Rules for diagrams in thermal field theories

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(Received 16 February 1993; revised manuscript received 11 November 1993)

Sets of rules are proposed that allow one to write down the amplitude associated with a diagram at temperature T once the energy running around each loop has been summed over, in the imaginary-time formalism. Alternative forms are given: one is based on tree diagrams, another one on possible intermediate states. A close analogy to the $T = 0$ case is obtained. The amplitude's analytic structure is explicit. A factorization property is found for the N-point imaginary-time Green functions.

PACS number(s): 11.10.Wx, 03.70.+k, 05.30.−d

I. INTRODUCTION

Perturbation theory for a relativistic field theory is more intricate at $T \neq 0$ than at $T = 0$. The formalism is well established [1,2], but there are few calculations beyond the one-loop order or beyond the two-point function [3]. On the other hand, the objectives are ambitious: resummation of infrared divergences in high temperature QCD [4] or generalization to $T \neq 0$ of the cancellation of the infrared divergences in physical processes [4]. Recently, there has been renewed interest in the N-point functions at $T \neq 0$ in the relationship between the time-ordered functions and the advanced ones, in the real-time formalism, and also in the connection to the functions arising in the imaginary-time formalism [5−7].

This paper is concerned with the amplitude associated to a diagram contributing to an N-point function at $T \neq 0$ in the imaginary-time formalism. The Φ^3 theory is chosen as an example.

It will be shown that the summation over the energy running around each loop is easily performed at any loop order. One is left with an integral over the space momentum, and the integrand is a sum of terms. The analytic properties of the resulting amplitude with respect to the external energies are explicit as they are the consequence of the presence of simple poles in those variables in the denominator of the integrand. The numerators have the full T dependence, stressing the similarity between the $T = 0$ and $T \neq 0$ cases.

More precisely, each diagram is written in the imaginary-time formalism. The summation over the energy running around each loop is performed, and the result is a form where the external energy variables p_j^0 are imaginary and the space momentum variables are real. This form is analytically continued to the real values of those p_j^0 variables. The properties of this analytical continuation have been considered recently by Evans [6] for a general N-point function.

One result is that the relevant analytical continuations may be characterized by the way one approaches the real values for the N external energy variables $p_j^0 \to p_j^0 + i\epsilon_j$, where the ϵ_j are infinitesimal real constants, appropriately chosen [6]. Different choices for the ϵ_j lead to different amplitudes.

The properties to be exhibited in this work are valid whatever the choice for the ϵ_j. The analytically continued N-point amplitude will be called the "full" amplitude.

In the forms to be written later on, the external energy variables only appear in the denominator factors of the integrand, in a linear way, and one may make the substitution $p_j^0 \to p_j^0 + i\epsilon_j$ everywhere, if one wishes. Moreover, as will be shown, those N-point amplitudes are the ones that do appear in the amplitude associated with a multiloop diagram contributing to the self-energy, when written in terms of intermediate states. That amplitude is insensitive to the choice of the ϵ_j.

Two forms are proposed for the integrand, and each one exhibits interesting features. One form is a very compact one, in terms of an expansion in powers of $(n+\frac{1}{2})$ (n is the Bose-Einstein factor) and of $T = 0$ tree diagrams. In the second form, the full amplitude is written as a sum of terms, each corresponding to a possible intermediate state. Those states are a simple extension of the familiar ones at $T = 0$; all energies' signs are allowed. The validity of those forms is proved, completely to the five-loop order, and partially to all-loop orders. They have been checked, by direct computation of diagrams, up to the four-loop order.

In Sec. II, it is shown how the coth method may be used to perform the sum over the energy running around each loop, at any loop order. It leads to compact forms for the resulting amplitude in terms of tree diagrams. In Sec. III, the amplitude is written as a sum of terms, each associated with an intermediate state. Rules are given to write down the result immediately; the analytic structure is exhibited. Conclusions are in Sec. IV.

II. THE COTH METHOD FOR THE DISCRETE SUMS

In the imaginary-time formalism, all the energy variables take on discrete imaginary values. There are several ways to perform the discrete sum over the energy running around a loop [1,2]. We want to show that the coth method [2,8], often used for the computation of one-loop amplitudes, but, to our knowledge, not for higher orders, can be extended to any-loop order and leads to concise forms for the resulting amplitudes.

A. Method

The summation over the energy may be performed in the following way:

$$T\sum_{n=-\infty}^{\infty} f(k_n^0 = in2\pi T) = \int_C \frac{dz}{2i\pi} f(z)\frac{1}{2}\coth\frac{\beta}{2}z\ , \quad (1)$$

where the contour C originally runs around the poles of the coth function and is deformed to two straight lines running on each side of the imaginary axis, provided $f(z)$ has no singularity along that axis. One may then further deform the contour in the complex plane to pick up the poles of $f(z)$, with coth going to ±1 as $|z| \to \infty$.

We use the method to compute a one-loop and a two-loop amplitude. One feature of our method is to write the propagator associated with an internal line

$$\frac{1}{k_0^2 - \mathbf{k}^2 - m^2} = \frac{1}{2E_k}\sum_{s=\pm1}\frac{s}{k_0 - sE_k}\ , \quad (2)$$

with $E_k = (\mathbf{k}^2 + m^2)^{\frac{1}{2}}$.

The one-loop self-energy in a Φ^3 theory is (see Fig. 1)

$$G(p_0,p) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4E_1E_2}\sum_{s_1s_2} I\ , \quad (3)$$

$$I = -T\sum_n \frac{s_1s_2}{(k_n^0 - s_1E_1)(p_0 - k_n^0 - s_2E_2)}\ , \quad (4)$$

where both k_n^0 and p_0 take on discrete imaginary values, and where E_1, E_2 are the on-shell energies associated with the propagators of lines 1 and 2, when written as in (2). With the use of Eq. (1) one picks up both poles of k_n^0 to obtain

$$I = \frac{s_1s_2}{2}\frac{\coth\frac{\beta}{2}s_1E_1 - \coth\frac{\beta}{2}(p_0 - s_2E_2)}{p_0 - s_1E_1 - s_2E_2}\ . \quad (5)$$

One now has to analytically continue the form towards the real values of p_0, in order to obtain the real part of the self-energy.

Before the analytic continuation, one substitutes $\beta p_0 = 2i\pi r$ in the numerator so that p_0 drops out of the argument of the coth. That substitution rule is common to all

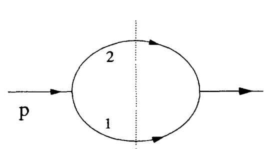

FIG. 1. A two-particle intermediate state.

methods of performing the discrete sum in the imaginary-time formalism [9–13,6]. One then recovers the familiar form $M_{s_1s_2}$ for the numerator [14,13]. Indeed

$$\frac{1}{2}\coth\frac{\beta}{2}s_iE_i = \frac{s_i}{2}\coth\frac{\beta}{2}E_i = s_i[n(E_i) + \tfrac{1}{2}]\ , \quad (6)$$

$$f_{s_i} = \frac{s_i}{2}\left(1 + \coth\frac{\beta}{2}s_iE_i\right)\ , \quad s_i = \pm1\ , \quad (7)$$

$$M_{s_1s_2\cdots s_n} = \prod_i f_{s_i} - \prod_i f_{-s_i}\ , \quad (8)$$

where $n(E)$ is the Bose-Einstein factor, f_+ and f_- the weights associated with the emission and absorption of a Bose particle in a thermal bath, and $M_{s_1s_2\cdots s_n}$ the statistical weight associated with an n-particle intermediate state in the bath.

In a Φ^4 theory, the next-to-lowest contribution to the self-energy is a two-loop diagram. With a straightforward generalization of (3) to this two-loop case, the summation to be performed is

$$I' = T^2\sum_{n,m}\frac{s_1}{(k_n^0 - s_1E_1)}\frac{s_2}{(l_m^0 - s_2E_2)} \times\frac{s_3}{(p_0 - k_n^0 - l_m^0 - s_3E_3)}\ , \quad (9)$$

where k_n^0, l_m^0, p_0 take on discrete imaginary values. For fixed m, one performs the sum over n with the use of (1), and one obtains

$$I' = -T\sum_m \frac{s_1s_2s_3}{l_m^0 - s_2E_2} \times\frac{1}{2}\frac{\coth\frac{\beta}{2}s_1E_1 - \coth\frac{\beta}{2}(p_0 - l_m^0 - s_3E_3)}{p_0 - l_m^0 - s_1E_1 - s_3E_3}\ . \quad (10)$$

Here, before considering the summation over m, one substitutes $\beta l_m^0 = 2i\pi m$, $\beta p_0 = 2i\pi r$ in the numerator, so that both p_0 and l_m^0 drop out of the argument of the coth. One then performs the sum over m, picking up both poles of l_m^0, with the result

$$I' = \frac{s_1s_2s_3}{4}\left(\coth\frac{\beta}{2}s_1E_1 + \coth\frac{\beta}{2}s_3E_3\right) \times\frac{[\coth\frac{\beta}{2}s_2E_2 - \coth\frac{\beta}{2}(p_0 - s_1E_1 - s_3E_3)]}{p_0 - s_1E_1 - s_2E_2 - s_3E_3}\ . \quad (11)$$

Dropping p_0 out of the coth, one obtains

$$I' = \frac{s_1s_2s_3}{4}\frac{1}{p_0 - s_1E_1 - s_2E_2 - s_3E_3} \times\left(\coth\frac{\beta}{2}s_1E_1\coth\frac{\beta}{2}s_2E_2 + \coth\frac{\beta}{2}s_2E_2\coth\frac{\beta}{2}s_3E_3 + \coth\frac{\beta}{2}s_1E_1\coth\frac{\beta}{2}s_3E_3 + 1\right)\ . \quad (12)$$

With (7) and (8), one recognizes in the numerator the statistical weight $M_{s_1 s_2 s_3}$ associated with a three-particle intermediate state. Note that the prescription, to drop out the terms in the coth, gives a result independent of the choice of the loop variable. Also the substitution in (10) introduces a pole in l_m^0, which is picked up at the next step.

Those features are general for any number of loops. One performs the summations in consecutive order. The strategic step is from (10) to (11). At each loop summation, the function $f(z)$ of (1) is a sum of terms. In each term, there is a one-to-one correspondence between the poles and the internal lines of one loop. Also the choice for the order of the summations is irrelevant, as the symmetric result, to be described later on, shows. The summations for a three-loop amplitude are performed in Sec. II of the Appendix.

For any number of loops, the *prescription* is (i) perform the sums in consecutive orders and (ii) at each step, drop out of the argument of the coth any possible term before considering any other summation. There are alternative methods for the summation over the energy running around a loop: (i) the Fourier transform of the propagator with respect to the energy variable [11,12]; (ii) the analytical continuation of the Kronecker δ function in the energy variable [2,9,10]. We have checked that the coth method gives the same answer as those two methods for the following diagrams in a Φ^3 theory: the two-loop contribution to the 2-point and 3-point functions, the three-loop contribution to the free energy. The algebra is much more involved in those other methods. Another easy comparison is the lowest contribution to the self-energy in a Φ^m theory for $m = 5, 6, \ldots$.

One central feature of the resulting forms is that the denominators are those of $T = 0$ tree diagrams. Indeed to perform the summation over the energy running around a loop is to pick up a pole of the integrand. A pole of a Feynman diagram corresponds to an internal line put on shell. For an l-loop diagram, to perform all the summations is to pick up successively l poles, i.e., to put l internal lines on shell; therefore, the resulting denominator is associated with a tree diagram. For example, for the diagram of Fig. 1, one tree corresponds to line 1 on shell and the denominator of form (5) is line 2's propagator (the numerator is then $\coth\frac{\beta}{2}s_1E_1$). In the other tree, the roles of lines 1 and 2 are interchanged. Similarly in form (12), two among the lines 1, 2, and 3 are on shell, and the denominator is the third one's propagator.

It follows from (12) a relation that allows one to write down an alternative form for any two-loop amplitude:

$$T^2\sum_{n,m}\prod_i\frac{1}{k_n^0-a_i}\prod_j\frac{1}{l_m^0-b_j}\prod_k\frac{1}{k_n^0+l_m^0+c_k}$$

$$=\sum_{i,j,k}\prod_{i'\neq i}\frac{1}{a_i-a_{i'}}\prod_{j'\neq j}\frac{1}{b_j-b_{j'}}\prod_{k'\neq k}\frac{1}{c_{k'}-c_k}\frac{1}{4}\left(\frac{Cta_iCtb_j+Cta_iCtc_k+Ctb_jCtc_k+1}{a_i+b_j+c_k}\right), \quad (13)$$

where $Cta_i = \coth\frac{\beta}{2}a_i$. To go from (12) to (13), one makes use of the expansion

$$\prod_i\frac{1}{k_n^0-a_i}=\sum_i\frac{1}{k_n^0-a_i}\prod_{i'\neq i}\frac{1}{a_i-a_{i'}}\,. \quad (14)$$

Another useful property follows. The function $f(z)$, to be used in (1), is a meromorphic function, decreasing faster than $|z|^{-1}$ at infinity, and the sum of the residues of $f(z)$ is zero, as follows from considering a contour integral of $f(z)$ along a large circle enclosing all the poles.

A similar method may be used for the fermion case, where the numerator in (2) has an extra factor $(\gamma_0 sE_k - \gamma\cdot\mathbf{k}+m)$, independent of k_0, and for example, tanh functions replace the coth ones, in the statistical weight associated with a two or three fermions intermediate state.

B. Results

We discuss now the resulting forms for the amplitudes in a Φ^3 theory. Examples are given in Sec. 1 of the Appendix for one, two, three, and four loops.

For the one-loop amplitudes, one has to pick up successively each pole of $f(z)$ in (1). As a result, the amplitude is written as a sum of terms [15,13]. To each internal line i $(k_i^0, \mathbf{k}_i)$, is associated a term whose numerator is $\coth\frac{\beta}{2}s_iE_i$ and whose denominator is given by the diagram's residue at the pole $k_i^0 = s_iE_i = s_i(\mathbf{k}_i^2+m^2)^{1/2}$. In other terms, the factor multiplying $\coth\frac{\beta}{2}s_iE_i$ is the $T = 0$ tree diagram obtained by cutting line i and giving it the four-momentum $(s_iE_i, \mathbf{k}_i)$ (see the explicit form of the one-loop vertex in the Appendix).

For the two-loop amplitudes, there will appear in the numerators a product of two coth and, possibly, a term 1, as in (12). The denominator associated with a factor $\coth\frac{\beta}{2}s_aE_a\ \coth\frac{\beta}{2}s_bE_b$ is given by the $T = 0$ tree diagram obtained from the original diagram by cutting lines a and b and giving them, respectively, the energy s_aE_a and s_bE_b. Indeed the presence of both coth means that one has picked up the residues of both poles. One has to sum over the terms corresponding to all ways of cutting two lines of the diagram so that a connected tree diagram is obtained. In addition, there are terms whose numerator is 1. They arise from lines which belong to both loops, as in (12), and are associated with a possible three-particle intermediate state of the diagram. They are obtained by modifying the numerators of the trees where one common line is cut, according to the following rule.

Rule for the ϵ_{ij} terms in two-loop diagrams. Let i be the momenta belonging to only one loop, j those belonging to the other loop, and k the momenta common to both loops. The substitution is, for all j and k, $\coth\frac{\beta}{2}s_jE_j \ \coth\frac{\beta}{2}s_kE_k$ is replaced by $(\coth\frac{\beta}{2}s_jE_j \ \coth\frac{\beta}{2}s_kE_k + \epsilon_{jk})$ ($\epsilon_{jk} = -1$ if j and k have the same orientation in the loop, $+1$ otherwise), and no change is made for all $\coth\frac{\beta}{2}s_iE_i \ \coth\frac{\beta}{2}s_kE_k$. Alternatively, one can make the substitution for all i, k and no change for all j, k . Both methods give the same total result; indeed, when line k is cut, one is left with a one-loop diagram, and the sum of the residues of the function $f(z)$ in (1) is zero for that loop. As an example, the integrand corresponding to the diagram of Fig. 2(a) is written down in the Appendix. To summarize, there are alternate choices for those class of trees, where a term 1 appears in addition to the product of two coth, in the numerator. Those alternate forms give the same total result.

At the three-loop level, numerators are a product of three coth, plus, possibly, terms with a single coth. The denominator associated with a three coth term is given by the tree diagram obtained by cutting those three lines. For one single coth, that line is cut, the resulting diagram is a tree with a two-loop diagram inserted in it, and one modifies the numerator of the associated trees according to the rule just given for this two-loop diagram. (See two examples in the Appendix.)

For a four-loop diagram, the trees are obtained by cutting four lines. One then cuts two of those lines, a two-loop diagram is obtained where one modifies the trees' numerator according to the rule for the ϵ_{ij} terms for this two-loop diagram. The leftover terms, with no coth, involve a few trees, they are obtained in the following way.

Rule for the $\epsilon_{ij}\epsilon_{kl}$ terms in four-loop diagrams. If the diagram possesses two loops which have no common line, add the term $\epsilon_{ij}\epsilon_{kl}$ to obtain $(\coth i \ \coth j + \epsilon_{ij})(\coth k \ \coth l + \epsilon_{kl})$ in the numerator of the relevant tree, where lines i, j belong to one of those loops, and lines k, l to the other one. Sum over all the i, j and k, l that are involved in the substitution rule for two loops. If the diagram does not have such pair of loops, one considers the five-particle intermediate states. The modification occurs in the related trees where three lines, common to two loops, are cut. For example, for the free energy, there are as many such new terms as there are ways of

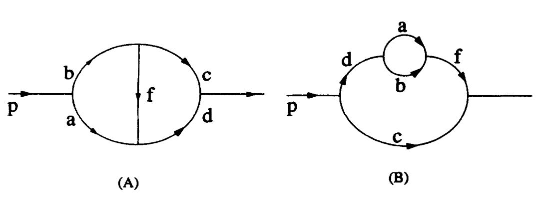

FIG. 2. Two-loop diagrams (A) and (B) for the self-energy.

cutting the diagram to obtain a five-particle intermediate state. (An example is in the Appendix.)

A new rule will be needed at the six-loop order; a general substitution rule probably exists to all orders; we have not found it.

In practice, the quickest way to obtain the form, is to compute the diagram, i.e., to perform the summations with the coth method, and, at the same time, to interpret each term as a tree, or a piece of it. Indeed, the trees organize themselves naturally into regular patterns. Before each loop summation, the merging of some terms is fairly apparent. The rules are useful to organize the many terms of the result, to check the systematics of the terms which have a definite power of coth, and to write down alternative forms for that class of terms.

An alternative form may be written that emphasizes the connection, at a given loop order, between the diagrams contributing to the free energy and those for an N-point function. That connection is of importance to two aspects, to be examined in Sec. III: (i) the possible multiparticle intermediate states, which control the ϵ_{ij} terms just discussed, (ii) the analytic properties of the amplitude. For example, for any two-loop amplitude, one may make use of Eq. (13) where a_i, b_j, and c_k contain both external and internal energies. For the diagram contributing to the free energy, i, j, k take one value each, and when the denominator vanishes in (13), so does the numerator; therefore, the form is regular, an expected feature for the free energy, a real quantity. For an N-point amplitude, the external energies only enter the denominator factors. In some terms those energies will not appear in the sum $a_i + b_j + c_k$, and the form will be regular when that sum vanishes. The case of a double pole, i.e., $a_i = a_{i'}$ in (13), is easily handled. A two-loop example is in the Appendix.

More generally, at a given-loop order, that form emphasizes that the basic loop pattern is given by the diagrams contributing to the free energy. The result of sticking external legs to the internal lines may then be obtained with the use of the partial fraction expansion (14). As a consequence, that alternate form has the following feature: it groups the terms in a way such that whenever a denominator involving only internal energies vanishes, so does the numerator in an explicit fashion.

We summarize the results of the section. For an l-loop diagram at temperature T, the summation over the energy running around each loop can be performed easily. One is left with the integration over the space components of the internal momenta. The integrand is written as the sum over all possible connected tree diagrams obtained by cutting l internal lines, and attributing the on-shell energy s_iE_i to a cut internal line i ($s_i = \pm 1$). For each tree (i) The $T = 0$ tree diagram gives the denominator. (ii) The numerator is an (even or odd) lth degree polynomial in $\frac{1}{2}\coth\frac{\beta}{2}s_iE_i = [n(E_i) + \frac{1}{2}]s_i$. The highest term is the product of the coth for the l cut lines. For the two next terms, a rule has been given, with multiple equivalent forms.

(iii) The whole T dependence is in the numerator.

(iv) The external energies p_i^0 only appear in the denominator, which is a product of factors that are linear in the p_i^0 variables. Those are initially discrete imaginary, and may be analytically continued towards real values.

(v) All possible cases for the signs s_m of the internal lines' energies are separate, as the propagator is written as in (2). There is a factor

$$\frac{1}{2^L}\prod_m \frac{s_m}{2E_m}$$

where m runs over all internal lines and L is the number of loops.

(vi) The sum over the loops' space momenta and over the signs s_m is made at the end.

That expansion in terms of $\frac{1}{2}\coth\frac{\beta}{2}E = n(E)+\frac{1}{2}$ has similarities with the expansion of the amplitudes in terms of $n(E)$ and of the $T=0$ scattering amplitudes [16,17]. It is much more concise.

III. SUM OVER INTERMEDIATE STATES

The forms obtained in the preceding section are well suited to study the analytic structure of the amplitude associated with a diagram at $T\neq 0$. Indeed, a cut of the integral arises from a pole of the integrand. At $T=0$ the analytic structure of the amplitude is expressed in terms of Lorentz invariants. At $T\neq 0$, a condition on a Lorentz invariant translates into a condition on the energy component of the Lorentz invariant (in the plasma rest frame), i.e., a condition involving the external energy variables. The Landau equations that give the location of the possible singularities are the same at $T=0$ and $T\neq 0$ [18]; however, only some of those are present on the physical sheet at $T=0$.

In this section, it will be shown that the full amplitude associated with a diagram may be written as a sum of terms, each one corresponding to a possible intermediate state of the diagram. Each term factorizes into three parts, one factor describes the intermediate state. The other ones are the amplitudes associated with the two pieces of the diagram, sitting on each side of the cutting plane, as computed in the previous section. That expansion emphasizes the similarity of the analytic structure for the $T=0$ and $T\neq 0$ cases. The intermediate states are a simple extension of the familiar ones at $T=0$, as all possible signs for the intermediate energies are allowed.

A new tool will be needed: the real part of the *integrand associated with* an n-particle *intermediate state* in a plasma. The statistical weight $M_{s_1s_2\cdots s_n}$ associated with that state enters.

The integrand may be obtained from the lowest-order contribution to the real part of the self-energy in a Φ^{n+1} theory and the generalization of Eqs. (3) and (5) is

$$G(p_0,p)=\int\prod_l\frac{d^3k_l}{(2\pi)^3}\left(\prod_i\frac{1}{2E_i}\right)\sum_{\text{all } s_i}\frac{M_{s_1s_2\cdots s_n}}{p_o-\sum_i s_iE_i} \tag{15}$$

with $M_{s_1s_2\cdots s_n}$ given by (8). For our purpose, p_0 will be replaced by the sum of the external energies situated on one side of the intermediate state.

In the following, it will be convenient to think in the following way, the particles appearing in the intermediate state are on mass shell; i.e., the particle i with space momentum $\mathbf{k}_i$ has the energy $s_iE_i=s_i(\mathbf{k}_i^2+m^2)^{1/2}$, and the line i will be called as usual a *cut line*; however, the incoming energy p_0 is not equal to the energy $\sum_i s_iE_i$ that we now associate with the intermediate state. Indeed the quantity $(p_0-\sum_i s_iE_i)$ precisely appears in the denominator of form (15).

We shall need

$$M_{s_1s_2}=\frac{s_1s_2}{2}\left(\coth\frac{\beta}{2}s_1E_1+\coth\frac{\beta}{2}s_2E_2\right),\; M_{s_1s_2s_3}=\frac{s_1s_2s_3}{4}\left(\coth\frac{\beta}{2}s_1E_1\coth\frac{\beta}{2}s_2E_2+\text{perm}+1\right),$$
$$M_{s_1s_2s_3s_4}=\frac{s_1s_2s_3s_4}{8}\left(\coth\frac{\beta}{2}s_1E_1\coth\frac{\beta}{2}s_2E_2\coth\frac{\beta}{2}s_3E_3+\text{perm}+\coth\frac{\beta}{2}s_1E_1+\text{perm}\right). \tag{16}$$

The $T=0$ limits may be written

$$M_{s_1s_2}=s_1\delta_{s_1s_2}\;,\quad M_{s_1s_2s_3}=s_1\delta_{s_1s_2}\delta_{s_1s_3}\;,\dots . \tag{17}$$

Care must be given to signs: (i) if the orientation of line 2 is reversed in Fig. 1, s_2 is changed into $-s_2$ in the integrand of Eq. (15), both in the denominator and in $M_{s_1s_2\cdots s_n}$; (ii) the denominator of the integrand in (15) is the sum of the energies entering the vertex situated at the left side of the cut (the rule is invariant by a rotation of π of the diagram).

The other building block is the propagator, as written in (2):

$$\frac{1}{k_0^2-\mathbf{k}^2-m^2}=\frac{1}{2E_k}\sum_{s=\pm1}\frac{s}{k_0-sE_k}\,.$$

Once the energy running around the loop has been integrated over, k_0 has a precise value to be specified later on. That value will be referred to as the *energy flowing through the propagator* and the propagator will be called an *uncut propagator* (or unthermalized propagator). As explained further on in this section, the flow will be oriented and the energy flowing through a propagator will be the *incoming flow*.

In Sec. III A, we will concentrate on the self-energy, whose properties at $T\neq 0$ are well understood. The expansion in terms of intermediate states will be obtained and the resulting properties, factorization and analyticity, will be exhibited.

A. Self-energy

We now show how the expansion in terms of intermediate states amounts to an expansion, in terms of simple poles of the incoming energy p_0, of the forms previously obtained. Consider for example the diagram of Fig. 2(a), written as a sum of trees in Sec. 1 of the Appendix. A factor $(p_0 - s_aE_a - s_bE_b)^{-1}$ can only appear in two types of trees: either line a is cut and it is line b's propagator, or line b is cut and it is line a's propagator. If one performs an expansion in terms of simple poles in p_0 , one has to set $p_0 - s_aE_a - s_bE_b = 0$ in the remaining factors associated with those trees; i.e., one puts an extra internal line on shell. As a result, the trees are split into two pieces, where both lines a and b are on shell. Those pieces are multiplied by

$$s_a s_b \frac{1}{2} \frac{\coth \frac{\beta}{2} s_a E_a + \coth \frac{\beta}{2} s_b E_b}{p_0 - s_aE_a - s_bE_b} = \frac{M_{s_a s_b}}{p_0 - s_aE_a - s_bE_b} ,$$

where $\frac{1}{2}\coth \frac{\beta}{2} s_aE_a$ comes from the trees where line a is cut, and $\frac{1}{2}\coth\frac{\beta}{2} s_bE_b$ from the trees where line b is cut. Summing over all the relevant trees, the pieces (with both lines a and b on shell) build up, on each side of the cutting plane, a 3-point Green function with external legs p_0, a, b. For the case of Fig. 2(a), one is treelike, the other is one loop. Those functions are expressed so that p_0 does not appear.

The argument generalizes to any two-particle intermediate state of any diagram contributing to the self-energy, at any loop order. The internal lines a and b must belong to the same loop and to no other loop. The two 3-point Green functions (p_0, a, b) are expressed as a sum of trees, whose numerators are those needed.

Similarly, considering again the diagram of Fig. 2(a), a factor $(p_0 - s_aE_a - s_fE_f - s_cE_c)^{-1}$ appears in the trees where two among the three lines a, f, and c are cut, and it is the third one's propagator. To pick up the residue of that pole splits those trees into two pieces where the three lines are on shell. Consequently those pieces are multiplied by $M_{s_a s_f s_c}(p_0 - s_aE_a - s_fE_f - s_cE_c)^{-1}$ with $M_{s_a s_f s_c}$ given by (16), where the extra term 1 in the numerator comes from the tree where line f (common to both loops) is cut and, say, line a as explained in Sec. II. Again the argument holds for the three-particle intermediate state a, f, c of any loop diagram. Such a state involves two nearby loops, with one line common to both loops and one line from each of those loops (not common to any other loop). When one sums over all the relevant trees and one puts lines a, f, c on shell, one builds up a 4-point Green function (p_0, a, f, c) on each side of the cutting plane, which is expressed so that p_0 does not appear. A three-loop example is in Sec. 1 of the Appendix.

We summarize the results that are of general validity. If one integrates over all the energies running around the loops, the amplitude is the sum over all possible trees made from the diagram. Under this form, an expansion in terms of intermediate states is an expansion into simple poles of the incoming energy p_0. If one is interested in an n-particle intermediate state $q_1, \ldots, q_n$ of the diagram, one selects the trees that contain the pole $(p_0 - \sum_{i=1}^{n} s_iE_i)^{-1}$. They are the trees where $n-1$ lines, among the n lines $q_1, \ldots, q_n$ are cut, i.e., on shell, the uncut line q_j carries the energy $p_0 - \sum_{i\neq j} s_iE_i$ and the looked-for pole is line q_j's propagator. To pick up the residue of the pole in the remaining factors of each tree amounts to splitting the tree into two pieces where the n particles are on shell. The strategic properties are (i) only a subset of trees contribute to an n-particle intermediate state and (ii) the n particles are on-shell in the residue of the pole.

With this result from the tree expansion, one is able to make a general proof, along the lines of the Appendix of Weldon's original paper [14]. One considers an l-loop diagram and the cutting plane that goes through the lines $q_1, \ldots, q_n$. One wants to isolate all the terms relevant to the n-particle intermediate state $q_1, \ldots, q_n$. With a parallel orientation of these momenta, the product of the factors associated with the n internal lines is

$$\frac{s_1 \ldots s_n}{(q_1^0 - s_1E_1)(q_2^0 - s_2E_2)\cdots(q_{n-1}^0 - s_{n-1}E_{n-1})(p_0 - \sum_{i\neq n} q_i^0 - s_nE_n)} .$$

First, one sums over the imaginary discrete energy q_1^0, which runs around some loop; this summation is made with the coth method of Sec. II, and one picks only the two poles corresponding to line q_1 and to line q_n. Then, one sums over q_2^0 and picks up only the two poles corresponding to line q_2 and line q_n (or q_1). One similarly performs the $n-1$ summations, leaving the other $(l-n+1)$ summations undone. The pole $(p_0 - \sum_{i=1}^{n} s_iE_i)^{-1}$ is now obtained as the common factor of 2^{n-1} terms. One computes the residue by setting on-shell the n particles q_i, $q_i^0 = s_iE_i \;\; i = 1, \ldots, n$ in the full factor that multiplies the pole.

One resulting factor is the product of the denominators from the remaining internal lines of the diagram, with $(l-n+1)$ undone loops. This factor splits into two pieces: one is the $(n+1)$-point Green function associated with the diagram piece on the left side of the cutting plane $G^A(p; q_1 \ldots q_n)$; the other one is associated with the right side $G^B(q_1 \ldots q_n; p)$; in both Green functions the n particles are on shell. The other factor collects the 2^{n-1} numerators resulting from the $n-1$ summations; it is

$$\frac{s_1 \cdots s_n}{2^{n-1}}(Cts_1E_1 + Cts_nE_n)[Cts_2E_2 + Ct(s_1E_1 + s_nE_n)] \cdots \left[Cts_{n-1}E_{n-1} + Ct\Big(\sum_{i\neq n-1} s_iE_i\Big)\right]$$

where Ct is for coth $\frac{\beta}{2}$; it may be written in a symmetric way and is $M_{s_1 \cdots s_n}$ defined in Eqs. (8), (16).

To summarize, the amplitude associated to a diagram contributing to the self-energy, may be written as a sum of terms. Each term corresponds to a possible intermediate state of the diagram, and the sum is over all possible intermediate states. To the intermediate state $q_1, \ldots, q_n$ is associated the term

$$G^{(2)}(p_0,p) = \sum_{s_i} \int \prod_{i=1}^{n} \left(\frac{d^3q_i}{(2\pi)^3}\frac{1}{2E_i}\right) \; G^A(p;q_1\ldots q_n) \; \frac{M_{s_1\ldots s_n}}{p_0 - \sum_i s_iE_i} \; G^B(q_1\ldots q_n;p) \tag{18}$$

where, in the Green funtions G^A and G^B, $q_i^0 = s_iE_i$ $i = 1, \ldots, n$, and $p_0 = \sum_i s_iE_i$. That property was known for the imaginary part [19]; it is true for the real part.

As an *example*, the forms for the two-loop diagrams, shown in Fig. 2, will be written down. In each term, the pieces on each side of the cutting plane involve treelike propagators, written as in (2), and possibly, a one-loop amplitude. An easy way to take into account that p_0 should not appear in those pieces is to define, in each piece, an oriented energy flow. It flows from the cut lines toward the external vertex, through the treelike propagators and into the loop amplitude. This prescription is valid at any loop order. The flow specifies which energies are entering the multiloop amplitude, i.e., which are the relevant energy variables, but this loop amplitude is a black box for the flow.

We use a simplified notation

$$s_aE_a = a \;\; s_bE_b = b \; . \tag{19}$$

For the diagram shown in Fig. 2(a), the integrand is the sum of four terms

$$\text{cut}(afc) + \text{cut}(bfd) + \text{cut}(ab) + \text{cut}(cd) \; .$$

Each cut divides the diagram into two pieces, where the energy flows towards the external vertex. Two terms correspond to a three-particle intermediate state; the two other terms correspond to a two-particle intermediate state multiplied by the one-loop vertex:

$$G^{(2)}(p_0,p) = \int \frac{d^3k}{(2\pi)^3}\frac{d^3l}{(2\pi)^3} \prod_i \frac{1}{2E_i} \sum_{s_i} D \tag{20}$$

with $i = a, b, c, d, f$ and

$$\begin{aligned} D^{(A)} = {} & \frac{M_{s_cs_as_f}}{p_0 - a - c - f}\frac{s_b}{c + f - b}\frac{s_d}{a + f - d} + \frac{M_{s_bs_d-s_f}}{p_0 - b - d + f}\frac{s_a}{d - f - a}\frac{s_c}{b - f - c} \\ & + \frac{M_{s_as_b}}{p_0 - a - b}V_{p_1^0=b,p_2^0=-a;s_fs_c-s_d} + \frac{M_{s_cs_d}}{p_0 - c - d}V_{p_1^0=-d,p_2^0=c;-s_f-s_as_b} \; , \end{aligned} \tag{21}$$

where $V_{p_1^0p_2^0;s_as_bs_c}$ is the one-loop vertex written in Eq. (A1) of the Appendix. The energies entering the vertex are the cut lines' ones.

The integrand corresponding to the diagram shown in Fig. 2(b) is the sum over the three possible intermediate states

$$\text{cut}(abc) + \text{cut}(cd) + \text{cut}(cf) \; .$$

In the two last cuts, a one-loop amplitude shows up, and the energy flow, as defined at the two-loop level, specifies that the energy incoming the one-loop amplitude is the cut line's one, but is irrelevant to the description of the interior of that loop:

$$\begin{aligned} D^{(B)} = {} & \frac{M_{s_as_bs_c}}{p_0 - a - b - c}\frac{s_d}{a + b - d}\frac{s_f}{a + b - f} \\ & + \frac{M_{s_ds_c}}{p_0 - d - c}\frac{s_f}{d - f}\frac{M_{s_as_b}}{d - a - b} \\ & + \frac{M_{s_fs_c}}{p_0 - f - c}\frac{s_d}{f - d}\frac{M_{s_as_b}}{f - a - b} \; . \end{aligned} \tag{22}$$

We now point out the remarkable analytic properties of the resulting forms, in connection with the *Cutkosky rules*. An important property follows. The total integrand, such as $D^{(A)}$ or $D^{(B)}$, is regular when any denominator involving only internal lines vanishes. The reason is that the integrand must be regular for any $\beta p_0 = n2i\pi$, i.e., when the diagram is fully Euclidean in the imaginary-time formalism. For example, consider the pole $a + b - d = 0$ in the form (22); its residue is a meromorphic function of the complex variable p_0 that must vanish for $\beta p_0 = n2i\pi$; it vanishes everywhere as that function decreases faster than $|p_0|^{-1}$ for large $|p_0|$ (it does along the imaginary axis). The same argument applies to any denominator involving only internal lines, for any diagram. An alternative argument is that most of those denominators arise from the partial fraction expansion, and the remaining ones are present in the corresponding form for the free energy (see the remark at the end of Sec. II B). The absence of the pole $a + b - d = 0$ in (22) is easily checked with the use of forms (16) in the numerators, and for the pole $f - d = 0$ the form with a derivative [19] is recovered.

Consequently, the only poles of the integrand are those involving p_0, and they are associated with the intermediate states. The integrand does not inherit from the singularities of the pieces on each side of the intermediate state. Now the cuts of the amplitude, as given by (20), arise from the poles of the integrand D. The discontinuity across a cut [20] is the difference between the two ways of avoiding the pole by distorting the integration contour, as it is summarized in

$$\frac{1}{p_0 - a \pm i\epsilon} = \mathrm{PP}\frac{1}{p_0 - a} \mp i\pi\delta(p_0 - a)\,. \tag{23}$$

Thus, from the amplitude's real part, we obtain its imaginary part. And we find a very simple generalization to the case $T \neq 0$ of the perturbative unitarity relation [21]. The discontinuity across a cut associated with an n-particle intermediate state involves the statistical weight associated with that state, multiplied by two $(n+1)$-point Green functions. One does not need a $\pm i\epsilon$ prescription to the internal lines of those Green functions, because, as just said, the whole integrand does not have those poles. So, provided one computes, for one value of p_0, the full imaginary part of the self-energy, i.e., the discontinuity across all cuts sitting at that p_0 value, the $\pm i\epsilon$ are unnecessary. The Cutkosky rules at $T = 0$ [21] or at $T \neq 0$ [22] are more precise, as they provide the discontinuity across each cut separately, and there, a $\pm i\epsilon$ prescription to the internal lines is necessary. For example, for the diagram shown in Fig. 2(a), if one considers the discontinuity across the cut (ab), the one-loop vertex $(a, b, p_{0,\mathrm{out}})$ has a two-particle cut, identified by a factor $(a + b - c - d)^{-1}$ in (21) and (A1); however, if one sums over the discontinuities across the cut (ab) and the cut (cd), those vertex' singularities cancel.

To conclude, for any diagram contributing to the self-energy, the expansion in terms of n-particle intermediate states provide an easy way to obtain both the real and the imaginary part of the amplitude. There enter the statistical weight associated with the intermediate state, and the $(n + 1)$-point thermal Green functions. These functions are obtained from the imaginary-time formalism by a straightforward analytical continuation of the external energies.

B. N-point Green functions

We now want to show how the properties, just found for the self-energy, generalize to any N-point Green function, as obtained in Sec. II. For simplicity, we shall restrict ourselves to planar diagrams. For these, the expansion in terms of intermediate states is an expansion in terms of a particular set of variables, well known at $T = 0$, the multiperipheral variables. For an N-point amplitude, with external momenta $p_1, p_2, p_3, \ldots$ all incoming, one such set of $N - 1$ variables is $p_1,\ p_1 + p_2,\ p_1 + p_2 + p_3, \ldots$; they are conveniently drawn from a polygon [23] whose sides are made from the external momenta, see Fig. 3(a). All other combinations of momenta

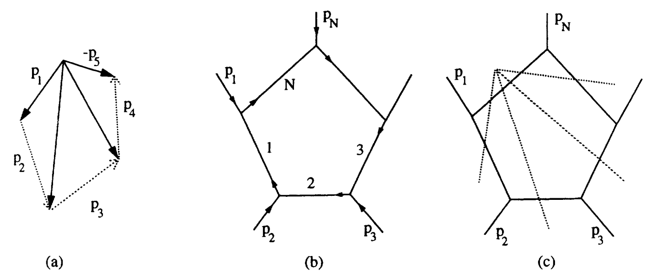

FIG. 3. N-point amplitudes. (a) The polygon and a set of multiperipheral variables. (b) One-loop amplitude. (c) The set of cutting planes associated to the set of variables drawn in (a).

intersect the drawn lines, and are called dual variables $p_2,\ p_2 + p_3, \ldots$. There are N equivalent sets of such variables. The analytical properties, written at $T = 0$ in terms of a Lorentz invariant $(p_1 + p_2 + \cdots + p_i)^2$, are expressed, in the plasma rest frame, in terms of its energy component $p_1^0 + p_2^0 + \cdots + p_i^0$.

We shall first show how those variables occur in the one-loop Green function, written in Sec. II as a sum of N trees. Consider the tree where the internal line on-shell is line N, with energy $s_N E_N$ [see Fig. 3(b)]. The associated factor depends on the set of multiperipheral variables $p_1^0,\ p_1^0 + p_2^0, \ldots,\ p_1^0 + p_2^0 + \cdots + p_{N-1}^0$, to be called set 1. The integrand has a pole in each of those variables, associated with the tree's internal lines. Similarly, each other tree diagram depends on a set of multiperipheral variables, but only one of them belongs to the set 1 [see the polygon in Fig. 3(a)], the other ones belong to the set of variables dual to the set 1. In order to write the full amplitude's integrand as a sum of terms, each one having a pole in only one of the variables of set 1, one just has to write the first tree diagram (whose numerator is $\frac{1}{2}\coth\frac{\beta}{2}s_N E_N$) as the sum of the other $(N - 1)$ $T = 0$ trees. The resulting form is the sum over $(N - 1)$ terms, each corresponding to a possible intermediate state of the diagram. For example, the tree where the internal line i is cut [see Fig. 3(b)] has now the numerator $\frac{1}{2}s_i s_N(\coth\frac{\beta}{2}s_i E_i - \coth\frac{\beta}{2}s_N E_N) = M_{-s_i, s_N}$, and the propagator of the internal line N is $(p_1^0 + p_2^0 + \cdots + p_i^0 + s_i E_i - s_N E_N)^{-1}$. The product of those two factors is the factor associated with the intermediate state (i N) with the following interpretation. The cutting plane goes through the lines i and N, and the total external energy on one side of the cutting plane is $p_1^0 + p_2^0 + \cdots + p_i^0$, while the intermediate state's energy is $s_N E_N - s_i E_i$. The factors associated with the other internal lines' propagators build up, on each side of the cutting plane, two $T = 0$ treelike amplitudes, which are expressed in terms of dual variables.

To summarize, the one-loop N-point amplitude may be written as a sum of $(N - 1)$ terms, each one associated with a possible intermediate state of the diagram. Each term factorizes into a factor associated with the two-particle intermediate state, and into two treelike amplitudes. The location of the $(N-1)$ cutting planes associated with the set 1 is shown in Fig. 3(c) . There exist N alternative expansions of that amplitude, one for each set of multiperipheral variables.

In a multiloop planar diagram, external vertices are ordered. Such a diagram may be obtained from the one-loop diagram by adding lines that join internal vertices. According to the rules of Sec. II, the denominators of the amplitude's integrand are those of all possible tree diagrams. Those trees depend on the external variables in the manner of the one-loop case; i.e., the multiperipheral variables are the relevant ones. It is easy to perform the expansion in terms of simple poles in each variable belonging to set 1, i.e., $p_1^0,\ p_1^0+p_2^0, \ldots,\ p_1^0+p_2^0+\cdots+p_{N-1}^0$. Indeed, it may be considered as an expansion in terms of simple poles in p_1^0 (the dual variables are expressed independently of p_1^0 and p_N^0).

In Sec. III A, an expansion in simple poles in p_0 was used for the proof on how a specific n-particle intermediate state was built up from the treelike form. That same proof applies to each variable belonging to the set 1. First, one selects the class of trees where there appears the denominator characteristic of the intermediate state. One picks up the residue of this pole and, as a result, each tree is split into two pieces. From those pieces, one builds up two Green functions, one on each side of the cutting plane, expressed so that p_1^0 and p_N^0 do not appear. A practical way to write down those Green functions is to define, on each side, an oriented energy flow, sinking at p_1^0 or p_N^0; it will prevent those variables from entering any propagator or loop amplitude. There are N alternate expansions, in terms of intermediate states, of such an amplitude, one for each set of multiperipheral variables, a remarkable planar duality property.

We now state the *rules* that allow to write down immediately the form for an N-point multiloop, one-particle irreducible, planar diagram, as an expansion in terms of intermediate states.

(1) Select one set of $(N-1)$ multiperipheral variables for the external energies, say $p_1^0,\ p_1^0+p_2^0, \ldots,\ p_1^0+p_2^0+\cdots+p_{N-1}^0$. The definition of the set of $(N-1)$ associated cutting planes is done as in Fig. 3(c). The two external vertices where p_1^0 and p_N^0 are attached are called the energy sinks.

(2) A cutting plane divides the diagram into two pieces. The associated term factorizes into three parts: one factor describes the intermediate state, the other ones are associated with the pieces on each side of the cutting plane.

(3) The factor associated to an n-particle intermediate state is defined in Eq. (15), where the relevant external energy p_0 is the sum of the external energies situated on one side of the cutting plane.

(4) An oriented energy flow is defined in each piece. The energy flows from the cut lines and the external lines towards the energy sink defined in rule 1.

(5) The factor associated with each piece is a one-particle reducible Green function made of treelike parts and of loop amplitudes. The total energy incoming a propagator, written as in (2), is, as usual in a tree diagram, the total energy of the cut and external lines situated on the appropriate side of the propagator.

(6) A loop-amplitude is a lower-loop amplitude, as computed with the same rules. The energies incoming that amplitude are specified by the orientation of the energy flow. The loop amplitude is a black box for that flow.

(7) The sum over all possible intermediate states is obtained by summing over all positions of the internal vertices with respect to the cutting planes, so that the diagram is cut into two, and only two, pieces [21].

(8) The sum over the loops' space momenta and over the signs $s_i=\pm 1$ of the internal lines' energies is made at the end.

We now turn to explicit examples.

The one-loop vertex is shown in Fig. 4(a), with p_1 incoming and p_2 outgoing. One possible set of variables is p_1^0, p_2^0, associated with [cut(ab)+cut(ac)]; the dual variable is $p_1^0-p_2^0$. Another set is $p_1^0, p_1^0-p_2^0$, linked to [cut(ab)+cut(bc)], with dual variable p_2^0 . The third set is $p_2^0, p_1^0-p_2^0$ linked to [cut(ac)+cut(bc)]. With the use of the rules, the three alternative forms are [13]

$$V(p_1^0,\mathbf{p}_1\ ;\ p_2^0,\mathbf{p}_2)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2^3E_aE_bE_c}\sum_{s_a,s_b,s_c}V_{p_1^0p_2^0;s_as_bs_c}\ , \tag{24}$$

$$V_{p_1^0p_2^0;s_as_bs_c}=\frac{1}{p_2^0-p_1^0+s_bE_b-s_cE_c}\left(s_c\frac{M_{s_as_b}}{p_1^0-s_aE_a-s_bE_b}-s_b\frac{M_{s_as_c}}{p_2^0-s_aE_a-s_cE_c}\right), \tag{25}$$

$$=\frac{1}{p_2^0-s_aE_a-s_cE_c}\left(s_c\frac{M_{s_as_b}}{p_1^0-s_aE_a-s_bE_b}+s_a\frac{M_{-s_b\ s_c}}{p_2^0-p_1^0+s_bE_b-s_cE_c}\right), \tag{26}$$

$$=\frac{1}{p_1^0-s_aE_a-s_bE_b}\left(s_b\frac{M_{s_as_c}}{p_2^0-s_aE_a-s_cE_c}+s_a\frac{M_{-s_b\ s_c}}{p_2^0-p_1^0+s_bE_b-s_cE_c}\right). \tag{27}$$

We now look at two-loop diagrams. For the planar diagram shown in Fig. 4(b), the amplitude may be written as the sum of four terms:

$$\text{cut}(ace)+\text{cut}(acf)+\text{cut}(ab)+\text{cut}(ad)\ .$$

The line a is always cut, and links the two energy sinks. In the two-particle intermediate states the full one-loop vertex enters, where the incoming energies are the cut line's one and $p_1^0-p_2^0$. As a result of the rules, the amplitude is written as a sum of simple poles in p_1^0 or p_2^0,

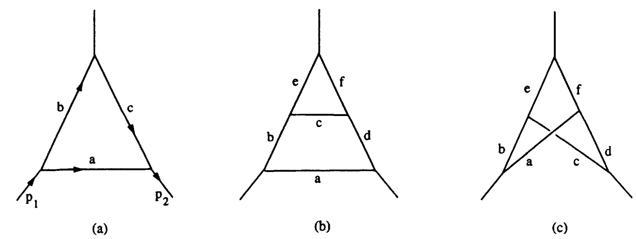

FIG. 4. Diagrams for the vertex. (a) One-loop. (b) Planar two-loop. (c) Non-planar two-loop.

whose factors are functions of $(p_1^0 - p_2^0)$, a form to be compared to the one-loop form (25). Alternatively, the amplitude may be written as the sum over the following intermediate states:

$$\text{cut}(ef) + \text{cut}(bd) + \text{cut}(ecd) + \text{cut}(bcf) + \text{cut}(eca) + \text{cut}(ba)\ .$$

The energy sinks are the vertices with leg p_1 or $p_1 - p_2$. The amplitude is now written as a sum of simple poles in p_1^0 or $p_1^0 - p_2^0$, whose factors are functions of p_2^0.

The nonplanar vertex, shown in Fig. 4(c), may be written as the sum of the cuts

$$\text{cut}(acf) + \text{cut}(ace) + \text{cut}(bdf) + \text{cut}(bde) + \text{cut}(ab) + \text{cut}(cd)\ .$$

Going from (ace) to (bdf) is switching the internal vertices with respect to the cutting plane. The energy sinks are the external vertices with a p_1 or p_2 leg. In the one-loop 4-point amplitude, the entering energies are the cut lines' ones and $p_1^0 - p_2^0$. As a result the amplitude is a sum of simple poles in p_1^0 or p_2^0, whose factors are functions of $p_1^0 - p_2^0$. The two alternate ways of writing the amplitude in terms of intermediate states are similar, because of the symmetry of the vertex with respect to the external vertices (the line joining the internal vertices is the symmetry axis).

Similarly for a planar diagram contributing to the 4-point function (see Fig. 5), the integrand may be written as a sum of simple poles in the variables p_1^0, $p_1^0+p_2^0$, $p_1^0+p_2^0+p_3^0 = -p_4^0$ and the factor multiplying each pole is a function of the two variables p_2^0 and p_3^0. Another set of compatible variables is p_2^0, $p_2^0+p_3^0$, $p_2^0+p_3^0+p_4^0 = -p_1^0$. The familiar singularities in the s channel (t channel) for the on-shell amplitude at $T = 0$, here appear in the $p_1^0 + p_2^0$ variable ($p_2^0 + p_3^0$ variable).

Generalizations may be made for nonplanar diagrams,

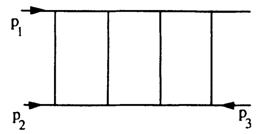

FIG. 5. A three-loop diagram for the 4-point function.

with more complicated rules, as they obey nonplanar duality.

We summarize the result of this Sec. III B. For a planar diagram at $T \neq 0$, contributing to an N-point Green function, a form has been exhibited, where the following remarkable features are explicit.

(i) Unitaritylike property. The full amplitude is written as a sum of terms, each corresponding to a possible intermediate state of the diagram. Such a term factorizes into two Green functions, associated with the two pieces of the diagram, and a factor describing the intermediate state.

(ii) Dualitylike property. There are N alternative expansions in terms of intermediate states.

(iii) $T = 0$-like property. The whole T dependence is in the statistical weight of the intermediate states, either of the amplitude or of lower-loop amplitudes.

In another context, those properties, factorization and duality, are the main ingredients of the dual resonance models [24]. The rules to write down those forms are simple, they make use of the $T = 0$ propagator and of the amplitude associated with an n-particle intermediate state. The demonstration of those properties relies on the treelike form of the integrand, once the energies running around the loops have been summed over. A complete proof has been given for an n-particle intermediate state, $n < 6$, and a partial proof for higher n.

All those properties have been checked by direct computation, and expansion, of the amplitudes associated with diagrams contributing to N-point functions, as far as the four-loop order.

We now comment on the analytic properties of the resulting amplitudes. The expansion, in terms of simple poles, of the integrand allows an easy reading of the analytic structure of the amplitude. The remarks made for the self-energy can be extended to the N-point Green functions. There is no singularity associated with the vanishing of any denominator involving only internal lines; i.e., the amplitude does not inherit from all the singularities of the pieces on each side of the cutting plane. Consider the diagram of Fig. 5 with the choice of variables p_1^0, $p_1^0+p_2^0$, $p_1^0+p_2^0+p_3^0$ and the rules for the energy flow. At $T = 0$, one may obtain from the resulting form the discontinuity across all cuts sitting at the same value of $p_1^0+p_2^0$, for example, and the discussion of the connection with the Cutkosky rules is similar to the self-energy case. At $T \neq 0$, the expansion, in terms of intermediate states, can be done when the external energy variables are discrete imaginary, as it amounts to a partial fraction expansion. Then one may compare various ways of approaching the real values of those variables. The analytic structure of the form is explicit and similar to the $T = 0$ case; one has separately the contribution from each type of cut, as each cut arises from a pole of the integrand. All the complexity of the $T \neq 0$ case is buried in the multiple cuts and in the discontinuity across those cuts. Those explicit forms should be useful to study the connection between those analytically continued N-point Green functions, and those which are defined in the real-time formalism [5,6,7], their real and imaginary parts, and also the relation to physically relevant quantities.

IV. CONCLUSION

For an l-loop diagram at $T \neq 0$, the summation over the energies running around the loops are easily performed in the imaginary-time formalism. One is left with an integral over the space momenta of the internal lines. Two alternative forms for the integrand have been proposed, their common features are the integrand is written as a sum of terms, the T dependence is in the numerators as $[n(E) + \frac{1}{2}]$ factors, and the external energies only appear in the denominator factors in a linear way. Rules have been given to write down the result immediately. The form in terms of intermediate states is unique, in contrast with the form in terms of trees, whose lowest powers of $n(E) + \frac{1}{2}$ have multiple forms.

Two new remarkable features of those imaginary-time N-point Green functions have been exhibited: their factorization property and their T dependence. It can be put entirely into statistical weights associated with intermediate states.

The analytic structure of the amplitude is explicit; the relevant variables are the multiperipheral ones, for planar diagrams. Those forms should be useful for a further investigation of the relationship between those Green functions and those of the retarded-advanced type, for $N \geq 3$.

It is to be noted that any diagram contributing to the self-energy has been written under a form where only those Green functions enter as real quantities.

ACKNOWLEDGMENTS

The author would like to thank M. Bergère, T. Bhattacharya, M. Le Bellac, and D. Poizat, for discussions and R. Pisarski for suggesting a look at the coth method. The nice hospitality of the theory group at Brookhaven, where part of this work was done, is gratefully acknowledged. Institut Non Linéaire de Nice is UMR 129 du CNRS.

APPENDIX A

1. Tree diagram examples

Notation : $a = s_a E_a$, $b = s_b E_b \cdots$ and $Cta = \coth \frac{\beta}{2} s_a E_a$. Each internal line's propagator is split according to Eq. (2); for example, $s_a(k_0 - s_a E_a)^{-1} = s_a(k_0 - \bar{a})^{-1}$ is for line a, with a overlined for an easy identification.

One- and two-loop examples. For the one-loop vertex of Fig. 4(a), the integrand in (24) is

$$V_{p_1^0 p_2^0; s_a s_b s_c} = \frac{s_a s_b s_c}{2} \left[\frac{Ctb}{(p_2^0 - p_1^0 + b - \bar{c})(p_1^0 - b - \bar{a})} + \frac{Cta}{(p_1^0 - a - \bar{b})(p_2^0 - a - \bar{c})} + \frac{Ctc}{(p_2^0 - c - \bar{a})(p_1^0 - p_2^0 + c - \bar{b})} \right] . \tag{A1}$$

In the first term, line b is cut and given the energy $s_b E_b = b$; the resulting tree diagram gives the associated factor.

For the two-loop self-energy diagram of Fig. 2(a), the integrand in (20) has two types of terms. There are eight terms whose numerators are product of two coth, corresponding to all possible ways of cutting the diagram so that a connected tree diagram is obtained; we write down half of them, as the other ones are similar. When lines b and c are cut and given respectively the energy $s_b E_b = b$ and $s_c E_c = c$, the resulting tree diagram gives the factor associated with $Ctb\ Ctc$. Dropping out a factor $s_a s_b s_c s_d s_f / 4$, one has

$$\begin{aligned} D_1 = & \frac{Ctb\ Ctc}{(p_0 - b - \bar{a})(b - c - \bar{f})(p_0 - c - \bar{d})} + \frac{Ctb\ Ctd}{(p_0 - b - \bar{a})(-p_0 + b + d - \bar{f})(p_0 - d - \bar{c})} \\ & + \frac{Ctf\ Cta}{(p_0 - a - \bar{b})(f + a - \bar{d})(p_0 - a - f - \bar{c})} + \frac{Ctf\ Ctb}{(p_0 - b - \bar{a})(b - f - \bar{c})(p_0 + f - b - \bar{d})} . \end{aligned} \tag{A2}$$

The terms whose numerator is 1 may be written in several ways. They can be obtained from (A2) by replacing $Cta \cdot Ctf$ and $Ctb \cdot Ctf$ respectively by $(Cta \cdot Ctf + 1)$ and $(Ctb \cdot Ctf - 1)$ (if one imagines having performed the abf loop first), or by replacing $Ctc \cdot Ctf$ and $Ctd \cdot Ctf$ respectively by $(Ctc \cdot Ctf + 1)$ and $(Ctd \cdot Ctf - 1)$ (if the loop cdf is performed first). Alternatively they are linked to the three-particle intermediate states and a suggestive form (see Sec. III A) is

$$D_2 = \frac{1}{(p_0 - a - f - c)(f + a - \bar{d})(f + c - \bar{b})} + \frac{(-1)}{(p_0 - b + f - d)(d - f - \bar{a})(b - f - \bar{c})} . \tag{A3}$$

For an alternative form for that two-loop self-energy diagram, one starts from the free-energy two-loop diagram and one sticks two external legs. The resulting form is given by (13) with $a_1 = a$, $a_2 = p_0 - b$, $b_1 = f$, $c_1 = -d$, $c_2 = c - p_0$. The sum over i, k gives four terms; no external energy enters the denominator $a_i + b + c_k$ for the two terms associated with three lines joining at an internal vertex.

Three-loop diagrams. A three-loop diagram contributing to the free energy is drawn on Fig. 6(a). The integrand is the sum over all possible tree diagrams. For example, when line g is cut, one is back to the two-loop self energy diagram of Fig. 2(a) just examined. The associated term is

$$Ctg\ \left[\frac{Ctb\ Ctc}{afd}+\frac{Ctb\ Ctd}{afc}+\frac{Ctf\ Cta+\epsilon_{fa}}{bcd}+\frac{Ctf\ Ctb+\epsilon_{fb}}{acd}+\frac{Cta\ Ctc}{bfd}+\frac{Cta\ Ctd}{bfc}+\frac{Ctf\ Ctc}{bad}+\frac{Ctf\ Ctd}{bac}\right], \tag{A4}$$

where the denominators have been written in a condensed way, a is for the propagator of line a and the first term in (A4) is for the first term in (A2) with $p_0 = g$ (for the appropriate orientations), and so on. A choice has been made between the alternative forms for the ϵ_{ij} terms.

There are similar terms where the role of g is played by any other line. Each tree occurs once, and the numerator is the product of the coth for the three cut lines. There are 16 trees. In addition, the full set of terms with one power of coth is

$$\begin{aligned}I_1 = {} & Ctg\ (\epsilon_{fa}+\epsilon_{fb}) + Ctb\ (\epsilon_{da}+\epsilon_{dg}) + Ctc\ (\epsilon_{ad}+\epsilon_{ag})\\ & + Ctf\ (\epsilon_{gd}+\epsilon_{ga}) + Cta\ (\epsilon_{cf}+\epsilon_{cd})\\ & + Ctd\ (\epsilon_{ba}+\epsilon_{bf})\ ,\end{aligned} \tag{A5}$$

where each term's denominator has not been written down as it is fixed precisely. Indeed the set of lines appearing in the denominator is the set complementary to the set of lines appearing in the numerator, as in (A4).

A diagram contributing to the self-energy may be obtained from the diagram in Fig. 6(a) by sticking two external legs, for example on lines b and c as shown on Fig. 6(b). There are now two more propagators b' and c' in each term of (A4), and there are new trees where line b' or (and) c' is cut. The extra terms with one single power of coth are

$$\begin{aligned}I_1' = {} & Ctg\ \ \epsilon_{fb'} + Ctb'\ (\epsilon_{dg}+\epsilon_{da}) + Ctc'\ (\epsilon_{ad}+\epsilon_{ag})\\ & + Cta\ (\epsilon_{c'f}+\epsilon_{c'd}) + Ctd\ (\epsilon_{b'a}+\epsilon_{b'f})\ .\end{aligned} \tag{A6}$$

The expansion of this integrand in terms of *intermediate states* is now considered. In each tree associated to Fig. 6(b), the external momentum p_0 flows through a string of internal lines, and one may perform an expansion of this string in terms of simple poles in p_0. For example, for the intermediate state ($g\ a\ f\ c'$) in Fig. 6(b), if the orientations of the momenta g, a, f, c' are chosen parallel to p, one looks for the denominator $(p_0 - g - a - f - c')$; it appears in the trees where three lines among the four $[g\ a\ f\ c']$ are cut and it is the remaining one's propagator. To take the residue of that p_0 pole is to split the tree into two pieces where the four lines $[g\ a\ f\ c']$ are on shell. The pieces are the same for all four trees so that the full numerator is $N_1 + N_2$ with

$$\begin{aligned}N_1 &= Ctg\ Cta\ Ctf + Ctg\ Cta\ Ctc' + Ctg\ Ctf\ Ctc'\\ &\quad + Cta\ Ctf\ Ctc'\ ,\\ N_2 &= Cta\ \epsilon_{c'f} + Ctf\ \epsilon_{ag} + Ctg\ \epsilon_{af} + Ctc'\ \epsilon_{ag}\ ,\end{aligned} \tag{A7}$$

where N_2 has been extracted from $I_1 + I_1'$. All the ϵ_{ij} are $+1$ for parallel orientations. $N_1 + N_2$ is the weight $M_{s_g s_a s_f s_{c'}}$ associated to the four-particle intermediate state ($gafc'$) as given in (16).

If one considers the intermediate state ($b'ag$) in Fig. 6(b), there appears on one side of the cutting plane a one-loop 4-point function ($b'\ a\ g\ p_{\rm out}$) written as a sum of trees (the loop is $d\ f\ c'\ c$). One selects the following trees: two lines among $[b'\ a\ g]$ are cut and one line among $[d\ f\ c'\ c]$ is cut. The residue of the pole $p_0 - a - g - b' = 0$ is such that the three lines $[b'\ a\ g]$ are on shell. The terms with one power of coth are extracted from $I_1 + I_1'$:

$$N_3 = \frac{1}{b}\ \epsilon_{ag}\ \left[\frac{Ctf}{dcc'}+\frac{Ctc}{fdc'}+\frac{Ctc'}{fdc}+\frac{Ctd}{fcc'}\epsilon_{ab'}\epsilon_{ag}\right]\ , \tag{A8}$$

where $\epsilon_{ag}^2 = 1$ has been inserted in the last term. Both ϵ are $+1$ for a parallel orientation of b', a, g. N_3 contributes the ϵ_{ag} term of the weight $M_{s_a s_g s_b'}$ associated to the intermediate state ($b'ag$), as given in (16).

Lastly, if one considers the intermediate state (bb') in Fig. 6(b), one selects the trees where one line is cut among $[b\ b']$. That line is multiplied by a two-loop self-energy diagram by construction.

One may stick other external legs to the diagram of Fig. 6(b) in order to build an N-point function. The previous analysis of the expansion in terms of intermediate states extends in a straightforward way to the case of planar diagrams. Indeed, as discussed in Sec. III B, it amounts to an expansion of the trees in terms of simple poles in one single variable. It is now clear that the germ of any intermediate state is in the integrand contributing to the free energy.

A Four loop diagram. A vacuum four-loop diagram is shown on Fig. 7. The trees are obtained by cutting

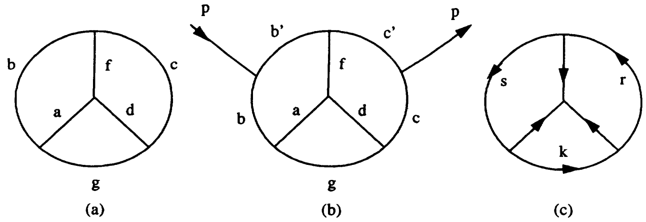

FIG. 6. Three-loop diagrams.

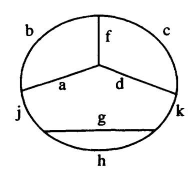

FIG. 7. A four-loop diagram.

four lines and the associated product of coth is in the numerator. There are terms with two coth, those lines are cut and the remaining diagram is a two-loop diagram where the ϵ_{ij} terms are written down as previously. In addition, there are two terms with no coth, $\epsilon_{gh}\ \epsilon_{fb}$ and $\epsilon_{gh}\ \epsilon_{fa}$, and those terms originate from

$$(\epsilon_{gh} + Ctg\ Cth) \left[\frac{\epsilon_{fb} + Ctf\ Ctb}{acdjk} + \frac{\epsilon_{fa} + Ctf\ Cta}{bcdjk}\right], \tag{A9}$$

as line g is common to two loops and line f is common to two other loops. In particular, those terms with no coth contributes to the weight of the five-particle intermediate states $(bfdgh)$ and $(cfagh)$, respectively. For any vacuum four-loop diagram, the summation over the energies is easily performed with the coth method, and the systematics of the terms is easy to interpret.

2. Computation of a three-loop diagram with the coth method

The notation is that of Sec. 1 of the Appendix. The three-loop diagram drawn on Fig. 6(a) is computed, the orientation of the momenta is defined in Fig. 6(c). The energies of the loop momenta are called k, r, s instead of k_0, r_0, s_0. The summation to be performed is

$$I'' = \sum_{k,r,s} \frac{1}{(s-b)(r-s-f)(s-k-a)(r-c)(k-r-d)(k-g)} . \tag{A10}$$

The result of the summation over s is

$$I'' = \sum_{k,r} \frac{1}{(r-c)(k-r-d)(k-g)} \left[\frac{Ctb}{(r-b-f)(b-k-a)} + \frac{Cta}{(k+a-b)(r-k-a-f)} + \frac{Ctf}{(r-f-b)(r-f-k-a)}\right]. \tag{A11}$$

The last term in each denominator factor identifies the associated internal line. The summation over r is performed and the result is

$$\begin{aligned} I'' = & \frac{Ctc}{(k-c-d)(k-g)} \left[\frac{Ctb}{(c-b-f)(b-k-a)} + \frac{Cta}{(k+a-b)(c-k-a-f)} + \frac{Ctf}{(c-f-b)(c-f-k-a)}\right] \\ & + \frac{Ctd}{(k-d-c)(k-g)} \left[\frac{Ctb}{(k-d-b-f)(b-k-a)} + \frac{Cta}{(k+a-b)(-d-a-f)} + \frac{Ctf}{(k-d-f-b)(-d-a-f)}\right] \\ & + \frac{Ct(b+f)}{(b+f-c)(k-b-f-d)(k-g)} [Ctb + Ctf] \frac{1}{(b-k-a)} \\ & + \frac{Ct(a+f)}{(k+a+f-c)(-a-f-d)(k-g)} [Cta + Ctf] \frac{1}{(k+a-b)} . \end{aligned} \tag{A12}$$

The last term in each denominator factor identifies the associated internal line, and each term of (A12) is associated to a k loop. The summation over k is then performed, and the terms are grouped that are residue of the same pole. Then the result is written in a symetric way. It is described in Sec. 1 of the Appendix, and parts of it are written in (A4) and (A5) [Note that the orientations of momenta in Fig. 6(c) differ from those of Fig. 2(a) and Eq. (A2)].

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