

Kac and new determinants for fractional superconformal algebras

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We derive the Kac and new determinant formulas for an arbitrary (integer) level K fractional superconformal algebra using the BRST cohomology techniques developed in conformal field theory. In particular, we reproduce the Kac determinants for the Virasoro ($K = 1$) and superconformal ($K = 2$) algebras. For $K \geq 3$ there always exist modules where the Kac determinant factorizes into a product of more fundamental new determinants. Using our results for general K , we sketch the nonunitarity proof for the $SU(2)$ minimal series; as expected, the only unitary models are those already known from the coset construction. We apply the Kac determinant formulas for the spin-4/3 parafermion current algebra (i.e., the $K = 4$ fractional superconformal algebra) to the recently constructed three-dimensional flat Minkowski space-time representation of the spin-4/3 fractional superstring.

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I. INTRODUCTION

The Kac determinant [1] for the Virasoro algebra is a very useful tool for analyzing the minimal unitary series in conformal field theory (CFT) [2] and for understanding the no-ghost theorem in bosonic string theory [3]. So far, the Kac determinants have been known only for the Virasoro and superconformal algebras [4]. These algebras are special cases of the so-called fractional superconformal algebras (FSCA's) [5,6] labeled by the level K of the $SU(2)_K$ current algebra [7]. (The level $K = 1$ and 2 FSCA's are precisely the Virasoro and superconformal algebras.) In this paper we derive the Kac and new determinant formulas for an arbitrary (integer) level K FSCA.

In Sec. II we classify the modules of the level K FSCA and present the Kac determinant formulas for each module. For $K \geq 3$ there always exist modules where the Kac determinant factorizes into a product of more fundamental new determinants. For these modules we also present the new determinant formulas. We recover the well-known Kac determinant formulas for the Virasoro and superconformal algebras, and elaborate on the case of the spin-4/3 parafermion current algebra (i.e., the $K = 4$ FSCA) of Zamolodchikov and Fateev [8]. This algebra has three modules (the so-called S , D , and R modules) as opposed to the superconformal algebra that has only two modules (i.e., the Neveu-Schwarz and Ramond modules). We discuss in detail the Kac determinant formulas for the S and D modules and the Kac and new determinant formulas for the R module. We also give a few low-lying levels to clarify notations.

In Sec. III we derive the Kac and new determinant formulas using a generalization [9,10] of the Becchi-Rouet-

Stora-Tyutin (BRST) operator of Felder [11]. We also derive the relation between the Kac and new determinants. The zeros of the Kac determinants are given in Refs. [5,9,10]. To derive the Kac and new determinant formulas, we need to fix the orders of these zeros, as well as the zero mode contribution. For level $K \geq 3$ FSCA's, the orders of some zeros of the Kac and new determinants for a given module come from the string functions that give the counting of states in a different module. This is never the case for the Virasoro and superconformal algebras.

Next, we turn to the applications of the Kac determinant formulas. In Sec. IV, using our results for general K , we deduce the values of the central charge c and conformal dimension h for which FSCA's can have unitary representations. In particular, we sketch the nonunitarity proof for the $SU(2)$ minimal series and argue that the only unitary models are precisely the known $SU(2)_K \otimes SU(2)_L / SU(2)_{K+L}$ coset models [12].

Recent evidence supports the existence of the so-called fractional superstrings (FSS's) [13]. A particular representation (with $c = 5$) of the spin-4/3 FSS's has bosons and fermions living in three-dimensional flat Minkowski space-time [14]. In Sec. V we discuss the no-ghost theorem for the space-time bosonic sector of this theory, using the Kac determinant formulas for the $K = 4$ FSCA.

We also use the Kac determinant formulas for the $K = 4$ FSCA to examine the physical null state structure of the critical spin-4/3 FSS ($c = 10$) and find extra sets of zero-norm physical states; this property is expected of any consistent string theory.

II. KAC AND NEW DETERMINANTS

In this section we present the Kac and new determinant formulas for an arbitrary level K fractional super-

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conformal algebra (FSCA). We recover the well-known Kac determinants for the Virasoro ($K = 1$) and superconformal ($K = 2$) algebras, and elaborate on the case of the spin-4/3 parafermion current algebra (i.e., the $K = 4$ FSCA) of Zamolodchikov and Fateev [8]. We derive the Kac and new determinant formulas in the next section.

A. Preliminaries

By definition the Kac determinants for a given algebra are representation independent. The simplest representation of the level K FSCA (convenient for derivation of the Kac and new determinants) is a noninteracting system of the \mathbf{Z}_K parafermion (PF) [15] and a free boson. The $SU(2)_K/U(1)$ coset model is a realization of the \mathbf{Z}_K PF theory. The chiral $SU(2)_K$ Wess-Zumino-Witten (WZW) theory [7] has the central charge

$$c_0 = \frac{3K}{K+2} \quad (2.1)$$

and consists of holomorphic Virasoro primary fields $\Phi_m^j(z)$ of conformal dimensions $\frac{j(j+1)}{(K+2)}$. The indices $j, m \in \mathbf{Z}/2$ label $SU(2)$ representations, where

$$0 \leq j \leq K/2 \text{ and } |m| \leq j \text{ with } j - m \in \mathbf{Z}. \quad (2.2)$$

When we factor a $U(1)$ subgroup out of $SU(2)_K$, we correspondingly factor the primary fields as

$$\Phi_m^j(z) = \phi_m^j(z) \exp \left\{ i \frac{m}{\sqrt{K}} \varphi(z) \right\}. \quad (2.3)$$

Here φ is the free $U(1)$ boson normalized so that $\langle \varphi(z)\varphi(w) \rangle = -2 \ln(z-w)$. $\phi_m^j(z)$ are the Virasoro primary fields in the \mathbf{Z}_K PF theory with conformal dimensions

$$\Delta_m^j = \frac{j(j+1)}{(K+2)} - \frac{m^2}{K} \text{ for } |m| \leq j. \quad (2.4)$$

The $SU(2)_K$ algebra has an automorphism

$$\phi_m^j = \phi_{m+K/2}^{K/2-j}. \quad (2.5)$$

With these identifications, we can consistently extend ϕ_m^j to $|m| > j$.

The fusion rules of the PF fields follow from those of the $SU(2)_K$ theory:

$$[\phi_{m_1}^{j_1}] \otimes [\phi_{m_2}^{j_2}] \sim \sum_{j=|j_1-j_2|}^r [\phi_{m_1+m_2}^j], \quad (2.6)$$

where $r = \min\{j_1 + j_2, K - j_1 - j_2\}$.

The $SU(2)_K$ currents are given by

$$J^\pm = \sqrt{K} \phi_{\pm 1}^0 \exp(\pm i\varphi/\sqrt{K}), \quad (2.7)$$

$$J^0 = \frac{i}{2} \sqrt{K} \partial\varphi, \quad (2.8)$$

where $\phi_{\pm 1}^0$ have conformal dimension $(K-1)/K$.

The $SU(2)_K$ primary fields read

$$\Phi_j^j(z) = \phi_j^j(z) \exp[ij\varphi(z)/\sqrt{K}]. \quad (2.9)$$

Note $\Phi_{-j}^j = (\Phi_j^j)^\dagger$ are also primary.

Next we consider the so-called fractional supercurrent [whose conformal dimension is $\Delta = (K+4)/(K+2)$]

$$G = J_{-1}^a \Phi_a^1 = \frac{\sqrt{K}}{2} \epsilon \partial\varphi + \frac{iK}{K+4} \eta. \quad (2.10)$$

Here J_{-1}^a are the conformal dimension 1 creation modes of the $J^a(z)$ currents; $\epsilon \equiv \phi_0^1$ is the so-called PF energy operator and η is its first PF descendent. The closed algebra generated by the currents $T(z)$ and $G(z)$ is the level K FSCA. [Since ϵ does not exist for $K=1$, $G(z)$ is absent in this case and we recover the Virasoro algebra.] The central charge of this algebra is c_0 . If we turn on the background charge of the φ boson, the algebra, generated by the appropriately modified $T(z)$ and $G(z)$ currents, remains closed. We note that there exist other FSCA's with additional fractional supercurrents [6].

The following operator product expansions (OPE's) define the level K FSCA with arbitrary central charge c :

$$\begin{aligned} T(z)T(w) &= \frac{(c/2)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots, \\ T(z)G(w) &= \frac{\Delta}{(z-w)^2} G(w) + \frac{1}{(z-w)} \partial G(w) + \dots, \end{aligned} \quad (2.11)$$

$$\begin{aligned} G(z)G(w) &= (z-w)^{-2\Delta} \left\{ \frac{c}{\Delta} + 2(z-w)^2 T(w) + \dots \right\} \\ &\quad + \lambda(c)(z-w)^{-\Delta} \\ &\quad \times \left\{ G(w) + \frac{1}{2}(z-w) \partial G(w) + \dots \right\}. \end{aligned}$$

The associativity condition for this algebra fixes the structure constant $\lambda^2(c)$:

$$\lambda^2(c) = \frac{2K^2 c_{111}^2}{3(K+4)^2} (c_1 - c). \quad (2.12)$$

Here c_{111} is the $SU(2)_K$ structure constant for the OPE of two chiral spin-1 primary fields to give a chiral spin-1 field and [6]

$$c_1 \equiv 24/K + c_0. \quad (2.13)$$

As mentioned earlier, for $K=1$ the G current is absent. For $K=2$, $c_{111} = 0$, $\lambda(c) = 0$, and G is the usual supercurrent. It is a Hermitian current of conformal dimension $3/2$. For $K \geq 3$, c_{111} is nonvanishing and

$$G^\dagger = \alpha G, \quad (2.14)$$

where

$$\alpha = \begin{cases} +1 & \text{if } c \leq c_1, \\ -1 & \text{otherwise.} \end{cases} \quad (2.15)$$

For definiteness we will take the branch of Eq. (2.12) that is positive for $c < c_1$.

The vertex operators $\chi_m^j(0)$, labeled by the $SU(2)_K$ quantum numbers j and m , create states in modules of the level K FSCA: $|\chi_m^j\rangle \equiv \chi_m^j(0)|0\rangle$. The fields primary with respect to $G(z)$ and $T(z)$ have the form $\chi_j^j(z)$ [$\chi_{-j}^j(z) = (\chi_j^j)^\dagger$ are also primary]. $G(z)$ has $j=1$ and

$m = 0$ and in terms of modings acts on $|\chi_m^j\rangle$ in the same manner as ϵ acts on the PF fields ϕ_m^j :

$$G(z)\chi_m^j(0) = \sum_n [z^{n+2j/(K+2)-1}G_{-n-2(j+1)/(K+2)} + z^{n-2/(K+2)-1}G_{-n} + z^{n-2(j+1)/(K+2)}G_{-n+2j/(K+2)-1}]\chi_m^j(0), \quad (2.16)$$

where the resulting states have spins $j+1$, j , and $j-1$, respectively. [The terms not allowed by the fusion rules (2.6) are absent.] The positive modes of the T and G currents annihilate the highest weight states created by the primary fields:

$$L_0|h\rangle = h|h\rangle, \quad L_n|h\rangle = G_r|h\rangle = 0, \quad n, r > 0. \quad (2.17)$$

Here r is in general a rational number.

The module $[\chi_j^j]$ built from the highest weight state $|\chi_j^j\rangle$ is spanned by the states

$$L_{-n_1}^{\lambda_1} \cdots L_{-n_p}^{\lambda_p} G_{-r_1}^{\rho_1} \cdots G_{-r_q}^{\rho_q} |\chi_j^j\rangle, \quad (2.18)$$

$$C_{2m}^{2j} = \frac{q^{d_m^j}}{\eta^3} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{s(s+1)/2+r(r+1)/2+sr(K+1)} \left(q^{s(j-m)+r(j+m)} - q^{K+1-2j+s[K+1-j+m]+r[K+1-j-m]} \right). \quad (2.20)$$

Here η is the Dedekind η function:

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (2.21)$$

and

$$d_m^j = \Delta_m^j + \frac{1}{4(K+2)}. \quad (2.22)$$

For later convenience we define the level of a state (2.18) as

$$N = N_0 + \Delta_j^j, \quad (2.23)$$

where Δ_j^j is given by (2.4) and

$$N_0 = \sum_{i=1}^p \lambda_i n_i + \sum_{k=1}^q \rho_k r_k \quad (2.24)$$

is (generically a rational number) the level of the state above the highest weight state $|\chi_j^j\rangle$. The number of its descendents (i.e., the number of states in the module $[\chi_j^j]$) at level N is given by $P_j(N)$, the latter being a coefficient in the power expansion

$$C_{2m}(q) \equiv \sum_j C_{2m}^{2j}(q) = q^{-\frac{c_0}{24}} \sum P_m(N) q^N. \quad (2.25)$$

The sum over j includes its values allowed by the fusion rules (2.6) up to identifications (2.5), i.e., each independent string function appears once and only once.

We note that the G current modings when acting on the highest weight states $|\chi_j^j\rangle$ and $|\chi_{K/2-j}^{K/2-j}\rangle$ ($0 < j <$

where the ordering is the same as for the states

$$a_{-n_1}^{\lambda_1} \cdots a_{-n_p}^{\lambda_p} \epsilon_{-r_1}^{\rho_1} \cdots \epsilon_{-r_q}^{\rho_q} \phi_j^j(0)|0\rangle, \quad n_i > 0, r_k \geq 0. \quad (2.19)$$

Here a_{-n_i} represent creation modes of the φ boson, while ϵ_{-r_k} stand for modes of the \mathbf{Z}_K PF energy operator. Note that λ_i can be any non-negative integers, while ρ_k can take only two values: zero and unity.

The naive counting of states $|\chi_m^j\rangle$ of the form (2.18) is identical to the counting of states (2.19) with the same quantum numbers j and m and is given by the level K string functions C_{2m}^{2j} [16]:

$K/4$) are the same. Hence the modules $[\chi_j^j]$ and $[\chi_{K/2-j}^{K/2-j}]$ are structurally identical.

We end this subsection with the following remark. An important feature of level $K \geq 3$ FSCA's is the appearance of cuts in the GG OPE (2.11). Since there are two different cuts on the right-hand side, upon continuation of a correlation function involving $G(z)G(w)$ OPE along a contour interchanging z and w in the complex plane, it is inconsistent for the correlator to pick up a simple phase; that is, the current G is non-Abelianly braided. There is only one FSCA, namely, the spin-4/3 ($K=4$) parafermion current algebra, where it is possible to split the fractional supercurrent so that the split currents are Abelianly braided. This makes for important technical simplifications. For example, it enables one to write down the generalized commutation relations between the modes of the split currents and to explicitly calculate the Kac and new determinants. For other FSCA's the generalized commutation relations have not yet been written down because of technical difficulties arising from their non-Abelian braiding properties. Hence, the spin-4/3 parafermion current algebra provides an important check on the Kac and new determinant formulas presented in the next subsection, and we consider it in detail in Sec. II C.

B. Kac and new determinant formulas

As mentioned earlier, so far the Kac determinant formulas have been known only for the Virasoro and superconformal algebras. These Kac determinants are polynomials in the highest weight h . One expects the determinants of inner products of states in modules of FCSA's

also be polynomials in h . This is the case for some modules, but not for all.

There are three different types of modules in the Fock space of the level K FSCA.

(i) In the module $[\chi_0^0]$, built from the highest weight state $|\chi_0^0\rangle$, the counting of states is given by C_0 defined in (2.25). The fractional supercurrent zero mode G_0 does not act on $|\chi_0^0\rangle$ and the determinant of inner products of states at any level is a polynomial in h .

(ii) For $0 < j \neq K/4$ the highest weight state $|\chi_j^j\rangle$ is an eigenstate of G_0 . The counting of states in the module $[\chi_j^j]$ is given by C_{2j} . The determinant of inner products of states at any level is a polynomial in the G_0 eigenvalue, but is *not* a polynomial in h . However, as mentioned earlier, the modules $[\chi_j^j]$ and $[\chi_{K/2-j}^{K/2-j}]$ ($0 < j < K/4$) are structurally identical. This allows us to construct the highest weight state $|h\rangle$ (as a linear combination of $|\chi_j^j\rangle$ and $|\chi_{K/2-j}^{K/2-j}\rangle$) that is no longer an eigenstate of G_0 . The determinant of inner products of states in the module $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$ built from $|h\rangle$ is a polynomial in h . The number of states is given by $C_{2j} + C_{K-2j} (\equiv 2C_{2j})$.

(iii) If K is even, there is the third type of modules, namely, the $[\chi_{K/4}^{K/4}]$ module. G_0 is not necessarily diagonal with respect to the highest weight state $|\chi_{K/4}^{K/4}\rangle$. However, the determinant of inner products of states is always a polynomial in h . The counting of states is given by $C_{K/2}$.

Now we define $\det[\mathcal{M}^{(j)}(N)]$ as the determinant of inner products of states (2.18) at level N . It is a generalization of the Kac determinants for the Virasoro and superconformal algebras. For the modules $[\chi_0^0]$, $[\chi_{K/4}^{K/4}]$, and $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$ ($0 < j < K/4$) the determinant is a polynomial in h , and we will refer to it as the Kac determinant. For the module $[\chi_j^j]$ ($0 < j \neq K/4$) the determinant is a polynomial in the G_0 eigenvalue, but is *not* a polynomial in h . We will refer to these determinants as the new determinants. The Kac determinant for the module $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$ factorizes into a product of the new determinants for the $[\chi_j^j]$ and $[\chi_{K/2-j}^{K/2-j}]$ modules.

First we present (up to positive normalization constants independent of the highest weight h and the central charge c) the Kac and new determinant formulas for the modules described above. Then we give explicit formulas for the zeros of the Kac and new determinants. Finally, we comment on some features of the determinants for FSCA's.

1. The module $[\chi_0^0]$

The Kac determinant reads

$$\det[\mathcal{M}^{(0)}(N_0)] = \alpha^{Q(0)} \prod_{r,s} [\alpha(h - h_{r,s})]^{P_\ell(N_0 - rs/K)}, \quad (2.26)$$

where $r, s \in \mathbf{N}$, $s - r = 0 \pmod K$ and $rs/K \leq N_0$; $2\ell = (s + r) \pmod K$ and

$$Q(0) = \sum_n P_0(N_0 - n^2), \quad (2.27)$$

where $n \in \mathbf{N}$, $n^2 \leq N_0$.

2. The module $[\chi_{K/4}^{K/4}]$, K is even

(a) G_0 is diagonal with respect to the highest weight state $|h; g^\pm\rangle$:

$$G_0|h; g^\pm\rangle = g^\pm|h; g^\pm\rangle. \quad (2.28)$$

With the appropriate normalization of G_0 , the eigenvalues g^\pm are given by

$$g^\pm = \pm\sqrt{h - c/24}. \quad (2.29)$$

The Kac determinants for the modules built from $|h; g^\pm\rangle$ read

$$\det[\mathcal{M}_{(\pm)}^{(K/4)}(N)] = \alpha^{Q(K/4)} \prod_{r,s} (h - h_{r,s})^{P_\ell(N - rs/K)}, \quad (2.30)$$

where $N \equiv N_0 + K/8(K + 2)$, $s - r = K/2 \pmod K$ and $rs/K \leq N$; $2\ell = (s + r) \pmod K$ and

$$Q(K/4) = P_{K/4}(N)/2 + \sum_n P_{K/4}(N - n^2). \quad (2.31)$$

Here $n^2 \leq N$.

(b) G_0 is *not* diagonal with respect to the highest weight state $|h\rangle$.

The Kac determinant reads

$$\det[\mathcal{M}^{(K/4)}(N)] = \alpha^{Q(K/4)} (h - c/24)^{P_{K/4}(N)/2} \times \prod_{r,s} (h - h_{r,s})^{P_\ell(N - rs/K)}, \quad (2.32)$$

where r, s, ℓ , and N are the same as in Eq. (2.30).

3. The module $[\chi_j^j]$, $0 < j \neq K/4$

(a) The highest weight state $|\chi_j^j\rangle \equiv |h; g^j\rangle$ is an eigenstate of G_0 :

$$G_0|h; g^j\rangle = g^j|h; g^j\rangle. \quad (2.33)$$

With the appropriate normalization of G_0 , the eigenvalue g^j is given by

$$g^j = a^j \sqrt{c_1 - c} + \text{sgn}(j - K/4) \sqrt{h - h_0^j}, \quad (2.34)$$

where

$$a^j = \frac{|K - 4j|}{\sqrt{24(K + 4)}}, \quad (2.35)$$

and we define, for arbitrary j ,

$$h_0^j \equiv \frac{c - c_0}{24} + \Delta_j^j = \frac{c}{24} - \frac{(K - 4j)^2}{8K(K + 2)}. \quad (2.36)$$

The new determinant reads

$$\det[\mathcal{M}^{(j)}(N)] = \alpha^{Q(j)} \prod_{r,s} [\alpha(g^j - g_{r,s}) \times (g^j - g_{s,r})]^{P_\ell(N-rs/K)}, \quad (2.37)$$

where $N = N_0 + \Delta_j^j$, $s - r = 2j \pmod K$ and $rs/K \leq N$; $2\ell = (s + r) \pmod K$ and

$$Q(j) = \sum_n P_j(N - n^2). \quad (2.38)$$

(b) The module $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$, $0 < j < K/4$.

The highest weight state $|h\rangle$ is a linear combination of $|\chi_j^j\rangle$ and $|\chi_{K/2-j}^{K/2-j}\rangle$ such that G_0 is *not* diagonal with respect to $|h\rangle$.

The Kac determinant reads

$$\det[\mathcal{M}_0^{(j)}(N)] = [\alpha(h - h_0^j)]^{P_j(N)} \times \prod_{r,s} [(h - h_{r,s})(h - h_{s,r})]^{P_\ell(N-rs/K)}, \quad (2.39)$$

where r, s, ℓ are the same as in (2.37).

The Kac determinant (2.39) factorizes as

$$\det[\mathcal{M}_0^{(j)}(N)] = [\alpha(h - h_0^j)]^{P_j(N)} \times \det[\mathcal{M}^{(j)}(N)] \det[\mathcal{M}^{(K/2-j)}(N)]. \quad (2.40)$$

The zeros of the Kac determinants (2.26), (2.30), (2.32), and (2.39) read [5]

$$h_{r,s} = \frac{c - c_0}{24} + \Delta_j^j + \frac{1}{96} \left[(r + s)\sqrt{c_0 - c} + (r - s)\sqrt{c_1 - c} \right]^2 = h_0^j + b_{r,s}^2, \quad (2.41)$$

where

$$b_{r,s} = \frac{1}{\sqrt{96}} \left[(r + s)\sqrt{c_0 - c} + (r - s)\sqrt{c_1 - c} \right]. \quad (2.42)$$

The zeros of the new determinants (2.37) are given by

$$g_{r,s} = a^j \sqrt{c_1 - c} + \text{sgn}(j - K/4) b_{r,s}. \quad (2.43)$$

Here $2j = (s - r) \pmod K$. The relation between the zeros of the Kac and corresponding new determinants reads

$$(g^j - g_{r,s})(g^{K/2-j} - g_{r,s}) = -(h - h_{r,s}). \quad (2.44)$$

Their orders $P_m(N)$ are given by

$$C_{2m}(q) \equiv \sum_j C_{2m}^{2j}(q) = q^{-\frac{K}{8(K+2)}} \sum P_m(N) q^N. \quad (2.45)$$

The sum over j includes its values allowed by the fusion rules (2.6) up to identifications (2.5), i.e., each independent string function appears once and only once.

We conclude this subsection with the following remarks.

(a) We define the Kac and new determinants for a linearly independent set of states where the total number of the T and G modes is minimal. If this number is not minimal, the determinant will be different by an overall factor (that can be either positive or negative) independent of h , but in general dependent on c .

(b) For the Virasoro algebra the G current is absent, while for the superconformal algebra it is always Hermitian. Therefore, for the $K = 1, 2$ cases $\alpha \equiv 1$ in all of the above formulas. For level $K \geq 3$ FSCA's, α flips sign as the central charge exceeds c_1 defined in (2.13). Hence in the Kac and new determinants there appears the factor of α to the power (defined mod 2) determined by the total number of the G current modes in a linearly independent set of states with minimal counting.

(c) For level $K \geq 3$ FSCA's, the orders of some zeros of the Kac and new determinants for a given module come from the string functions that give the counting of states in a different module. This is never the case for the Virasoro and superconformal algebras.

The examples in the next subsection illuminate these issues.

C. Examples

Now we consider some concrete cases in order to clarify our conventions and notations. We give the Kac and new determinant formulas up to normalization constants independent of the highest weight h and the central charge c .

1. The Virasoro algebra ($K = 1$)

We have only one module $[\chi_0^0]$ and the counting of states is given by the string function

$$C_0 \equiv C_0^0 = 1/\eta(q) = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \quad (2.46)$$

Equation (2.26) (with $\alpha \equiv 1$) reduces to the well-known Kac determinant formula for the Virasoro algebra [1].

2. The superconformal algebra ($K = 2$)

(a) The Neveu-Schwarz module, or the $[\chi_0^0]$ module. According to Eq. (2.25) the counting of states is given by

$$C_0 \equiv C_0^0 + C_0^2 = q^{-1/16} \prod_{n=1}^{\infty} \frac{1 + q^{n-1/2}}{1 - q^n}, \quad (2.47)$$

Eq. (2.26) (again with $\alpha \equiv 1$) now reduces to the Kac determinant formula for the Neveu-Schwarz module.

(b) The Ramond module, or the $[\chi_{1/2}^{1/2}]$ module. This is the simplest $[\chi_{K/4}^{K/4}]$ module. The counting is given by the string function

$$C_1 \equiv C_1^1 = q^{-1/16} \left(q^{1/16} \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} \right). \quad (2.48)$$

The level of a state in this module according to Eq. (2.23) is defined as $N = N_0 + 1/16$, where N_0 is the level of the state above the highest weight state. Equation (2.32) is then the Kac determinant formula for the Ramond module [4].

To illuminate the Kac determinant formula (2.30) for the $[\chi_{K/4}^{K/4}]$ module, we consider the Ramond module built from an eigenstate $|h; g^\pm\rangle$ of the supercurrent zero mode G_0 . Because of the zero mode relation

$$G_0^2 = L_0 - c/24, \quad (2.49)$$

we have

$$g^\pm = \pm \sqrt{h - c/24}. \quad (2.50)$$

A linearly independent set of states at level $17/16 = 1 + 1/16$ reads

$$|\phi_1^\pm\rangle = L_{-1}|h; g^\pm\rangle, \quad |\phi_2^\pm\rangle = G_{-1}|h; g^\pm\rangle. \quad (2.51)$$

The Kac determinants of these states can be computed using the superconformal mode algebra:

$$\det(\mathcal{M}_{(\pm)}) = 4(h - h_{1,2})(h - h_{2,1}), \quad (2.52)$$

where $h_{1,2}$ and $h_{2,1}$ are given by Eq. (2.41) at $K = 2$ and $j = 1/2$. Note that at this level the Kac determinant is a polynomial in h . This holds at all levels for all the $[\chi_{K/4}^{K/4}]$ modules.

3. The spin-4/3 parafermion current algebra ($K = 4$)

The chiral fractional supercurrent $G(z)$, whose conformal dimension is $4/3$, can be split into two pieces:

$$G(z) = \begin{cases} G^+(z) + G^-(z) & \text{if } c \leq 8, \\ i[G^-(z) - G^+(z)] & \text{otherwise.} \end{cases} \quad (2.53)$$

The currents $G^\pm(z)$ are Abelianly braided (i.e., parafermionic), and with the $T(z)$ current form the closed spin-4/3 parafermion current algebra of Zamolodchikov and Fateev [8]:

$$G^\pm(z)G^\pm(w) = \frac{\lambda^\pm}{(z-w)^{4/3}} \times \left\{ G^\mp(w) + \frac{1}{2}(z-w)\partial G^\mp(w) + \dots \right\},$$

$$G^+(z)G^-(w) = \frac{1}{(z-w)^{8/3}} \left\{ \frac{3c}{8} + (z-w)^2 T(w) + \dots \right\}, \quad (2.54)$$

$$T(z)G^\pm(w) = \frac{(4/3)G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{(z-w)} + \dots,$$

$$T(z)T(w) = \frac{(c/2)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots,$$

where

$$\lambda^+ = \alpha\lambda^- = \lambda = \frac{\sqrt{|8-c|}}{\sqrt{6}}, \quad \alpha = \begin{cases} +1 & \text{if } c \leq 8, \\ -1 & \text{otherwise.} \end{cases} \quad (2.55)$$

The G^\pm currents have the following Hermiticity properties:

$$(G^\pm)^\dagger = \alpha G^\mp. \quad (2.56)$$

The algebra (2.54) obeys a \mathbf{Z}_3 symmetry. The $G^\pm(z)$ and $T(z)$ currents have \mathbf{Z}_3 charges $q = \pm 1$ and zero, respectively. The \mathbf{Z}_3 charge q is defined mod 3.

The modes of the currents

$$T(z) = \sum_n z^{-n-2} L_n, \quad G^\pm(z) = \sum_r z^{-r-4/3} G_r^\pm \quad (2.57)$$

satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n},$$

$$[L_m, G_r^\pm] = \left(\frac{m}{3} - r\right) G_{m+r}^\pm. \quad (2.58)$$

The modings of the currents $G^\pm(z)$ and the generalized commutation relations (GCR's) for them depend on a representation of the algebra. First we consider the representations that correspond to the integer spin j modules. The Fock space falls into sectors \mathcal{H}_q labeled by their \mathbf{Z}_3 charge. The currents G^\pm act on the Fock space sectors according to the rules

$$G^\pm : \mathcal{H}_q \rightarrow \mathcal{H}_{q\pm 1}. \quad (2.59)$$

With these actions, the mode expansions of the G^\pm currents are defined as

$$G^\pm(z)\chi_q(0) = \sum_n z^{n\mp q/3} G_{-1-n-(1\mp q)/3}^\pm \chi_q(0), \quad (2.60)$$

where χ_q is an arbitrary state in \mathcal{H}_q . These mode expansions can be inverted to give

$$G_{n-(1\mp q)/3}^\pm \chi_q(0) = \oint_\gamma \frac{dz}{2\pi i} z^{n\pm q/3} G^\pm(z)\chi_q(0). \quad (2.61)$$

Here, γ is a contour encircling the origin once, where $\chi_q(0)$ is inserted.

The G_r modes satisfy the GCR's

$$\sum_{\ell=0}^{\infty} c_{\ell}^{(1/3)} \left[G_{\pm \frac{q}{3}+n-\ell}^{\pm} G_{2\pm \frac{q}{3}+m+\ell}^{\pm} + G_{\pm \frac{q}{3}+m+1-\ell}^{\pm} G_{2\pm \frac{q}{3}+n-1+\ell}^{\pm} \right] = \lambda^{\pm} G_{2\pm \frac{2q}{3}+n+m}^{\mp}, \quad (2.62)$$

$$\sum_{\ell=0}^{\infty} c_{\ell}^{(-1/3)} \left[G_{\frac{1+q}{3}+n-\ell}^{+} G_{-\frac{1+q}{3}+m+\ell}^{-} + G_{-\frac{2+q}{3}+m-\ell}^{-} G_{\frac{2+q}{3}+n+\ell}^{+} \right] = L_{n+m} + \frac{3c}{16} \left(n+1 + \frac{q}{3} \right) \left(n + \frac{q}{3} \right) \delta_{n+m}, \quad (2.63)$$

when acting on a state with the \mathbf{Z}_3 charge q . Here $c_{\ell}^{(\alpha)}$ are the binomial coefficients

$$(1-x)^{\alpha} = \sum_{\ell=0}^{\infty} c_{\ell}^{(\alpha)} x^{\ell}. \quad (2.64)$$

According to the Hermiticity assignments (2.56) and the mode expansions (2.60), the Hermiticity properties of the G_r^{\pm} modes are $(G_r^{+})^{\dagger} = \alpha G_{-r}^{-}$.

The highest weight states $|h; q\rangle$ with the \mathbf{Z}_3 charge q satisfy the conditions

$$\begin{aligned} L_0|h; q\rangle &= h|h; q\rangle, \quad L_n|h; q\rangle = 0, \\ G_{n-(1\mp q)/3}^{\pm}|h; q\rangle &= 0, \quad n > 0. \end{aligned} \quad (2.65)$$

Next we consider the representations of the algebra corresponding to the half odd integer spin j modules. The fields $z^{4/3} G^{\pm}(z) \chi_R(0)$ (where χ_R is an arbitrary state in the Fock space) are double-valued analytic functions, i.e., the fields $\chi_R(z)$ create square root cuts in the complex plane. For the purpose of calculating the

$$\sum_{\ell=0}^{\infty} D_{\left(\frac{1}{3}, -\frac{1}{3}\right)}^{(\ell)} \left[\tilde{G}_{\frac{n-\ell}{2}} \tilde{G}_{\frac{m+\ell}{2}} + \tilde{G}_{\frac{m-\ell}{2}} \tilde{G}_{\frac{n+\ell}{2}} \right]$$

$$= \frac{\lambda}{2} (-1)^{n+m} \tilde{G}_{\frac{n+m}{2}} + \alpha [(-1)^n + (-1)^m] \left\{ L_{\frac{n+m}{2}} + \frac{3c}{8} \left(\frac{n^2}{8} - \frac{5}{48} \right) \delta_{n+m} \right\}, \quad (2.68)$$

where $D_{(\alpha, \beta)}^{(\ell)}$ are the binomial coefficients

$$(1-x)^{\alpha} (1+x)^{\beta} = \sum_{\ell=0}^{\infty} D_{(\alpha, \beta)}^{(\ell)} x^{\ell}. \quad (2.69)$$

According to the Hermiticity assignments (2.56) and the mode expansions (2.66), the Hermiticity properties of the $\tilde{G}_{n/2}$ modes are $(\tilde{G}_{n/2})^{\dagger} = (-1)^n \tilde{G}_{-n/2}$.

The highest weight states satisfy the conditions

$$\begin{aligned} L_0|h\rangle_R &= h|h\rangle_R, \quad L_n|h\rangle_R = 0, \quad \tilde{G}_{n/2}|h\rangle_R = 0, \\ n &> 0. \end{aligned} \quad (2.70)$$

Now we turn to the description of the spin-4/3 parafermion current algebra modules where the counting of states is given by the $K=4$ string functions [16,17]:

Kac and new determinants it suffices to consider only the states $\chi_R(0)$ that are \mathbf{Z}_3 singlets. The modings of the G^{\pm} currents then are given by the OPE's

$$G^{\pm}(z) \chi_R(0) = 2^{-\frac{2}{3}} \sum_{n=-\infty}^{\infty} z^{n/2-4/3} G_{-n/2}^{\pm} \chi_R(0),$$

$$n \in \mathbf{Z}, \quad (2.66)$$

where the overall normalization is chosen for later convenience. These mode expansions can be inverted to give

$$G_{n/2}^{\pm} \chi_R(0) = 2^{-\frac{1}{3}} \oint_{\gamma} \frac{dz}{2\pi i} z^{1/3+n/2} G^{\pm}(z) \chi_R(0). \quad (2.67)$$

Here, γ is a contour encircling the origin twice, where $\chi_R(0)$ is inserted.

The modes $G_{n/2}^{\pm}$ are related to each other via $G_{n/2}^{+} = (-1)^n \alpha G_{n/2}^{-}$ due to the single-bypass relations discussed in Ref. [8]. Defining $\tilde{G}_{n/2} \equiv G_{n/2}^{-}$, we find that the modes $\tilde{G}_{n/2}$ satisfy the GCR

$$C_0^0 + C_0^4 = \frac{1}{\eta^2(q)} \left(\sum_{n=-\infty}^{\infty} q^{3n^2} \right),$$

$$C_2^2 = \frac{q^{1/12}}{\eta^2(q)} \left(\sum_{n=-\infty}^{\infty} q^{3n^2+n} \right),$$

$$C_0^2 = \frac{q^{1/3}}{\eta^2(q)} \left(\sum_{n=-\infty}^{\infty} q^{3n^2+2n} \right), \quad (2.71)$$

$$C_2^4 = C_2^0 = \frac{q^{3/4}}{\eta^2(q)} \left(\sum_{n=0}^{\infty} q^{3n^2+3n} \right),$$

$$C_1^1 + C_1^3 = C_3^3 + C_3^1 = 1/\eta(q^{\frac{1}{2}}).$$

According to Eq. (2.25) we define

$$\begin{aligned}
C_0 &\equiv C_0^0 + C_0^2 + C_0^4 = q^{-1/12} \sum P_0(N)q^N = q^{-1/12}(1 + q^{1/3} + \dots), \\
C_2 &\equiv C_2^0 + C_2^2 + C_2^4 = q^{-1/12} \sum P_1(N)q^N = q^{-1/12}(q^{1/12} + 2q^{3/4} + \dots), \\
C_1 &\equiv C_1^1 + C_1^3 = q^{-1/12} \sum P_{1/2}(N)q^N = q^{-1/12}(q^{1/16} + q^{9/16} + \dots), \\
C_3 &\equiv C_3^1 + C_3^3 = q^{-1/12} \sum P_{3/2}(N)q^N = C_1.
\end{aligned} \tag{2.72}$$

We also define the following combination of the string functions:

$$C_0 + C_2 = q^{-1/12} \sum P(N)q^N. \tag{2.73}$$

(a) The S module, or the $[\chi_0^0]$ module. This module is built from the highest weight state $|h; 0\rangle$. We define the level of a state as $N = N_0$, where N_0 is the level of the state above $|h; 0\rangle$. Equation (2.60) gives $N \in \mathbf{Z}$ or $N \in \mathbf{Z} + 1/3$. The S module falls into three submodules, $S^{(0)}$ and $S^{(\pm)}$, labeled according to the \mathbf{Z}_3 charge of the states. The $S^{(0)}$ submodule contains the states (with quantum numbers $j = 0, 2$ and $m = 0$) at integer levels, while the $S^{(\pm)}$ submodules consist of the states ($j = 1$ and $m = 0$) at levels $N \in \mathbf{Z} + 1/3$. The number of states at level N in each submodule is given by $P_0(N)$ defined in (2.72).

For the S module we define the Kac determinant at integer levels as that of the $S^{(0)}$ submodule, and at levels $N \in \mathbf{Z} + 1/3$ as that of the $S^{(-)}$ submodule. The Kac determinants for the $S^{(+)}$ and $S^{(-)}$ submodules are identical. We choose the normalization $\langle h; 0|h; 0\rangle = 1$.

The Kac determinant formula for the S module is given by

$$\det(\mathcal{M}_N^S) = \alpha^{Q(S)} \prod_{r,s} [\alpha(h - h_{r,s}^S)]^{P(N - rs/4)}, \tag{2.74}$$

where $r, s \in \mathbf{N}$, $s - r = 0 \pmod{4}$, $rs/4 \leq N$, and

$$Q(S) = \sum_n P_0(N - n^2). \tag{2.75}$$

Here $n \in \mathbf{N}$ and $n^2 \leq N$. The zeros $h_{r,s}^S$ of the Kac determinant (2.74) are given by Eq. (2.41):

$$h_{r,s}^S = \frac{c-2}{24} + \frac{1}{96} [(r+s)\sqrt{2-c} + (r-s)\sqrt{8-c}]^2. \tag{2.76}$$

Note that the order of the zeros $P(N - rs/4)$ comes from the sum $C_0 + C_2$ defined in Eq. (2.73), not just from C_0 .

We present a few low-lying levels explicitly to clarify the notation. There is $P_0(1/3) = 1$ state in the $S^{(-)}$ submodule at level $1/3$:

$$|\phi\rangle = G_{-\frac{1}{3}}^- |h; 0\rangle. \tag{2.77}$$

All the other states at this level are linearly dependent on (2.77). For instance, using the GCR (2.62) we have $G_0^+ G_{-\frac{1}{3}}^+ |h; 0\rangle = (\lambda^+/2) G_{-\frac{1}{3}}^- |h; 0\rangle$, and we choose the state (2.77), since it has the minimal number of the G

creation operators. The Kac determinant can be calculated using (2.58) and (2.63):

$$\det(\mathcal{M}_{1/3}^S) = \alpha(h - h_{1,1}). \tag{2.78}$$

At level 1 there are $P_0(1) = 2$ states in the $S^{(0)}$ submodule:

$$|\phi_1\rangle = L_{-1}|h; 0\rangle, \quad |\phi_2\rangle = G_{-\frac{2}{3}}^- G_{-\frac{1}{3}}^+ |h; 0\rangle. \tag{2.79}$$

The Kac determinant can be obtained using the commutation relations (2.58) and GCR's (2.62) and (2.63):

$$\det(\mathcal{M}_1^S) = \frac{4}{3}(h - h_{1,1})^2(h - h_{2,2}). \tag{2.80}$$

At the next level $N = 4/3$ there are $P_0(4/3) = 3$ states:

$$|\phi_1\rangle = L_{-1} G_{-\frac{1}{3}}^- |h; 0\rangle, \tag{2.81}$$

$$|\phi_2\rangle = G_{-1}^+ G_{-\frac{1}{3}}^+ |h; 0\rangle, \quad |\phi_3\rangle = G_{-\frac{4}{3}}^- |h; 0\rangle,$$

and the Kac determinant is given by

$$\det(\mathcal{M}_{4/3}^S) = \frac{4}{3}(h - h_{1,1})^2(h - h_{2,2})(h - h_{1,5})(h - h_{5,1}). \tag{2.82}$$

(b) The D module, or the $[\chi_1^1]$ module. This module (of the $[\chi_{K/4}^{K/4}]$ type) is built from the highest weight state $|h; -1\rangle$. We define the level of a state as $N = N_0 + 1/12$, where N_0 is the level of the state above $|h; -1\rangle$. Equation (2.60) gives $N \in \mathbf{Z} + 1/12$ or $N \in \mathbf{Z} + 3/4$. The D module falls into three submodules: $D^{(0)}$ and $D^{(\pm)}$. The $D^{(\pm)}$ submodules consist of the states ($j = 1$ and $m = 1$) at levels $N \in \mathbf{Z} + 1/12$, while the $D^{(0)}$ submodule contains the states ($j = 0, 2$ and $m = 1$) at levels $N \in \mathbf{Z} + 3/4$. The number of states at level N in each submodule is given by $P_1(N)$ defined in (2.72).

For the D module we define the Kac determinant at levels $N \in \mathbf{Z} + 1/12$ as that of the $D^{(-)}$ submodule, and at levels $N \in \mathbf{Z} + 3/4$ as that of the $D^{(0)}$ submodule. The Kac determinants for the $D^{(+)}$ and $D^{(-)}$ submodules are identical for $N > 1/12$. We choose the normalization $\langle h; -1|h; -1\rangle = 1$, $G_0^- |h; -1\rangle = \sqrt{h - c/24} |h; +1\rangle$.

The Kac determinant formula for the D module is given by

$$\begin{aligned}
\det(\mathcal{M}_N^D) &= \alpha^{Q(D)} (h - c/24)^{P_1(N)/2} \\
&\times \prod_{r,s} (h - h_{r,s}^D)^{P(N - rs/4)},
\end{aligned} \tag{2.83}$$

where $s - r = 2 \pmod{4}$, $rs/4 \leq N > 1/12$, and

$$Q(D) = P_1(N)/2 + \sum_n P_1(N - n^2). \quad (2.84)$$

Here $n^2 \leq N$. The zeros $h_{r,s}^D$ are given by

$$h_{r,s}^D = \frac{c}{24} + \frac{1}{96} [(r+s)\sqrt{2-c} + (r-s)\sqrt{8-c}]^2. \quad (2.85)$$

For example, at level $3/4 = 2/3 + 1/12$ there are $P_1(3/4) = 2$ states in the $D^{(0)}$ submodule:

$$|\phi_1\rangle = G_{-\frac{3}{2}}^+ |h; -1\rangle, \quad |\phi_2\rangle = G_{-\frac{3}{2}}^- G_0^- |h; -1\rangle, \quad (2.86)$$

and

$$\det(\mathcal{M}_{3/4}^D) = \alpha(h - c/24)(h - h_{1,3})(h - h_{3,1}). \quad (2.87)$$

At level $13/12 = 1 + 1/12$ there are $P_1(13/12) = 2$ states in the $D^{(-)}$ submodule:

$$|\phi_1\rangle = L_{-1} |h; -1\rangle, \quad |\phi_2\rangle = G_{-1}^+ G_0^- |h; -1\rangle, \quad (2.88)$$

and the Kac determinant reads

$$\det(\mathcal{M}_{13/12}^D) = \frac{4}{3}(h - c/24)(h - h_{1,3})(h - h_{3,1}). \quad (2.89)$$

We also give the first nontrivial level in the D module built from the highest weight states that are the eigenstates of the fractional supercurrent zero mode G_0 :

$$G_0 |h; g^\pm\rangle = g^\pm |h; g^\pm\rangle. \quad (2.90)$$

With our normalization of G_0 in Eq. (2.29),

$$G_0 = \begin{cases} G_0^+ + G_0^- & \text{if } c \leq 8, \\ i(G_0^- - G_0^+) & \text{otherwise.} \end{cases} \quad (2.91)$$

The eigenstates of G_0 read

$$|h; g^\pm\rangle = \begin{cases} \frac{1}{\sqrt{2}}(|h; -1\rangle \pm |h; +1\rangle) & \text{if } c \leq 8, \\ \frac{1}{\sqrt{2}}(|h; -1\rangle \pm i|h; +1\rangle) & \text{otherwise.} \end{cases} \quad (2.92)$$

They correspond to the eigenvalues $g^\pm = \pm\sqrt{h - c/24}$. At level $3/4$ we have the states

$$|\phi_1^\pm\rangle = G_{-\frac{3}{2}}^+ |h; g^\pm\rangle, \quad |\phi_2^\pm\rangle = G_{-\frac{3}{2}}^- G_0^- |h; g^\pm\rangle. \quad (2.93)$$

The corresponding Kac determinants read

$$\det(\mathcal{M}_{3/4}^{(\pm)}) = \alpha(h - h_{1,3})(h - h_{3,1}). \quad (2.94)$$

(c) The R module, or the $[\chi_{1/2}^{1/2} \oplus \chi_{3/2}^{3/2}]$ module. This module is built from the highest weight state $|h\rangle_R$ that is *not* an eigenstate of the G_0 operator. We define the level of a state in this module as $N = N_0 + 1/16$, where N_0 is the level of the state above $|h\rangle_R$. Equation (2.66) gives $N \in \mathbf{Z}/2 + 1/16$. (At integer levels $j = 1/2$ and $m = 1/2$, or $j = 3/2$ and $m = 3/2$, while at half odd

integer levels $j = 3/2$ and $m = 1/2$, or $j = 1/2$ and $m = 3/2$.) The number of states at level N is given by $P_{1/2}(N) + P_{3/2}(N) = 2P_{1/2}(N)$ defined in (2.72). We choose the normalization ${}_R\langle h|h\rangle_R = 1$.

The Kac determinant formula for the R module is given by

$$\det(\mathcal{M}_N^R) = [\alpha(h + 1/48 - c/24)]^{P_{1/2}(N)} \times \prod_{r,s} [(h - h_{r,s}^R)(h - h_{s,r}^R)]^{P_{1/2}(N - rs/4)}, \quad (2.95)$$

where $s - r = 1 \pmod{4}$. The zeros $h_{r,s}^R$ read

$$h_{r,s}^R = \frac{c}{24} - \frac{1}{48} + \frac{1}{96} [(r+s)\sqrt{2-c} + (r-s)\sqrt{8-c}]^2. \quad (2.96)$$

We give a few low-lying states in the R module. At level $1/16$ there are $2P_{1/2}(1/16) = 2$ states,

$$|\phi_1\rangle = |h\rangle_R, \quad |\phi_2\rangle = \tilde{G}_0 |h\rangle_R, \quad (2.97)$$

and the Kac determinant can be calculated using Eq. (2.58) and GCR (2.68):

$$\det(\mathcal{M}_{1/16}^R) = \alpha(h + 1/48 - c/24). \quad (2.98)$$

At level $9/16 = 1/2 + 1/16$ there are $2P_{1/2}(9/16) = 2$ states

$$|\phi_1\rangle = \tilde{G}_{-\frac{1}{2}} |h\rangle_R, \quad |\phi_2\rangle = \tilde{G}_{-\frac{1}{2}} \tilde{G}_0 |h\rangle_R. \quad (2.99)$$

The Kac determinant at this level is given by

$$\det(\mathcal{M}_{9/16}^R) = \frac{16}{9}\alpha(h + 1/48 - c/24) \times (h - h_{1,2})(h - h_{2,1}). \quad (2.100)$$

At level $17/16 = 1 + 1/16$ we have $2P_{1/2}(17/16) = 4$ states:

$$|\phi_1\rangle = L_{-1} |h\rangle_R, \quad |\phi_2\rangle = L_{-1} \tilde{G}_0 |h\rangle_R, \quad (2.101)$$

$$|\phi_3\rangle = \tilde{G}_{-1} |h\rangle_R, \quad |\phi_4\rangle = \tilde{G}_{-1} \tilde{G}_0 |h\rangle_R, \quad (2.102)$$

and the Kac determinant

$$\det(\mathcal{M}_{17/16}^R) = \frac{256}{81}(h + 1/48 - c/24)^2 (h - h_{1,2}) \times (h - h_{2,1})(h - h_{1,4})(h - h_{4,1}). \quad (2.103)$$

(d) The $R^{(\pm)}$ modules, or the $[\chi_{3/2}^{3/2}]$ and $[\chi_{1/2}^{1/2}]$ modules. $P_{1/2}(N)$ gives the number of states at level N in the $R^{(\pm)}$ modules built from the highest weight states $|h; g^\pm\rangle$:

$$G_0 |h; g^\pm\rangle = g^\pm |h; g^\pm\rangle. \quad (2.104)$$

With our normalization of G_0 in Eq. (2.34),

$$G_0 = \begin{cases} \tilde{G}_0 & \text{if } c \leq 8, \\ i\tilde{G}_0 & \text{otherwise.} \end{cases} \quad (2.105)$$

The eigenvalues g^\pm are given by

$$g^\pm = \frac{\sqrt{8-c}}{8\sqrt{6}} \pm \sqrt{h + 1/48 - c/24} . \quad (2.106)$$

The corresponding new determinant formulas read

$$\begin{aligned} \det(\mathcal{M}_N^{R(\pm)}) &= \alpha^{Q(R)} \prod_{r,s} [\alpha(g^\pm - g_{r,s})(g^\pm - g_{s,r})]^{P_{1/2}(N-rs/4)} , \\ & \quad (2.107) \end{aligned}$$

where r, s are the same as in (2.95) and

$$Q(R) = \sum_n P_{1/2}(N - n^2) . \quad (2.108)$$

Here $n^2 \leq N$.

The zeros of the determinants are given by

$$\begin{aligned} g_{r,s} &= \frac{\sqrt{8-c}}{8\sqrt{6}} \\ &+ \frac{1}{\sqrt{96}} \operatorname{sgn}(j-1) [(r+s)\sqrt{2-c} + (r-s)\sqrt{8-c}] , \\ & \quad (2.109) \end{aligned}$$

where $2j = (s-r) \bmod 4$.

As an example, we consider the $R^{(\pm)}$ -modules at the lowest nontrivial level $N = 9/16$. There is one state in each module:

$$|\phi^\pm\rangle = \tilde{G}_{-\frac{1}{2}} |h; g^\pm\rangle . \quad (2.110)$$

The new determinants can be calculated using Eq. (2.58) and the GCR (2.68)

$$\det(\mathcal{M}_{9/16}^{R(\pm)}) = \frac{4}{3} \alpha (g^\pm - g_{1,2})(g^\pm - g_{2,1}) . \quad (2.111)$$

The Kac determinant (2.100) is related to the new determinants (2.111) via

$$\begin{aligned} \det(\mathcal{M}_{9/16}^R) &= \alpha(h + 1/48 \\ &- c/24) \det(\mathcal{M}_{9/16}^{R(+)} \det(\mathcal{M}_{9/16}^{R(-)}) . \\ & \quad (2.112) \end{aligned}$$

This factorization generalizes to all levels.

Some of the examples of the Kac and new determinants presented in this section will be useful later for the discussion of the null state structure of the critical $K = 4$ fractional superstring theory. The Kac spectrum of highest weights $h_{r,s}$ for the $K = 4$ case was first given in Ref. [8].

III. DERIVATION OF KAC AND NEW DETERMINANT FORMULAS

In this section we derive the Kac and new determinant formulas for an arbitrary (integer) level K FSCA using

the BRST operator [9,10] in a particular representation of FSCA's via a noninteracting theory of the \mathbf{Z}_K PF and a single boson with the background charge. Since by definition the Kac and new determinants for a given algebra are representation independent, our derivation is valid for all the representations of FSCA's with modules classified in Sec. II.

In Sec. III A we deduce the highest weights of the primary fields for which there are null states in the modules. The Kac and new determinants vanish for these values of h . In Sec. III B we derive the conformal dimensions of the null states. The difference between these conformal dimensions and the highest weights of the primary fields gives the lowest levels at which the null states appear in the corresponding modules. There we also derive the multiplicities of the null states at higher levels. This fixes the orders of the zeros of the Kac and new determinants. In Sec. III C we point out the relation between the Kac and new determinants, and via this relation derive the contributions in the Kac determinants due to the fractional supercurrent zero mode.

A. BRST operators

We consider the representation of the level K FSCA constructed by turning on the background charge of the φ boson in the $SU(2)_K$ WZW model discussed in Sec. II. After the background charge is turned on, the energy-momentum tensor of the φ boson reads

$$T_\varphi = -\frac{1}{4}(\partial\varphi)^2 + i\alpha_0\partial^2\varphi . \quad (3.1)$$

The total energy-momentum tensor of the theory is given by

$$T = T_\varphi + T_{\text{PF}} , \quad (3.2)$$

where T_{PF} is the energy-momentum tensor of the \mathbf{Z}_K PF. The total central charge is

$$c = c_0 - 24\alpha_0^2 \leq c_0 . \quad (3.3)$$

Once we turn on the background charge, the $SU(2)_K$ symmetry is broken. However, the off-diagonal currents $J^\pm(z)$ [see Eq. (2.7)] can be modified so that they remain the spin-1 screening currents:

$$S^\pm(z) = \phi_{\pm 1}^0(z) \exp[i\alpha_\pm \varphi(z)] , \quad (3.4)$$

where

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + \frac{1}{K}} . \quad (3.5)$$

The only remaining symmetry that survives the presence of the background charge is the fractional superconformal symmetry. The fractional supercurrent G commutes with the screening charges $S^{(\pm)}$, i.e.,

$$S^\pm(z)G(w) \sim \frac{W(w)}{(z-w)^2} + \text{reg} , \quad (3.6)$$

where the single pole term is absent [$W(w)$ is some op-

erator] and G is given by

$$G = \frac{\sqrt{K}}{2} \left\{ [\epsilon \partial \varphi - i \alpha_0 (K + 2) \partial \epsilon] + \frac{iK(\alpha_+ - \alpha_-)}{K + 4} \eta \right\}. \quad (3.7)$$

If we take α_0 to zero we will recover Eq. (2.10).

The currents $T(z)$ and $G(z)$, defined in (3.2) and (3.7), generate the level K FSCA (2.11) for the values of the central charge $c \leq c_0$ [6].

Now we consider the BRST operators [9,10]

$$Q_p^{(\pm)} \equiv \prod_{i=1}^p \oint \frac{dz_i}{2\pi i} S^{\pm}(z_i), \quad (3.8)$$

where the z_j integration contour is inside of the z_i contour for $j > i$, and all contours start and end at z_1 . Since the screening currents, $S^{\pm}(z)$, are dimension-one operators and their OPE with $G(z)$ is given by (3.6), we conclude that $Q_p^{(\pm)}$ commute with $T(z)$ and $G(z)$:

$$\prod_{i=1}^p \exp[i\alpha_+ \varphi(z_i)] \exp[i\beta_{n,n'} \varphi(0)]$$

$$= \prod_{i < k} (z_i - z_k)^{2\alpha_+^2} \prod_{i=1}^p z_i^{2\alpha_+ \beta_{n,n'}} : \exp \left(i\alpha_+ \sum_{i=1}^p \varphi(z_i) + i\beta_{n,n'} \varphi(0) \right) : , \quad (3.12)$$

$$\Psi_1(z_1) \cdots \Psi_1(z_p) \phi_j^j(0) = \prod_{i < k} (z_i - z_k)^{-2/K} \prod_{i=1}^p z_i^{-2j/K} \phi_{j+p}^j(0) + \cdots . \quad (3.13)$$

[The latter equation follows from the fusion rules (2.6) and the conformal dimensions of the PF fields given in (2.4).] Then we make the change of variables $z_i \rightarrow z_1 \cdot u_i$ for $i = 2, 3, \dots, p$ and demand the exponent of z_1 be an integer (so that the z_1 contour closes). This gives $p = n$ and the above condition (3.11).

The conformal weight of the field $\chi_{j,n,n'}^j$ is given by

$$\Delta_{n,n'}^{(j)} = \frac{c - c_0}{24} + \frac{1}{96} \left[(n + n') \sqrt{c_0 - c} + (n - n') \sqrt{c_1 - c} \right]^2 + \Delta_j^j. \quad (3.14)$$

In the next subsection we show that for $n = r > 0$ and $n' = s > 0$ there are null states in the modules built from the highest weight states $|\chi_{j,r,s}^j\rangle$. Therefore, Eq. (3.14) gives the values of the highest weight h at which the Kac and new determinants vanish. Thus, for $n = r > 0$ and $n' = s > 0$ Eq. (3.14) gives (2.41).

B. Conformal weights and multiplicities of null states

To deduce the conformal weights and multiplicities of the null states appearing in the modules, we consider the

$$[Q_p^{(\pm)}, T(z)] = 0, \quad [Q_p^{(\pm)}, G(z)] = 0. \quad (3.9)$$

From the requirement that the primary fields of Eq. (2.9) remain primary with respect to $G(z)$ and $T(z)$, we find the following expression for the possible primary fields:

$$\chi_{j,n,n'}^j(z) = \phi_j^j(z) \exp[i\beta_{n,n'} \varphi(z)], \quad (3.10)$$

where $\beta_{n,n'} = \frac{1}{2}(1 - n) \alpha_+ + \frac{1}{2}(1 - n') \alpha_-$ and n, n' are integers satisfying the condition

$$2j = (n' - n) \bmod K. \quad (3.11)$$

This condition follows from the requirement that the BRST operators $Q_p^{(\pm)}$ be well defined on the Fock space. (There is no conceptual difference between $Q_p^{(+)}$ and $Q_p^{(-)}$, so from now on we concentrate only on $Q_p^{(+)}$.) The operation $Q_p^{(+)} |\phi_j^j \cdot \exp[i\beta_{n,n'} \varphi(0)]\rangle$ is well defined if and only if the outer z_1 contour closes. By the standard method we have

BRST mapping of the primary states onto the null states. The following diagram illustrates the BRST mapping:

$$\begin{array}{ccc} \chi_{-j,-r,-s}^j & & \\ \downarrow & & \\ \chi_{-j,-r,-s}^\ell & \xleftarrow{Q_r^{(+)}} & \chi_{-\ell,r,-s}^\ell \\ \downarrow & & \downarrow \\ \chi_{-j,-r,-s}^{j'} & \xleftarrow{Q_r^{(+)}} & \chi_{-\ell,r,-s}^{j'} \end{array}$$

The vertical axis measures the conformal dimension of the states: The vertical arrows indicate the action of the T and G current modes. The T current has $j = m = 0$ quantum numbers, while G has $j = 1$ and $m = 0$. Hence in the vertical direction only the spin j changes. The BRST operator $Q_r^{(+)}$ has $j = 0$ and $m = r$. Therefore, in the horizontal direction only the magnetic quantum number m varies. In particular, the action of $Q_r^{(+)}$ on a field increases its m quantum

number by r up to periodicity (2.5). In the diagram, $\chi_{m,n,n'}^j \equiv \phi_m^j(0) \cdot \exp[i\beta_{n,n'}\varphi(0)]|0\rangle$.

Consider the highest weight state $\chi_{-\ell,r,-s}^\ell$ with $r, s > 0$. According to Eq. (3.11) $2\ell = (s+r) \bmod K$. The BRST operator $Q_r^{(+)}$ maps this state onto the state $\chi_{-j,-r,-s}^\ell$ that is a descendent of the highest weight state $\chi_{-j,-r,-s}^j$:

$$\chi_{-j,-r,-s}^\ell = Q_r^{(+)} \chi_{-\ell,r,-s}^\ell. \quad (3.15)$$

Here $-j = (-\ell + r) \bmod K = -\binom{s-r}{2} \bmod (K/2)$ in agreement with the condition $2j = \binom{s-r}{2} \bmod K$ following from (3.11). Since the operator $Q_r^{(+)}$ commutes with T and G , the positive modes of T and G annihilate $\chi_{-j,-r,-s}^\ell$, because they annihilate the primary state $\chi_{-\ell,r,-s}^\ell$. This, in particular, means that $\chi_{-j,-r,-s}^\ell$ is a primary and descendent at once, hence it is null. The dimension of this null state is the same as that of $\chi_{-\ell,r,-s}^\ell$ and equals $\Delta_{r,-s}^{(\ell)}$.

The modules $[\chi_{-j,-r,-s}^j]$ and $[\chi_{j,r,s}^j]$ [$2j = (s-r) \bmod K$], built from the highest weight states $\chi_{-j,-r,-s}^j$ and $\chi_{j,r,s}^j$, are dual. The duality means that each state in $[\chi_{-j,-r,-s}^j]$ has a dual state in $[\chi_{j,r,s}^j]$ and structurally these modules are identical. In particular, the highest weight state $\chi_{j,r,s}^j$ is dual to $\chi_{-j,-r,-s}^j$ in the sense that the two-point correlation function $\langle \chi_{-j,-r,-s}^j | \chi_{j,r,s}^j \rangle = 1$: The highest weights of the states $\chi_{-j,-r,-s}^j$ and $\chi_{j,r,s}^j$ are equal. Thus, the null state $\chi_{-j,-r,-s}^\ell$ appearing in the module $[\chi_{-j,-r,-s}^j]$ is dual to the null state $\chi_{j,r,s}^\ell$ of the module $[\chi_{j,r,s}^j]$. These null states have the same conformal dimensions. The level of the state $\chi_{j,r,s}^\ell$ above the highest weight state $\chi_{j,r,s}^j$, i.e., the lowest level in the module $[\chi_{j,r,s}^j]$ at which this null state appears with multiplicity one, is given by

$$N_0^{r,s} = \Delta_{r,-s}^{(\ell)} - \Delta_{r,s}^{(j)} = \frac{rs}{K} + \Delta_\ell^\ell - \Delta_j^j. \quad (3.16)$$

The modified level of this state, that we defined in Eq. (2.23), is then

$$N_{r,s} = \frac{rs}{K} + \Delta_\ell^\ell. \quad (3.17)$$

The descendents of the null state $\chi_{-j,-r,-s}^\ell$, which is primary, are null states as well. However, those are not primary. There is a one-to-one correspondence between the descendents of the primary state $\chi_{-\ell,r,-s}^\ell$ and the descendents of the null state $\chi_{-j,-r,-s}^\ell$ manifested in the BRST mapping. Thus, the number of the descendents of $\chi_{-j,-r,-s}^\ell$ with spin j' at some level N in the module $[\chi_{-j,-r,-s}^j]$ is the same as the number of the descendents of $\chi_{-\ell,r,-s}^\ell$ with the same spin j' at the level $\tilde{N}_0 = N - N_{r,s}$ above the highest weight state $\chi_{-\ell,r,-s}^\ell$ in the module $[\chi_{-\ell,r,-s}^\ell]$. The modified level, defined in Eq. (2.23), corresponding to \tilde{N}_0 is then $\tilde{N} = \tilde{N}_0 + \Delta_\ell^\ell = N - rs/K$. The number of descen-

dents of $\chi_{-\ell,r,-s}^\ell$ with spin j' at the level \tilde{N} is given by $P_\ell^{j'}(\tilde{N}) [\equiv P_\ell^{j'}(\tilde{N})]$, where $P_m^j(N)$ are defined in Eq. (2.25). Because of duality, the counting of descendents of the null state $\chi_{j,r,s}^\ell$ in the module $[\chi_{j,r,s}^j]$ is the same as the counting of descendents of the null state $\chi_{-j,-r,-s}^\ell$ in the module $[\chi_{-j,-r,-s}^j]$. The orders of the zeros of the Kac and new determinants are precisely given by this counting.

Note that only the highest weight states in the BRST cohomology are the true highest weight states [10,11]. In this sense, only the states $\chi_{j,r,s}^j$ and $\chi_{-j,-r,-s}^j$ with $r, s > 0$ are the highest weight states, whereas the state $\chi_{-\ell,r,-s}^\ell$ is not. The highest weight states $\chi_{j,r,s}^j$ and $\chi_{-j,-r,-s}^j$ are dual to each other, that is, if $\chi_{j,r,s}^j$ is a bra vector in the Fock space, then $\chi_{-j,-r,-s}^j$ is its corresponding ket vector. The φ momenta of these states add up to $2\alpha_0$ to make up for the presence of the background charge so that their two-point correlation function is nonvanishing.

Now we turn to the Kac determinant formulas deriving whose explicit form is our primary goal here. We are ready to write down the Kac determinant formula (up to a positive normalization constant independent of \hbar and c) for the module $[\chi_0^0]$ of an arbitrary level K FSCA. Since the fractional supercurrent zero mode G_0 does not act on the highest weight state in this module, the determinant of inner products of states is a polynomial in \hbar at the lowest nontrivial level. Then by induction it is a polynomial at all the higher levels. This immediately follows from the counting of the null states derived using the BRST operator. Thus, we arrive at (2.26).

In other modules, however, the zero mode G_0 acts on the highest weight states. As pointed out in Sec. II, the determinants of inner products of states are polynomials in \hbar only for some modules, but not for all. In the next subsection we resolve this issue and derive the contribution of the zero mode G_0 into the Kac determinant via the relation between the Kac and more fundamental new determinants.

C. Zero mode contribution

In this subsection we derive the zero mode contribution. This will complete our derivation of the Kac determinant formulas for the $[\chi_{K/4}^{K/4}]$ and $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$ modules ($0 < j < K/4$) and the new determinant formulas for the $[\chi_j^j]$ modules ($0 < j \neq K/4$).

Consider the module $[\chi_j^j]$, $0 < j \neq K/4$. The form of the zero mode algebra is fixed by the OPE's (2.11) and the structure constant $\lambda(c)$ (2.12),

$$\left[G_0^2 - 2a^j \sqrt{c_1 - c} G_0 - (L_0 - \tilde{h}^j) \right] |\chi_j^j\rangle = 0, \quad (3.18)$$

where we choose the normalization \mathcal{N} of the G_0 operator

$$G(z)\chi_j^j(0) = \mathcal{N} z^{-(K+4)/(K+2)} G_0 \chi_j^j(0) + \dots, \quad (3.19)$$

so that in (3.18) there are only two parameters a^j and

\tilde{h}^j . The G_0 mode is diagonal with respect to the highest weight state $|\chi_j^j\rangle$,

$$G_0|\chi_j^j\rangle = g^j|\chi_j^j\rangle, \quad (3.20)$$

and Eq. (3.18) becomes

$$(g^j)^2 - 2a^j g^j \sqrt{c_1 - c} - (h - \tilde{h}^j) = 0, \quad (3.21)$$

where h is the highest weight of $|\chi_j^j\rangle$.

To fix the quantities a^j and \tilde{h}^j , we consider the eigenvalues $g_{r,s}$ of the G_0 operator corresponding to the highest weight states $|\chi_{j,r,s}^j\rangle$ created by the vertex operators (3.10) in the representation (3.7) of FSCA via the \mathbf{Z}_K PF and a single boson with background charge. We have

$$G_0|\chi_{j,r,s}^j\rangle = g_{r,s}|\chi_{j,r,s}^j\rangle. \quad (3.22)$$

We determine the form of $g_{r,s}$ from the explicit representation of the $G(z)$ current via (3.7) and the primary fields $\chi_{j,r,s}^j$ via (3.10):

$$g_{r,s} = u\sqrt{c_1 - c} + vb_{r,s}. \quad (3.23)$$

Here we introduced c -independent quantities u and v , and

$$b_{r,s} \equiv \frac{1}{\sqrt{96}} [(r+s)\sqrt{c_1 - c} + (r-s)\sqrt{c_1 - c}]. \quad (3.24)$$

The quantities (3.23) must satisfy Eq. (3.21) with $h = \Delta_{r,s}^{(j)}$, where $\Delta_{r,s}^{(j)}$ is the conformal dimension of the primary field $\chi_{j,r,s}^j$ and is given by Eq. (3.14). This condition completely determines a^j and \tilde{h}^j :

$$u = a^j = \frac{|K - 4j|}{\sqrt{24(K+4)}}, \quad (3.25)$$

and

$$\tilde{h}^j = \frac{c}{24} - c(a^j)^2 = \frac{c(j+1)(K+2-2j)}{3(K+4)^2}. \quad (3.26)$$

For the $[\chi_j^j]$ module, $0 < j \neq K/4$, we have $v = \text{sgn}(j - K/4)$ and we arrive at Eq. (2.43). For the $[\chi_{K/4}^{K/4}]$ module $v = \pm 1$ for each pair r, s :

$$g_{r,s}^{(1,2)} = \pm b_{r,s}. \quad (3.27)$$

Once we have derived a^j and \tilde{h}^j , we can solve Eq. (3.21):

$$g^j = a^j \sqrt{c_1 - c} + \text{sgn}(j - K/4) \sqrt{h - h_0^j}, \quad (3.28)$$

where

$$h_0^j = \frac{c - c_0}{24} + \Delta_j^j = \frac{c}{24} - \frac{(K-4j)^2}{8K(K+2)}. \quad (3.29)$$

Note that if $j = K/4$, the coefficient a^j vanishes and

$$h_0^{K/4} = \tilde{h}^{K/4} \equiv c/24. \quad (3.30)$$

Thus, in the module $[\chi_{K/4}^{K/4}]$,

$$g^\pm = \pm \sqrt{h - c/24}. \quad (3.31)$$

The determinants of inner products of states, corresponding to the eigenvalues g^\pm , read

$$\begin{aligned} \det[\mathcal{M}_{(\pm)}^{(K/4)}(N)] \\ = \alpha^{Q(K/4)} \prod_{r,s} [(g^\pm - g_{r,s}^{(1)})(g^\pm - g_{r,s}^{(2)})]^{P_\ell(N - r_s/K)}, \end{aligned} \quad (3.32)$$

where r, s, ℓ, N , and $Q(K/4)$ are the same as in Eq. (2.30).

Note that $(g^\pm - g_{r,s}^{(1)})(g^\pm - g_{r,s}^{(2)}) = (g^\pm)^2 - (g_{r,s}^{(1)})^2 = h - h_{r,s}$. This means that the determinants

$$\det[\mathcal{M}_{(\pm)}^{(K/4)}(N)] = \alpha^{Q(K/4)} \prod_{r,s} (h - h_{r,s})^{P_\ell(N - r_s/K)} \quad (3.33)$$

are the Kac determinants since they are polynomials in the highest weight h . If G_0 is not diagonal with respect to the highest weight state, there is the additional zero mode contribution in the Kac determinant. This contribution is given by $(h - c/24)^{P_{K/4}(N)/2}$. Thus, we arrive at Eq. (2.32).

The new determinant for the module $[\chi_j^j]$ ($0 < j \neq K/4$) is a polynomial in the G_0 eigenvalue g^j and is given by Eq. (2.37).

Consider the $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$ module ($0 < j < K/4$) built from the highest weight state

$$|h\rangle = \gamma|h; g^j\rangle + \delta|h; g^{K/2-j}\rangle, \quad (3.34)$$

where $\gamma\delta \neq 0$ and

$$\langle h; g^j | h; g^{j'} \rangle = \delta_{j,j'}. \quad (3.35)$$

We label the operators creating the states at level N by V_i , $i = 1, \dots, P_j(N)$, where $P_j(N)$ is the number of states at this level. Thus, the states in the $[\chi_j^j]$ module have the form

$$|\phi_i\rangle = V_i |h; g^j\rangle. \quad (3.36)$$

The new determinant reads

$$\det[\mathcal{M}^{(j)}(N)] = \det(\langle \phi_i | \phi_k \rangle) = \det(\langle h; g^j | Z_{ik} | h; g^j \rangle)$$

$$Z_{ik} \equiv V_i^\dagger V_k. \quad (3.37)$$

The Kac determinant for the $[\chi_j^j \oplus \chi_{K/2-j}^{K/2-j}]$ module is given by the $[2P_j(N)] \times [2P_j(N)]$ matrix

$$\begin{aligned}
\det[\mathcal{M}_0^{(j)}(N)] &= \begin{vmatrix} \langle h|Z_{ik}|h\rangle & \langle h|Z_{ik'}G_0|h\rangle \\ \langle h|G_0^\dagger Z_{i'k}|h\rangle & \langle h|G_0^\dagger Z_{i'k'}G_0|h\rangle \end{vmatrix} \\
&= [|\gamma\delta|^2 \alpha(g^j - g^{K/2-j})^2]^{P_j(N)} \det[\mathcal{M}^{(j)}(N)] \det[\mathcal{M}^{(K/2-j)}(N)] \\
&= [2|\gamma\delta|^2 \alpha(h - h_0^j)]^{P_j(N)} \det[\mathcal{M}^{(j)}(N)] \det[\mathcal{M}^{(K/2-j)}(N)].
\end{aligned} \tag{3.38}$$

This completes the derivation of the Kac and new determinant formulas.

Note, that if $h = h_0^j$, the state $G_0|h\rangle$ becomes null and the Kac determinant vanishes, whereas the new determinants are nonzero.

IV. MINIMAL UNITARY SERIES IN FSCA

In this section we deduce the values of c and h for which FSCA's can have unitary representations. Statistical mechanical systems near the second-order phase transition are always expected to be described by an effective unitary field theory with a local order parameter. For the remainder in this section we will thus confine our attention to unitary theories. We analyze unitary representations of FSCA's using the Kac determinants (2.26), (2.32), and (2.39). If for a given module the Kac determinant is negative at any level, it means that there are negative-norm states at that level and the representation is not unitary. If the determinant is greater than or equal zero, further investigation is needed to determine whether or not the representation is unitary. We sketch the nonunitarity proof for FSCA's. Our proof is closely parallel to that for the Virasoro and superconformal minimal series [2,4].

We consider an arbitrary level K FSCA. In the region $c_0 < c < c_1$, all $h_{r,s}$ with $r \neq s$ have nonvanishing imaginary parts, while all $h_{r,r} < 0$. For $c = c_1$ all zeros of the Kac determinant are real and satisfy the inequality

$$\begin{aligned}
h_{r,s} &\leq -j(j+1)(K+4)/K(K+2) \leq 0, \\
2j &= (s-r) \bmod K. \tag{4.1}
\end{aligned}$$

This means that in the region $c_0 < c \leq c_1$, $h > 0$ the Kac determinant is nonvanishing and that all of the Kac matrix eigenvalues are positive. Indeed, as $h \rightarrow \infty$, the matrix becomes dominated by its diagonal elements that are strictly positive. On the boundary $c = c_0$ we have

$$h_{r,s} = h_{s,r} = (r-s)^2/4K + \Delta_j^j. \tag{4.2}$$

Therefore, for $c = c_0$ and $h > 0$ the Kac determinant vanishes at the points $h = n^2/4K + \Delta_j^j$, $n \in \mathbf{N}$, but does not become negative. Thus, the Kac determinant poses no obstacle in principle to having unitary representations of FSCA in the region $c_0 \leq c \leq c_1$, $h > 0$. The only unitary representation with $h = 0$ is the trivial one with $c = 0$, while there are no unitary representations with $h < 0$.

When $c > c_1$ (2.11) shows that the G current is anti-Hermitian and the structure constant $\lambda(c)$ is imaginary. Therefore, all FSCA's with $c > c_1$ are necessarily nonunitary unless c_{111} , defined in Sec. III, vanishes. When

$K = 1$ or 2 this is exactly the case and there are no analogs of nonunitarity proof for $c > 25$ representations of the Virasoro algebra and $c > 27/2$ representations of the superconformal algebra.

In the region $0 < c < c_0$, $h > 0$ the Kac determinant is definitely negative at some level except for the points (c, h) lying on the vanishing curves $h = h_{r,s}(c)$ where the determinant becomes zero. Even on these curves, however, all points, but the ones where they intersect, have ghosts. This discrete set of intersection points, where unitary representations of the FSCA are not excluded, occur at the following values of the central charge:

$$c_p = c_0 - \frac{6K}{p(p+K)} = \frac{3K}{K+2} \left(1 - \frac{2(K+2)}{p(p+K)} \right), \tag{4.3}$$

$p = 3, 4, \dots$

($p = 2$ is the trivial theory with $c = 0$). For each p the allowed values of h are given by

$$h_{r,s} = \frac{[(p+K)r - ps]^2 - K^2}{4Kp(p+K)} + \Delta_j^j, \tag{4.4}$$

where $1 \leq r < p$, $1 \leq s < p+K$.

Thus, we see that the necessary conditions for unitary highest weight representations of the level K FSCA are either (for $K = 1, 2$ there is no upper bound on the central charge)

$$\frac{3K}{K+2} \leq c \leq \frac{3(K+4)^2}{K(K+2)}, \quad h \geq 0, \tag{4.5}$$

or (4.3) and (4.4). That the latter two conditions are also sufficient, i.e., that there indeed exist unitary representations of FSCA's for these discrete values of c and h , was shown in Ref. [10] via the Goddard-Kent-Olive (GKO) coset space construction [12]. The fractional superconformal unitary minimal model with the central charge c_p , $p \geq 3$, can be realized by the $SU(2)_K \otimes SU(2)_{p-2}/SU(2)_{K+p-2}$ coset model. An example of a statistical mechanical system with $K = 4$ fractional supersymmetry is the tricritical 3-state Potts model.

V. NO-GHOST THEOREM AND NULL STATE STRUCTURE OF FRACTIONAL SUPERSTRING

A. No-ghost theorem for subcritical $K = 4$ FSS

In this subsection we discuss the no-ghost theorem for the space-time bosonic sector of the subcritical spin-4/3 fractional superstring (FSS). We generalize the Brower-Thorn proof of the no-ghost theorem for the bosonic string [3] using the Kac determinant formulas for the

$K = 4$ FSCA presented in Sec. II. Our discussion, based on a reasonable assumption, closely parallels that of Ref. [3], so we shall be brief.

The physical state conditions for the space-time bosonic sector of the spin-4/3 FSS read

$$L_n|\text{phys}; q\rangle = G_{n-(1\mp q)/3}^\pm|\text{phys}; q\rangle = 0, \quad (5.1)$$

$$L_0|\text{phys}; q\rangle = v_q|\text{phys}; q\rangle, \quad n > 0,$$

where v_q is the intercept of the physical states $|\text{phys}; q\rangle$ with the \mathbf{Z}_3 charge q . Since

$$|\text{phys}; +1\rangle = G_0^-|\text{phys}; -1\rangle, \quad (5.2)$$

we have $v_{+1} = v_{-1}$.

Our primary interest in this section is the $c = 5$ representation of the spin-4/3 FSS realized by three free bosons and the $\text{SO}(2,1)_2$ WZW theory [14]. This representation has three-dimensional flat Minkowski space-time as its target space, i.e., $\text{SO}(2,1)$ global Lorentz symmetry, allowing for the particle interpretation of its scattering amplitudes. The tree level scattering of the physical states does not couple them to the spurious states. In this section we argue that the bosonic physical spectrum of this theory is free of negative norm states. This, in particular, would mean that the tree level scattering is unitary.

The Fock space of this string theory consists of the states

$$\begin{aligned} & | \{ \lambda \}, \{ \rho \}, \chi \rangle \\ & \equiv (\alpha_{-n_1}^0)^{\lambda_1} \dots (\alpha_{-n_p}^0)^{\lambda_p} (\epsilon_{-r_1}^{(\pm)0})^{\rho_1} \dots (\epsilon_{-r_s}^{(\pm)0})^{\rho_s} | \chi; q \rangle, \end{aligned} \quad (5.3)$$

where the ordering is the same as in (2.19). Here $\alpha_{-n_i}^0$, $i = 1, \dots, p$, are the timelike ($\mu = 0$) creation operators of the world-sheet boson $X^\mu(z)$, $\mu = 0, 1, 2$, while $\epsilon_{-r_k}^0$, $k = 1, \dots, s$, are the timelike creation operators of the spin-1/3 world-sheet field $e^\mu(z)$. The state $|\chi\rangle$ is created from the state $\exp[ik \cdot X(0)]|0\rangle$ with the momentum k^μ by the spacelike creation operators of the fields $X^i(z)$ and $\epsilon^i(z)$, $i = 1, 2$. Even though the spacelike components of $\epsilon^\mu(z)$ couple to the timelike component, $|\chi\rangle$ is assumed to be a positive norm state since $\text{SO}(3)_2$ is free of ghosts and the spacelike components do not change when we rotate $\text{SO}(3)_2$ to $\text{SO}(2,1)_2$ (the bosonic spacelike creation operators certainly do not spoil unitarity). The timelike operators α_{-n}^0 and ϵ_{-r}^0 create negative norm states, i.e., ghosts, in the Fock space of the FSS. The goal of the no-ghost theorem is to prove that there are no ghosts among the physical states that satisfy the above physical state conditions.

The states (5.3) are at the level

$$N_0 = \sum_{i=1}^p \lambda_i n_i + \sum_{k=1}^s \rho_k r_k \quad (5.4)$$

above the state $|\chi\rangle$. The metric tensor of these states is given by

$$\begin{aligned} & T_{N_0}(\{ \lambda' \}, \{ \rho' \}; \{ \lambda \}, \{ \rho \}) \\ & = \delta_{\lambda_1, \lambda'_1} \dots \delta_{\lambda_p, \lambda'_p} \delta_{\rho_1, \rho'_1} \dots \delta_{\rho_s, \rho'_s} (-1)^{\nu(p,s)}, \end{aligned} \quad (5.5)$$

where

$$\nu(p, s) = \sum_{i=1}^p \lambda_i + \sum_{k=1}^s \rho_k. \quad (5.6)$$

The Fock space \mathcal{F} that consists of the states (5.3) can be decomposed into the subspace \mathcal{R} of the primary states $|h; q\rangle$, defined as

$$\begin{aligned} & L_0|h; q\rangle = h|h; q\rangle, \quad L_n|h; q\rangle = G_{n-(1\mp q)/3}^\pm|h; q\rangle = 0, \\ & n > 0, \end{aligned} \quad (5.7)$$

and the subspace of spurious states \mathcal{S} , defined as the orthogonal complement of \mathcal{R} in \mathcal{F} . Thus, $\mathcal{F} = \mathcal{R} \oplus \mathcal{S}$. The subspace \mathcal{S} is spanned by the states

$$L_{-n_1}^{\lambda_1} \dots L_{-n_p}^{\lambda_p} (G_{-r_1}^\pm)^{\rho_1} \dots (G_{-r_s}^\pm)^{\rho_s} |h; q\rangle \quad (5.8)$$

[where the ordering is the same as in (5.3) and at least one of the numbers λ_i or ρ_k is nonzero] provided that their Kac determinant is nonvanishing (the latter condition ensures that these states are linearly independent). The physical states then are the states with $h = v_q = N_0 + \Delta_\chi$, where Δ_χ is the L_0 eigenvalue of $|\chi; q\rangle$.

Consider the physical states with the intercept v_q and the subspace \mathcal{S}_q of the spurious state space \mathcal{S} that consists of the states with the L_0 eigenvalue v_q and the \mathbf{Z}_3 charge q . According to the Kac determinant formulas for the $K = 4$ FSCA, as $h \rightarrow -\infty$, the metric tensor of the states in \mathcal{S}_q at level N_0 coincides with (5.5). If we now require the condition

$$\det(\mathcal{M}_N) \neq 0 \quad \text{for all } h < v_q - N_0 \quad (5.9)$$

be satisfied at any level $N_0 > 0$, then the counting of states in \mathcal{S}_q at level N_0 , given by (5.4), is the same as the counting of states (5.3) with the L_0 eigenvalue v_q . Moreover, the metric of spurious states in \mathcal{S}_q coincides with (5.5) at any level since the Kac determinant is a polynomial in h and the condition (5.9) is satisfied. This, in particular, means that the number of negative norm states in the Fock space with the L_0 eigenvalue v_q is the same as the number of negative norm states in \mathcal{S}_q . The latter subspace by definition consists only of spurious states, therefore, there are no ghosts among the above physical states. In addition to the above condition we should also check the level zero for the absence of ghosts because the G_0^\pm zero modes may create negative-norm physical states. Note that the above analysis can be applied to any representation of the spin-4/3 FSS with only one timelike direction. Equation (5.9) is then a sufficient condition for the absence of ghosts in the physical spectrum.

The \mathbf{Z}_3 charge q is a quantum number conserved in the tree level scattering processes. We refer to the physical

states with $q = \pm 1$ as the V sector, and the physical states with $q = 0$ as the T sector. The lowest-lying V -sector state is a massless vector particle, whereas its T -sector counterpart is a tachyon.

The states that are both spurious and physical are the so-called null states, i.e., they have zero norm. The null states in the V sector come from the $S^{(\pm)}$ and $D^{(\pm)}$ submodules; the null states in the T sector are those in the $S^{(0)}$ and $D^{(0)}$ submodules.

Now we turn to the ghost structure in the V sector. If $c < 8$ the only real zeros of the Kac determinant relevant to the V sector appear in the $S^{(\pm)}$ submodules and are given by

$$h_{r,r} = \frac{c-2}{24}(1-r^2) > \frac{1}{4} - \frac{r^2}{4}, \quad (5.10)$$

where r is an odd integer. On the other hand, $N_0 \geq N^{r,r} = r^2/4 + 1/12$ and $v_{\pm 1} - N_0 < v_{\pm 1} - 1/12 - r^2/4$. Therefore, taking into account the inequality (5.10) we conclude that in the V sector the condition (5.9) is always satisfied for intercepts $v_{\pm 1} < 1/3$ in representations with $c < 8$. However, in this sector the zero modes may spoil unitarity. From (5.2) we find that if $|\text{phys}; -1\rangle$ is a positive-norm physical state, then $|\text{phys}; +1\rangle$ has the norm

$$\langle \text{phys}; +1 | \text{phys}; +1 \rangle = (v - c/24) \langle \text{phys}; -1 | \text{phys}; -1 \rangle, \quad (5.11)$$

that is non-negative if and only if $v \geq c/24$. Thus, we have the absence of ghosts in the V sector for the following range of $v \equiv v_{\pm 1}$ and c :

$$V \text{ sector: } c/24 \leq v \leq 1/3, \quad c \leq 8. \quad (5.12)$$

The case $v = 1/3$ and/or $c = 8$ was included by continuity.

Next we consider the T sector. If $c < 8$, the only real zeros of the Kac determinant relevant to the T sector appear in the $S^{(0)}$ submodule and are given by (5.10) with r being an even integer. We have the absence of ghosts in the T sector for the following values of $v \equiv v_0$ and c :

$$T \text{ - sector: } v \leq (10 - c)/8, \quad c \leq 8. \quad (5.13)$$

For $c < 8$ the condition (5.9) is automatically satisfied in the R sector, because none of the zeros of the Kac determinant are real in this region. The zero mode contribution restricts the intercept to be greater than $c/24 - 1/48$. Thus, we have the absence of ghosts in the R sector for the following values of v_R and c :

$$R \text{ - sector: } v_R \geq c/24 - 1/48, \quad c \leq 8. \quad (5.14)$$

B. Null state structure of critical FSS

Any consistent string theory is expected to have extra sets of physical null states at the critical central charge. It is yet unclear if fractional superstrings exist as con-

sistent string theories. Nonetheless, the Kac determinant formulas for FSCA's allow us to examine the null state structure of plausible critical FSS with an arbitrary level K world-sheet fractional supersymmetry. In particular, we recover the well-known results for the bosonic ($K = 1$) and superstring ($K = 2$) theories. We analyze the null state structure of the spin-4/3 (i.e., $K = 4$) FSS at the critical central charge and find extra sets of zero-norm physical states.

First we review the discussion of Ref. [9]. The zeros of the Kac determinants for the level K FSCA are given by

$$h_{r,s} = \frac{c - c_0}{24} + \frac{1}{96} [(r+s)\sqrt{c_0 - c} + (r-s)\sqrt{c_1 - c}]^2 + \Delta_j^j. \quad (5.15)$$

Here $r, s \in \mathbf{N}$; $c_0 = 3K(K+2)$ and $c_1 = 24/K + c_0$; $\Delta_j^j = j(K-2j)/K(K+2)$, where $2j = (s-r) \bmod K$. The Kac determinant for the modules with $j \neq 0$ also vanishes at $h_0^j = (c - c_0)/24 + \Delta_j^j$ (see Sec. III).

The zero $h = h_{r,s}$ (i.e., the null state) first appears at the level

$$N_0^{r,s} = \frac{rs}{K} + \Delta_\ell^\ell - \Delta_j^j, \quad (5.16)$$

where $2\ell = (s+r) \bmod K$. The intercept of a physical null state is given by

$$v = h_{r,s} + N_0^{r,s}. \quad (5.17)$$

Regardless of the central charge, there always is a null state at level $N_0^{1,1}$ above the highest weight state $|h_{1,1}\rangle = |0\rangle$. For $K > 1$ the intercept of this state is

$$v = \frac{2}{K+2}; \quad (5.18)$$

for $K = 1$ we have $v = 1$. This physical null state belongs to the sector of the FSS theory that contains the massless vector particle (in open FSS) or the graviton (in closed FSS).

The second set of physical null states that belong to the same sector is built from the highest weight state $|h_{1,K+1}\rangle$. Their intercept must be the same as that of Eq. (5.18). On the other hand, according to Eq. (5.17)

$$v = h_{1,K+1} + N_0^{1,K+1}. \quad (5.19)$$

This set of constraints has a solution only at the critical central charge [9]

$$c_{\text{critical}} = c_1 + c_0 = \frac{6K}{K+2} + \frac{24}{K}. \quad (5.20)$$

For the bosonic string theory $c_{\text{critical}} = 26$, $v = 1$. For the space-time bosonic sector of the superstring theory we also obtain the well-known result: $c_{\text{critical}} = 15$, $v = 1/2$. In the fermionic, or Ramond sector of the superstring theory there also are two sets of physical null states. The first appears at level zero due to the vanishing of the Kac determinant at $h = c/24$. This gives the Ramond sector intercept $v_R = c/24 = 5/8$. The second set with the same intercept v_R appears at the level

$N_0^{1,2} = 1$ above the highest weight state $|h_{1,2} = -3/8\rangle$ ($c = 15$).

Equation (5.20) gives $c_{\text{critical}} = 10$ for the $K = 4$ FSS. According to Eq. (5.18) the intercept of the V -sector physical states is $v = 1/3$.

In the T sector at $c = 10$ there also are two sets of physical null states with the intercept $v = 1/3$. The first one occurs at level $N_0^{1,3} = 2/3$ in the $D^{(0)}$ submodule built from the highest weight state $|h_{1,3} = -1/3\rangle$. However, the second set appears at a rather high level $N_0^{2,6} = 3$ in the $S^{(0)}$ submodule. It is unclear if this will pose a problem for unitarity. The no-ghost theorem for the critical $K = 4$ FSS is needed to answer this question. In any case, it is likely that the T sector (that contains the tachyonic ground state) should be projected out in any consistent string model.

There is another possibility if the T and V sectors can have different intercepts. There are two sets of physical null states in the T sector at $c = 10$ and $v = 0$. The first set appears at level $N_0^{2,2} = 1$ in the $S^{(0)}$ submodule built from the highest weight state $|h_{2,2} = -1\rangle$, and the other one occurs at level $N_0^{1,7} = 5/3$ in the $D^{(0)}$ submodule built from the highest weight state $|h_{1,7} = -5/3\rangle$. The ground state in this case is no longer tachyonic, but massless.

Extra sets of physical null states also occur in the R sector at $c = 10$ and the intercept $v = 3/8$. The first set appears at level $N_0^{1,2} = 1/2$ above the highest weight state $|h_{1,2} = -1/8\rangle$, and the other one occurs at level $N_0^{1,4} = 1$ above the highest weight state $|h_{1,4} = -5/8\rangle$.

The above analysis of the physical null state structure of the $K = 4$ FSS indicates that at least the V and R

sectors can be expected to be free of ghosts at the critical central charge.

VI. CONCLUSIONS

In this paper we presented and derived the Kac and new determinant formulas for an arbitrary integer level K fractional superconformal algebra. Thus, although complicated, the FSCA's can be studied using tools developed in conformal field theories. Now we know the Kac and new determinant formulas for infinitely many algebras, and only for three of them these determinants can be explicitly calculated. These are the (super)Virasoro and the spin-4/3 parafermion current algebras, for which the (generalized) (anti)commutation relations are known. For the rest of the FSCA's the generalized commutation relations have not yet been written down because of the complications due to the non-Abelian braiding properties of these algebras.

Since all of the necessary tools for the rational level K FSCA's have been worked out [18] (namely, the $\mathbf{Z}_{p/q}$ parafermion theory, the rational level K string functions, and the BRST operators), it is straightforward to generalize the Kac and new determinants to those algebras.

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- [1] V.G. Kac, in *Group Theoretical Methods in Physics*, edited by W. Beiglbock *et al.* (Springer, New York, 1979), p. 441.
 - [2] D. Friedan, Z. Qiu, and S. Shenker, *Phys. Rev. Lett.* **52**, 1575 (1984).
 - [3] R.C. Brower and C.B. Thorn, *Nucl. Phys.* **B31**, 163 (1971); C.B. Thorn, *ibid.* **B248**, 551 (1984).
 - [4] D. Friedan, Z. Qiu, and S. Shenker, *Phys. Lett.* **151B**, 37 (1985); M.A. Bershadsky, V.G. Knizhnik, and M.G. Teitelman, *ibid.* **151B**, 31 (1985).
 - [5] D. Kastor, E. Martinec, and Z. Qiu, *Phys. Lett. B* **200**, 434 (1988); J. Bagger, D. Nemeschansky, and S. Yankielowicz, *Phys. Rev. Lett.* **60**, 389 (1988); F. Ravanini, *Mod. Phys. Lett. A* **3**, 397 (1988).
 - [6] P.C. Argyres, J. Grochocinski, and S.-H.H. Tye, *Nucl. Phys.* **B367**, 217 (1991); **B391**, 409 (1993).
 - [7] V.G. Knizhnik and A.B. Zamolodchikov, *Nucl. Phys.* **B247**, 83 (1984).
 - [8] A.B. Zamolodchikov and V.A. Fateev, *Theor. Math. Phys.* **71**, 451 (1987).
 - [9] P.C. Argyres, A. LeClair, and S.-H.H. Tye, *Phys. Lett. B* **253**, 306 (1991).
 - [10] S. Chung, E. Lyman, and S.-H.H. Tye, *Int. J. Mod. Phys. A* **7**, 3339 (1992).
 - [11] G. Felder, *Nucl. Phys.* **B317**, 215 (1989).
 - [12] P. Goddard, A. Kent, and D. Olive, *Commun. Math. Phys.* **103**, 105 (1986).
 - [13] P.C. Argyres and S.-H.H. Tye, *Phys. Rev. Lett.* **67**, 3339 (1991); K.R. Dienes and S.-H.H. Tye, *Nucl. Phys.* **B376**, 297 (1992); P.C. Argyres, K. Dienes, and S.-H.H. Tye, *Commun. Math. Phys.* **154**, 471 (1993); P. Frampton and J. Liu, *Phys. Rev. Lett.* **70**, 130 (1993); P.C. Argyres and K.R. Dienes, *ibid.* **71**, 819 (1993); G.B. Cleaver and P.J. Rosenthal, "Aspects of Fractional Superstrings," Caltech Report No. CALT-68-1756, 1993 (unpublished).
 - [14] P.C. Argyres and S.-H.H. Tye, *Phys. Rev. D* (to be published).
 - [15] A.B. Zamolodchikov and V.A. Fateev, *Zh. Eksp. Teor. Fiz.* **89**, 380 (1985) [*Sov. Phys. JETP* **62**, 215 (1985)].
 - [16] V.G. Kac, *Adv. Math.* **35**, 264 (1980); V.G. Kac and D. Peterson, *Bull. Am. Math. Soc.* **3**, 1057 (1980); *Adv. Math.* **53**, 125 (1984); J. Distler and Z. Qiu, *Nucl. Phys.* **B336**, 553 (1990).
 - [17] S.K. Yang, *Nucl. Phys.* **B285**, 183 (1987).
 - [18] C. Ahn, S. Chung, and S.-H.H. Tye, *Nucl. Phys.* **B365**, 191 (1991).