq. (1). In this contribution the inner loop can be calculated separately, which we denote as $\Pi_1(k)$. The calculation of $\Pi_1(k)$ is well known and it can be expressed as

$$\Pi_1(k)=-\frac{i\pi}{4}\frac{1}{(2\pi T)^2}\int\frac{d^3q}{(2\pi)^3}\left[\frac{X}{j+i(x-y)}-\frac{Y}{j+i(x+y)}+\frac{Y}{j-i(x+y)}-\frac{X}{j-i(x-y)}\right]\ , \tag{4}$$

where $k_0=2\pi ijT$ and

$$x=E_{k-q}/2\pi T\ ,\quad y=E_q/2\pi T\ ,\quad X=\coth(\pi x)-\coth(\pi y)\ ,\quad Y=\coth(\pi x)+\coth(\pi y)\ .$$

Here it should be noted that, when we do the analytical continuation of $k_0$ to real continuous values, $\Pi_1(k)$ is nonanalytic in the $k\to 0$ limit, but at this stage of calculation we do not need such a continuation. Inserting this $\Pi_1$ into $\Pi_a$ and splitting each propagator into partial fractions, we obtain the $j$ summation

$$\sum_j\left[\frac{1}{j-iz}-\frac{1}{j+iz}\right]^2\left[\frac{1}{j-n-iw}-\frac{1}{j-n+iw}\right]\left[\frac{X}{j+i(x-y)}-\frac{Y}{j+i(x+y)}+\frac{Y}{j-i(x+y)}-\frac{X}{j-i(x-y)}\right]\ , \tag{5}$$

where

$$z=E_k/2\pi T\ ,\quad w=E_{p-k}/2\pi T\ .$$

Thus we have 24 terms in all, and classify these terms into 6 classes $A,B,\ldots,F$ according to the 6 terms obtained from the first two sets of parentheses in Eq. (5):

$$A:\quad\left[\frac{1}{j-iz}\right]^2\frac{1}{j-n-iw}\frac{Z}{j+iu}\ ,$$

$$B:\quad\left[\frac{1}{j-iz}\right]^2\frac{1}{j-n+iw}\frac{Z}{j+iu}\ ,$$

$C$: $\dfrac{2}{(j-iz)(j+iz)}\dfrac{1}{j-n-iw}\dfrac{Z}{j+iu}$ ,

$D$: $\dfrac{2}{(j-iz)(j+iz)}\dfrac{1}{j-n+iw}\dfrac{Z}{j+iu}$ ,

$E$: $\dfrac{1}{(j+iz)^2}\dfrac{1}{j-n-iw}\dfrac{Z}{j+iu}$ ,

$F$: $\dfrac{1}{(j+iz)^2}\dfrac{1}{j-n+iw}\dfrac{Z}{j+iu}$ ,

where $u=\pm(x+y)$ or $\pm(x-y)$ and $Z=\pm Y$ or $\pm X$ correspondingly. Each class contains four terms.

After the $j$ summation, terms of each class are given as follows. Here we redefine new notation:

$(a,b,c,d,f)=2\pi T(x,y,z,w,u)$

$$A:\ \pi Z\left[\frac{1}{(c-d+p_0)(c+f)2T\sinh^2(c/2T)}-\frac{\coth(c/2T)+\coth(f/2T)}{(c+f)^2(d+f-p_0)}+\frac{\coth(c/2T)-\coth(d/2T)}{(c-d+p_0)^2(d+f-p_0)}\right], \quad (6)$$

$$B:\ \pi Z\left[\frac{1}{(c+d+p_0)(c+f)2T\sinh^2(c/2T)}+\frac{\coth(c/2T)+\coth(f/2T)}{(c+d)^2(d-f+p_0)}-\frac{\coth(c/2T)+\coth(d/2T)}{(c+d+p_0)^2(d-f+p_0)}\right], \quad (7)$$

$$C:\ 2\pi Z\left[-\frac{\coth(c/2T)}{(c+d-p_0)(c-f)c}+\frac{\coth(c/2T)-\coth(f/2T)}{(c-f)(c+f)(d+f-p_0)}-\frac{\coth(c/2T)-\coth(d/2T)}{(c+d-p_0)(c-d+p_0)(d+f-p_0)}\right], \quad (8)$$

$$D:\ 2\pi Z\left[-\frac{\coth(c/2T)}{(c-d-p_0)(c-f)c}-\frac{\coth(c/2T)+\coth(f/2T)}{(c-f)(c+f)(d-f+p_0)}+\frac{\coth(c/2T)-\coth(d/2T)}{(c-d-p_0)(c+d+p_0)(d-f-p_0)}\right], \quad (9)$$

$$E:\ \pi Z\left[-\frac{1}{(c+d-p_0)(c-f)2T\sinh^2(c/2T)}+\frac{\coth(c/2T)-\coth(f/2T)}{(c-f)^2(d+f-p_0)}-\frac{\coth(c/2T)+\coth(d/2T)}{(c+d-p_0)^2(c+f-p_0)}\right], \quad (10)$$

$$F:\ \pi Z\left[\frac{1}{(c-d-p_0)(c-f)2T\sinh^2(c/2T)}-\frac{\coth(c/2T)-\coth(f/2T)}{(c-f)^2(d-f+p_0)}+\frac{\coth(c/2T)-\coth(d/2T)}{(c-d-p_0)(c-d+p_0)(d-f+p_0)}\right]. \quad (11)$$

The self-energy $\Pi_a$ is expressed with these $A,B,\ldots,F$ as

$$\Pi_a(p)\sim\frac{-\lambda^4}{2^7\pi}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{A-B-C+D+E-F}{E_k{}^2E_qE_{k-q}E_{p-k}},$$

where $A,B,\ldots,F$ represent sums of four terms in each class.

Next we consider the behavior of these terms in the $p\to 0$ limit after analytical continuation of $p_0$ to real and continuous values. In each term, in the limit $p\to 0$, $d$ approaches $c$ and $a,b,c,f$ are independent of $p$. From the inspection of Eq. (6) to Eq. (11) we can easily draw the results: the first terms of $A,D,F$ class diverge in the limit $p\to 0$ both in spacelike and timelike limits; the third terms of $A$ and $F$ class diverge in the spacelike limit but approach zero in the timelike limit; the third terms of $C$ and $D$ class approach finite values in the spacelike limit (differentiation of hyperbolic cotangent) but go to zero in the timelike limit. All of these anomalous terms are temperature dependent as can be seen. Other terms do not show such anomalies.

The contribution of Fig. 2(b) is given as

$$\Pi_b(p)\sim\lambda^4T^2\sum_{j,l}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{k^2-m^2}\frac{1}{(p-k)^2-m^2}\frac{1}{q^2-m^2}\frac{1}{(k-q)^2-m^2}\frac{1}{(p-k+q)^2-m^2}. \quad (13)$$

We split each propagator into partial fractions, as before, and obtain 32 terms in all. However, they have an analogous structure and we present typical calculations of the double sum in the Appendix. After summations, each of the 32 terms can be expressed by the general form

$$\frac{i\pi^2}{(a-b-p_0)(d-f-p_0)}\{F(a,d,c,)-F(a,f,c,p_0)+F(b,f,c)-F(b,d,c,-p_0)\}, \quad (14)$$

where

$$a=\pm E_k,\quad b=\pm E_{p-k},\quad c=\pm E_q,\quad d=\pm E_{k-q},\quad f=\pm E_{p-k+q},$$

and

$$F(r,s,t)=\frac{\{\coth(r/2T)-\coth(s/2T)\}\{\coth(t/2T)-\coth((r-s)/2T)\}}{r-s-t},$$

$$F(r,s,t,p_0)=F(r,s,t)\frac{r-s-t}{r-s-t-p_0} .$$

All of the 32 terms can be classified as follows: $A$ class, $a$ and $b$ same sign, $d$ and $f$ same sign; $B$ class, $a$ and $b$ same sign, $d$ and $f$ different sign; $C$ class, $a$ and $b$ different sign, $d$ and $f$ same sign; $D$ class, $a$ and $b$ different sign, $d$ and $f$ different sign.

In these classes $A$, $B$, $C$, and $D$, $c=-E_q$. Terms with $c=E_q$ and have the same structure as $A,B,C,D$ are classified as $\overline{A},\overline{B},\overline{C},\overline{D}$, respectively. Each class $A,B,\ldots,\overline{D}$ contains four terms. Then,

$$\Pi_b(p)\sim\frac{i\lambda^4}{2^7\pi^2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{(A-\overline{A})-(B-\overline{B})-(C-\overline{C})+(D-\overline{D})}{E_kE_{p-k}E_qE_{k-q}E_{p-k+q}} , \tag{15}$$

where $A,\overline{A},\ldots,\overline{D}$ represent the sum of four terms of each class.

Next we consider the zero momentum limits. We show the spacelike limit of one of the $A$ class terms, denoting it as $A_1$, as an example. In this term we take $a,b,d,f$ as positive and $c=-E_q$:

$$A_1\underset{p_0\to 0}{\to}\frac{i\pi^2}{(a-b)(d-f)}\{F(a,d,c)-F(a,f,c)+F(b,f,c)-F(b,d,c)\}$$

$$\underset{p\to 0}{\to} i\pi^2\frac{\partial^2}{\partial r\partial s}F(r,s,c)\bigg|_{r=E_k,\ s=E_{k-q}} .$$

Thus we obtain the spacelike limit of the whole $A$ class terms:

$$A:\ i\pi^2\left\{\left[\frac{\partial^2}{\partial r\partial s}F(r,s,c)\right]_{r=-E_k,\ s=-E_{k-q}}+\left[\frac{\partial^2}{\partial r\partial s}F(r,s,c)\right]_{r=E_k,\ s=-E_{k-q}}\right.$$
$$\left.+\left[\frac{\partial^2}{\partial r\partial s}F(r,s,c)\right]_{r=-E_k,\ s=E_{k-q}}+\left[\frac{\partial^2}{\partial r\partial s}F(r,s,c)\right]_{r=E_k,s=E_{k-q}}\right\} . \tag{16}$$

Analogously we obtain

$$B:\ \frac{2i\pi^3T}{E_{k-q}}\left[\left[\frac{\partial}{\partial r}F(r,E_{k-q},c)\right]_{r=-E_k}+\left[\frac{\partial}{\partial r}F(r,E_{k-q},c)\right]_{r=E_k}-\left[\frac{\partial}{\partial r}F(r,-E_{k-q},c)\right]_{r=-E_k}\right.$$
$$\left.-\left[\frac{\partial}{\partial r}F(r,-E_{k-q},c)\right]_{r=E_k}\right] , \tag{17}$$

$$C:\ \frac{2i\pi^3T}{E_k}\left[\left[\frac{\partial}{\partial s}F(E_k,s,c)\right]_{s=-E_{k-q}}+\left[\frac{\partial}{\partial s}F(E_k,s,c)\right]_{s=E_{k-q}}-\left[\frac{\partial}{\partial s}F(-E_k,s,c)\right]_{s=-E_{k-q}}\right.$$
$$\left.-\left[\frac{\partial}{\partial s}F(-E_k,s,c)\right]_{s=E_{k-q}}\right] , \tag{18}$$

$$D:\ \frac{4i\pi^4T^2}{E_kE_{k-q}}\{F(E_k,E_{k-q},c)-F(E_k,-E_{k-q},c)-F(-E_k,E_{k-q},c)+F(-E_k,-E_{k-q},c)\} . \tag{19}$$

$\overline{A},\overline{B},\overline{C},\overline{D}$, classes are obtained by changing $c=-E_q$ to $c=E_q$.

The timelike limit of $A_1$, as an example, is

$$A_1\underset{p\to 0}{\sim}\frac{4i\pi^3T^2}{{p_0}^2}\{2F(a,d,c)-F(a,d,c,p_0)-F(a,d,c,-p_0)\} .$$

This goes to an indefinite form when $p_0\to 0$. So we must use differentiation with respect to $p_0$. Then we obtain $A$ class terms:

$$A:\ -8i\pi^4T^2\left[\frac{F(-E_k,-E_{k-q},c)}{(-E_k+E_{k-q}-c)^2}+\frac{F(-E_k,E_{k-q},c)}{(-E_k-E_{k-q}-c)^2}+\frac{F(E_k,-E_{k-q},c)}{(E_k+E_{k-q}-c)^2}+\frac{F(E_k,E_{k-q},c)}{(E_k-E_{k-q}-c)^2}\right] . \tag{20}$$

Analogously,

$$B:\ \frac{4i\pi^4T^2}{E_{k-q}}\left[\frac{F(-E_k,-E_{k-q},c)}{-E_k+E_{k-q}-c}-\frac{F(-E_k,E_{k-q},c)}{-E_k-E_{k-q}-c}+\frac{F(E_k,-E_{k-q},c)}{E_k+E_{k-q}-c}-\frac{F(E_k,E_{k-q},c)}{E_k-E_{k-q}-c}\right] , \tag{21}$$

$$C:\ \frac{4i\pi^4T^2}{E_k}\left[-\frac{F(-E_k,-E_{k-q},c)}{-E_k+E_{k-q}-c}-\frac{F(-E_k,E_{k-q},c)}{-E_k-E_{k-q}-c}+\frac{F(E_k,-E_{k-q},c)}{E_k+E_{k-q}-c}+\frac{F(E_k,E_{k-q},c)}{E_k-E_{k-q}-c}\right].$$

$D$ class terms are equal to Eq. (19) also in the timelike limit.

Thus, it is evident that the two limits are finite but do not coincide. It seems rather difficult to separate the temperature-dependent part in these expressions.

## IV. CONCLUSION

Finite temperature self-energy in $\phi^4$ coupling theory is analytic in the zero momentum limit, but, in $\phi^3$ coupling theory, it is nonanalytic and diverges in the same limit, at least up to two loop order in the imaginary time formalism. This result may suggest, more generally, that the analytic property at the zero momentum limit depends on the kind of interactions.

## APPENDIX

Here we present the details of the calculation of the double summation of a product of five propagators in Fig. 2(b) as an example.

We split each propagator into partial fractions as was shown in the text. Then we obtain the following typical form of double summations:

$$S=\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{1}{j+a}\frac{1}{j-n+b}\frac{1}{l+c}\frac{1}{j-l+d}\frac{1}{j-l-n+f},$$

where $a,b,\ldots,f$ are constants and $n$ is an integer. We split a product of the first two fractions into partial fractions and the likely last two fractions:

$$\frac{1}{j+a}\frac{1}{j-n+b}=\frac{1}{-n+b-a}\left[\frac{1}{j+a}-\frac{1}{j-n+b}\right],$$

$$\frac{1}{j-l+d}\frac{1}{j-l-n+f}=\frac{1}{-n+f-d}\left[\frac{1}{j-l+d}-\frac{1}{j-l-n+f}\right].$$

Then

$$S=\frac{1}{(a-b+n)(d-f+n)}\sum_{j,l}\left[\frac{1}{j+a}\frac{1}{j-l+d}-\frac{1}{j+a}\frac{1}{j-l-n+f}-\frac{1}{j-n+b}\frac{1}{j-l+d}+\frac{1}{j-n+b}\frac{1}{j-l-n+f}\right]\frac{1}{l+c}.$$

We perform the $j$ summation:

$$S=\frac{\pi}{(a-b+n)(d-f+n)}\sum_{l}\left[-\frac{\cot(\pi a)-\cot[\pi(d-l)]}{l+a-d}+\frac{\cot(\pi a)-\cot[\pi(f-l-n)]}{l+n+a-f}+\frac{\cot[\pi(b-n)]-\cot[\pi(d-l)]}{l-n+b-d}-\frac{\cot[\pi(b-n)]-\cot[\pi(f-l-n)]}{l+b-f}\right]\frac{1}{l+c}.$$

Here, because of the periodicity of cot, $n$ and $l$ in the variable can be dropped and all cot's are independent of $n$ and $l$, so we can perform $l$ summations. Thus we obtain the result

$$S=\frac{-\pi^2}{(a-b+n)(d-f+n)}\left[\frac{[\cot(\pi a)-\cot(\pi d)][\cot(\pi c)-\cot(\pi(a-d))]}{a-d-c}-\frac{[\cot(\pi a)-\cot(\pi f)][\cot(\pi c)-\cot(\pi(a-f))]}{a-f-c+n}-\frac{[\cot(\pi b)-\cot(\pi d)][\cot(\pi c)-\cot(\pi(b-d))]}{b-d-c-n}+\frac{[\cot(\pi b)-\cot(\pi f)][\cot(\pi c)-\cot(\pi(b-f))]}{b-f-c}\right].$$

[1] For a review and references of the preceding works, see P. S. Gribosky and B. R. Holstein, Z. Phys. C 47, 205 (1990).
[2] H. A. Weldon, Phys. Rev. D 47, 594 (1993).
[3] P. F. Bedaque and A. Das, Phys. Rev. D 47, 601 (1993).