Evaluation of the one-loop effective action at zero and finite temperature: Scalar fields

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A method for determining the leading quantum contributions to the effective action for both zero and finite temperatures is presented. While it is described in the context of a scalar field theory, it can be straightforwardly extended to include fermions. An extrapolation procedure which can significantly enhance the computational efficiency is introduced. This formalism is used to investigate quantum corrections to the nucleation rate in first-order phase transitions.

I. INTRODUCTION

The evaluation of quantum corrections to classical solutions is an important problem which pervades much of modern theoretical physics. However, while effective potentials have been studied extensively, methods for determining the effective action are less well developed. Moreover, the actual evaluation of such effects for realistic systems has often been hampered by their general intractability to analytical solution and the lack of efficient computational methods, although there have been efforts to address this problem [1–5].

In the effective potential approximation to the effective action, quantum fluctuations are integrated out about a constant classical field—but this is not expected to be adequate because the classical field is generally an inhomogeneous configuration. The derivative expansion [6] improves on this by accounting for spatially varying background fields; its leading term is the effective potential. The expansion is a perturbative approximation which extracts the dominant contribution of short-distance quantum effects on long-distance physics. When it converges, it provides an efficient means for performing calculations. However, when it diverges, one must often resort to brute-force techniques which entail an explicit, computationally intensive evaluation. Furthermore, the derivative expansion fails whenever the potential V is nonconvex ($V'' < 0$) in some region of space, which includes an important class of perturbatively calculated potentials [7]. It is clear that a general method, which is also applicable to such cases, is needed.

In this paper, a method for calculating the quantum effects arising from the effective action is presented. The next section contains the general formalism for evaluating the effective action. Section III discusses the exact formulation of the computational method as well as extrapolation techniques which improve its convergence properties. This formalism is applied to the analysis of phase transitions in Sec. IV [7].

II. GENERAL FORMALISM

Consider a scalar field theory with the Lagrangian density

$$\mathcal{L}(\phi) = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) , \tag{2.1}$$

where V is the tree-level potential which has a (classical) vacuum at ϕ_v. Since this paper considers $3+1$ dimensions exclusively, renormalizability constrains V to have no polynomials in ϕ of higher power than a quartic. The classical field $\tilde{\phi}$ is determined by the equation

$$\partial_\mu\partial^\mu\tilde{\phi} = -V'(\phi) . \tag{2.2}$$

The contribution of one-loop quantum effects to the effective action can be written as

$$\mathrm{Tr}\mathcal{L}^{(1)} = \frac{i}{2}\mathrm{Tr}\ln\left|\frac{\partial^2 + V''(\tilde{\phi})}{\partial^2 + \mu^2}\right| ,$$

where $\mu^2 = V''(\phi_v)$ and the trace runs over space-time coordinates.[1] Part of this trace can be evaluated as $\mathrm{Tr}\mathcal{L}^{(1)} = \mathrm{Tr}' \int \langle t|\mathcal{L}^{(1)}|t\rangle dt$, where Tr' runs over the remaining spatial degrees of freedom. Since this paper deals only with time-independent $\tilde{\phi}$ fields, specializing to this case means that states in the energy basis $|\omega\rangle$ are eigenstates of the operator in $\mathcal{L}^{(1)}$. So inserting a complete set of such states and performing a partial integration yields

$$\mathrm{Tr}\mathcal{L}^{(1)} = -i\int \mathrm{Tr}'\int\left[\frac{1}{-\omega^2 - \nabla^2 + V''(\tilde{\phi})}\right.$$
$$\left. - \frac{1}{-\omega^2 - \nabla^2 + \mu^2}\right]\omega^2\frac{d\omega}{2\pi}dt .$$

Observe the nonlocality of this expression; this generic feature of loop corrections makes exact analytical treatments difficult. The remaining trace can be conveniently performed over the eigenstates of the operators in $\mathcal{L}^{(1)}$: if ψ_j^0 and ψ_j are chosen such that

$$[-\nabla^2 + \mu^2]\psi_j^0 = (\omega_j^0)^2\psi_j^0 , \tag{2.3}$$

$$[-\nabla^2 + V''(\tilde{\phi})]\psi_j = (\omega_j)^2\psi_j , \tag{2.4}$$

where the subscript j indexes the eigenstates, then

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[1] The trace excludes possible negative and zero modes of the operator $\partial^2 + V''(\tilde{\phi})$. When such modes arise, they must be explicitly removed and treated differently [7].

$$\mathrm{Tr}\mathcal{L}^{(1)}=-\frac{1}{2}\int\sum_j(\omega_j-\omega_j^0)dt \ .$$

Hence the one-loop effective action can be written as

$$S_{\mathrm{eff}}(\tilde{\phi})=\int[\mathcal{L}(\tilde{\phi})-\mathcal{L}(\phi_v)]d^4x$$

$$+\int\left[-\frac{1}{2}\sum_{\omega_j^0<\Lambda}(\omega_j-\omega_j^0)+\int\mathcal{L}_{\mathrm{ct}}(\tilde{\phi},\Lambda)d^3x\right]dt \ .$$

(2.5)

The bare sum in $\mathrm{Tr}\mathcal{L}^{(1)}$ is divergent; it is regulated in Eq. (2.5) by a momentum cutoff Λ, and a counterterm $\mathcal{L}_{\mathrm{ct}}(\tilde{\phi},\Lambda)$ has been added to render it finite.

For time-independent fields $\tilde{\phi}$, it is more convenient to focus on the energy E of the system which is related to S_{eff} through

$$S_{\mathrm{eff}}(\tilde{\phi})=-E(\tilde{\phi})\int dt \ .$$

Then

$$E(\tilde{\phi})=E_{\mathrm{cl}}(\tilde{\phi})+E^{(1)}(\tilde{\phi},\Lambda)+E_{\mathrm{ct}}(\tilde{\phi},\Lambda) \ ,$$

(2.6)

where E_{cl} is the energy of the classical field configuration

$$E_{\mathrm{cl}}(\tilde{\phi})=-\int[\mathcal{L}(\tilde{\phi})-\mathcal{L}(\phi_v)]d^3x \ ,$$

(2.7)

$E^{(1)}$ is the one-loop contribution

$$E^{(1)}(\tilde{\phi},\Lambda)=\frac{1}{2}\sum_{\omega_j^0<\Lambda}(\omega_j-\omega_j^0) \ ,$$

(2.8)

and E_{ct} is the energy due to the one-loop counterterms

$$E_{\mathrm{ct}}(\tilde{\phi},\Lambda)=-\int\mathcal{L}_{\mathrm{ct}}(\tilde{\phi},\Lambda)d^3x \ .$$

(2.9)

At finite temperature T, the free energy F replaces E [7]:

$$F_{\mathrm{cl}}=E_{\mathrm{cl}} \ , \quad \Delta F_1=E^{(1)} \ , \quad F^{\mathrm{ct}}=E_{\mathrm{ct}} \ ;$$

(2.10)

for bosons, $\tilde{\phi}(\mathbf{x},\tau)$ is periodic in Euclidean time τ with period T^{-1}, and there is an additional contribution due to one-loop effects [7]:

$$\Delta F_T=T\sum_j\ln\left[\frac{1-e^{-\omega_j/T}}{1-e^{-\omega_j^0/T}}\right] \ .$$

(2.11)

Observe that no additional counterterms need to be added to F^{ct} because finite temperatures do not change the short-distance behavior of the theory.

In the following section, we describe the method developed to evaluate the quantum corrections ΔF_1 and ΔF_T formally given by Eqs. (2.8) and (2.11). While for some special situations, the ω_j can be obtained analytically, this is unfortunately not possible for a general potential $V'''(\tilde{\phi})$. Instead the eigenvalues must be found numerically; then, for ΔF_1, the bare sum $\sum_{\omega_j^0<\Lambda}(\omega_j-\omega_j^0)$ is computed explicitly, and finally the counterterm subtracted; for ΔF_T, the sum in Eq. (2.11) must be performed term by term. To attain reasonable accuracy this subtraction has to be done at a large cutoff Λ (to achieve convergence) when both the bare sum and the counter-

term (which individually diverge as a function of the cutoff) are numerically very large. Since the final result is much smaller, each term has to be determined very precisely, resulting in a heavy computational burden. Furthermore, the straightforward approach of evaluating the free energy by a "brute-force" term-by-term summation of the expressions in Eqs. (2.8) and (2.11) until convergence is reached is also computationally inefficient.

III. METHOD OF COMPUTATION

To circumvent the above-described problem of having to compute both the regulated bare sum and its counterterm to very high numerical accuracy, the three-dimensional problem is first decomposed into channels of definite angular momentum. Then for each channel, the divergent part of the bare sum is analytically removed through subtraction with the corresponding divergence in the counterterm, leaving a much smaller finite piece. Since the contribution of higher partial-wave channels decreases rapidly, this procedure overcomes the problem.

An improved computational method is then presented. It is based in part on the observation that the higher-energy modes in the spectrum of Eq. (2.4) are less perturbed by the potential $V'''(\tilde{\phi})$ due to the nonuniform background field $\tilde{\phi}$ than the lower-energy ones. This allows us to formulate an approximation method which accounts for the contribution of the high-energy modes accurately (where the accuracy of the approximation increases with the energy) so that only some of the lower-energy modes need to be treated exactly [2].

A. Exact formulation

The difference in the eigenenergies ω_j and ω_j^0 of the unbound states $(\omega>\mu)$ can be characterized by the phase shift between the (asymptotic forms) of the corresponding continuum state eigenfunctions ψ_j and ψ_j^0, as was first shown in one dimension [1]. Since the phase shift is generally a well-behaved, smoothly varying function of the energy, it is relatively easy to calculate. Hence it is convenient to express the free energy in terms of this quantity. To determine the phase shift, we consider Eq. (2.4) which determines the fluctuations about the classical field configuration.

Since most classical solutions $\tilde{\phi}$ exhibit spherical symmetry $[\tilde{\phi}=\tilde{\phi}(r)]$, we will restrict our analysis to such systems. Then the solution to Eq. (2.4) can be separated into radial and angular parts by choosing an eigenfunction of the form

$$\psi_{nlm}(r,\theta,\phi)=\frac{1}{r}u_{nl}(r)Y_{lm}(\theta,\phi) \ ,$$

(3.1)

where the radial wave function is determined by

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+V'''(\tilde{\phi}(r))-\omega_{n,l}^2\right]u_{n,l}(r)=0$$

(3.2)

with the boundary condition $u_{n,l}(0)=0$. The Y_{lm} are the spherical harmonics corresponding to a state with total angular momentum l and z-component m.

The corresponding equation for $u_{n,l}^0$ where $V''(\bar{\phi})$ is replaced by μ^2,

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \mu^2 - (\omega_{n,l}^0)^2 \right] u_{n,l}^0(r) = 0 , \quad (3.3)$$

has an exact analytical solution:

$$u_{nl}^0(r) = \sqrt{2} k_n r j_l(k_n r) , \quad (3.4)$$

where j_l is a spherical Bessel function and $k_n^2 = \omega_n^2 - \mu^2$. These solutions have the asymptotic form

$$u_{nl}^0(r) \rightarrow \sqrt{2} \sin\left[k_n r - \frac{l\pi}{2} \right] , \quad r \rightarrow \infty . \quad (3.5)$$

The potentials we consider behave asymptotically as $V''(\bar{\phi}(r)) \rightarrow \mu^2$ when $r \rightarrow \infty$ (which corresponds to those with finite action). For such potentials, the asymptotic behavior of the solution to Eq. (3.2) will be

$$u_{nl}(r) \rightarrow \sqrt{2} \sin\left[k_n r - \frac{l\pi}{2} + \delta_l(k_n) \right] , \quad r \rightarrow \infty . \quad (3.6)$$

These equations serve to define the phase shift δ_l for each angular momentum channel l. Note also that both $u_{n,l}$ and $u_{n,l}^0$ are $(2l+1)$-fold degenerate.

To facilitate the counting of states, it is convenient to discretize the eigenvalue spectrum. This procedure can be achieved by enclosing the system in a box of radius L (where L is much greater than the range of the potential V'') and imposing the boundary condition

$$u_{n,l}(L) = 0 , \quad (3.7)$$

which requires that

$$k_n L - \frac{l\pi}{2} + \delta_l(k_n) = n\pi . \quad (3.8)$$

Note that such a discretization is implicit in the formal sums in Eqs. (2.8) and (2.11). The values attained by ω^0 (before discretization) as defined by the energy eigenvalue of Eq. (3.3) is a continuous spectrum ranging from an energy of μ to infinity. The corresponding spectrum for ω determined by Eq. (3.2) will generally consist of some discrete bound states with energies $\omega_j^2 < \mu^2$ and a continuous spectrum with energies $\omega_j^2 > \mu^2$. The difference in structure between the continuum spectra of the two systems manifests in a difference in the respective density of states. Hence it is appropriate to express the sum over eigenenergies for the states in the continuum as an integral over the density of states:

$$\sum_j \omega_j^0 = \sum_l (2l+1) \int_\mu^\infty \omega n_l^0(\omega) \, d\omega , \quad (3.9)$$

$$\sum_j \omega_j = \sum_{\omega_{nl}^2 < \mu^2} (2l+1)\omega_{nl} + \sum_l (2l+1) \int_\mu^\infty \omega n_l(\omega) \, d\omega , \quad (3.10)$$

where $(2l+1)n_l(\omega)$ is the density of states of angular momentum l for the potential $V''(\bar{\phi})$ with an analogous definition for n_l^0. On taking the continuum limit $(L \rightarrow \infty)$, it follows from Eq. (3.8) that the densities of states are related to the continuum phase shift through

$$n_l(\omega) = n_l^0(\omega) + \frac{1}{\pi} \frac{d\delta_l(\omega)}{d\omega} . \quad (3.11)$$

Now if Eq. (2.4) has N bound states, then since Eqs. (2.4) and (2.3) must have the same total number of states,

$$N + \sum_l (2l+1) \int_\mu^\infty n_l(\omega) \, d\omega = \sum_l (2l+1) \int_\mu^\infty n_l^0(\omega) \, d\omega . \quad (3.12)$$

For a finite potential, this implies $N\pi = \delta(\mu)$.

It is convenient to define the free energy in each angular momentum channel such that

$$\Delta F_1(\Lambda) = \sum_l (2l+1)\Delta F_1^l(\Lambda) , \quad (3.13)$$

$$\Delta F_T = \sum_l (2l+1)\Delta F_T^l , \quad (3.14)$$

and to similarly partition the counterterm energy as

$$F^{\text{ct}}(\Lambda) = \sum_l (2l+1)F_l^{\text{ct}}(\Lambda) ; \quad (3.15)$$

then, from the above equations,

$$\Delta F_1^l(\Lambda) = \frac{1}{2} \sum_{\omega_{nl}^2 < \mu^2} (\omega_{nl} - \mu) - \frac{1}{2\pi} \sum_l \int_\mu^\Lambda \delta_l(\omega) \, d\omega , \quad (3.16)$$

and

$$\Delta F_T^l = T \sum_{\omega_{nl}^2 < \mu^2} \ln\left[\frac{1 - e^{-\omega_{nl}/T}}{1 - e^{-\mu/T}} \right] - \frac{1}{\pi} \int_\mu^\infty \frac{\delta_l(\omega)}{e^{\omega/T} - 1} \, d\omega . \quad (3.17)$$

The Appendix discusses the renormalization of the Lagrangian given by Eq. (2.1). It is shown there that the contribution of the counterterms to the energy are of the general form

$$\int \left[g(\omega^2 - p^2 - \mu^2) \int h(x) d^3x \right] \frac{d^3p}{(2\pi)^3} \frac{d\omega}{2\pi}$$

$$\equiv \int_{-\infty}^\infty \text{Tr}'(gh) \frac{d\omega}{2\pi} , \quad (3.18)$$

where g is a power of the propagator, h is a function of ϕ and its derivatives, and Tr' is a trace over the spatial variables. The partial wave decomposition of these contributions is achieved by taking the trace with respect to the eigenstates of Eq. (2.3) denoted here by $|nlm\rangle$:

$$\mathrm{Tr}'(gh) = \sum_{nlm} \sum_{n'l'm'} \langle nlm | g(\omega^2 + \nabla^2 - \mu^2) | n'l'm' \rangle \langle n'l'm' | h(r) | nlm \rangle$$

$$= \frac{1}{\pi} \sum_l (2l+1) \int_0^\infty \left[g(\omega^2 - p^2 - \mu^2) \int_0^\infty h(r) | u_{pl}^0(r) |^2 dr \right] dp \ . \tag{3.19}$$

From the Appendix, the counterterm contribution to the free energy is

$$F^{\mathrm{ct}}(\Lambda) = -\frac{i}{2} \int_{-\infty}^\infty \mathrm{Tr}' \{ \Delta_0(\omega, p) [m^2(r) - \mu^2] + \tfrac{1}{2} \Delta_0(\omega, p)^2 [m^2(r) - \mu^2]^2 \} \frac{d\omega}{2\pi} \ , \tag{3.20}$$

where

$$\Delta_0(\omega, p) = \frac{1}{\omega^2 - p^2 - \mu^2 + i\epsilon} \ ,$$

and $m^2(r) = V''(\tilde{\phi}(r))$. Evaluating the trace using Eq. (3.19) yields

$$F_l^{\mathrm{ct}}(\Lambda) = \int_0^{\Lambda_p} \left\{ -\frac{1}{4\pi} \frac{1}{(p^2 + \mu^2)^{1/2}} \right.$$

$$\times \int_0^\infty | u_{pl}^0(r) |^2 [m^2(r) - \mu^2] dr$$

$$+ \frac{1}{16\pi} \frac{1}{(p^2 + \mu^2)^{3/2}}$$

$$\left. \times \int_0^\infty | u_{pl}^0(r) |^2 [m^2(r) - \mu^2]^2 dr \right\} dp \ , \tag{3.21}$$

where $\Lambda_p = \sqrt{\Lambda^2 - \mu^2}$ is a three-momentum cutoff.

This completes the formulation of the method for the exact calculation of the free energy. However, as we have remarked above, the convergence of such an exact computation can be sufficiently slow so that extrapolation techniques can be useful. Among the various such procedures, we consider in particular the WKB approximation, which provides an analytic expression for the phase shift that is valid at high energies and hence can significantly reduce the effort required to evaluate the phase shift integral [2].

B. WKB-improved method

A differential equation of the form

$$\left[\frac{d^2}{dx^2} + k^2(x) \right] f(x) = 0 \tag{3.22}$$

has an approximate WKB solution given by

$$f_{\mathrm{WKB}}(x) = \frac{\exp \left[i \int_0^x k(\omega, y) dy \right]}{\sqrt{k(\omega, x)}} \ , \tag{3.23}$$

which is valid when the wavelength is much less than the distance scale over which k varies:

$$\frac{1}{k^2} \frac{dk}{dx} \ll 1 \ ,$$

where $k(\omega, x)$ is the local wave number

$$k(\omega, x) = \sqrt{\omega^2 - V''(\tilde{\phi}(x))} \ .$$

Hence the accuracy of the WKB approximation increases with energy. The phase shift for such solutions is given by

$$\delta^{\mathrm{WKB}}(\omega) = \int_{-\infty}^\infty [k(\omega, x) - \lim_{y \to \infty} k(\omega, y)] dx \ . \tag{3.24}$$

Explicitly,

$$\delta_l^{\mathrm{WKB}}(\omega) = \int_{a(\omega)}^\infty \left[\omega^2 - m^2(r) - \frac{l(l+1)}{r^2} \right]^{1/2} dr$$

$$- \int_{a_0(\omega)}^\infty \left[\omega^2 - \mu^2 - \frac{l(l+1)}{r^2} \right]^{1/2} dr \ , \tag{3.25}$$

where a and a_0 denote the classical turning points defined by

$$\omega^2 - m^2(a) - \frac{l(l+1)}{a^2} = 0 \ , \quad \text{and} \quad \omega^2 - \mu^2 - \frac{l(l+1)}{a_0^2} = 0 \ .$$

Applying this method to Eq. (3.2) yields an analytic expression for the energy integral of the phase shift:

$$\int_\mu^\Lambda \delta_l^{\mathrm{WKB}}(\omega) d\omega = \int_0^\infty \left[\int_{\Omega(r)}^\Lambda \left[\omega^2 - m^2(r) - \frac{l(l+1)}{r^2} \right]^{1/2} \theta(\Lambda - \Omega(r)) d\omega \right.$$

$$\left. - \int_{\Omega_0(r)}^\Lambda \left[\omega^2 - \mu^2 - \frac{l(l+1)}{r^2} \right]^{1/2} \theta(\Lambda - \Omega_0(r)) d\omega \right] dr \tag{3.26}$$

with

$$\Omega(r)=\left[m^2(r)+\frac{l(l+1)}{r^2}\right]^{1/2} ,$$

$$\Omega_0(r)=\left[\mu^2+\frac{l(l+1)}{r^2}\right]^{1/2} ,$$

and $\theta(x)$ is the unit step function. Observe that since the high-energy behavior of the phase shift is independent of the angular momentum, the energy of each angular

momentum channel is logarithmically divergent:

$$\Delta F_1^l(\Lambda)=\frac{1}{4\pi}\ln\left[\frac{\Lambda}{\mu}\right]\int_0^\infty[m^2(r)-\mu^2]dr+O(\Lambda^0) .$$

$$(3.27)$$

Now the divergent piece in ΔF_1^l can be analytically combined with the infinite part of F_l^{ct} in Eq. (3.21) to leave only finite terms. Performing this subtraction and taking the limit $\Lambda\to\infty$ gives the final expression for the WKB improved, temperature-independent renormalized free energy:

$$\Delta F_{1,\text{ren}}^l=\lim_{\Lambda\to\infty}[\Delta F_1^l(\Lambda)+F_l^{ct}(\Lambda)]$$

$$=\frac{1}{2}\sum_{\omega_{nl}^2<\mu^2}(\omega_{nl}-\mu)-\frac{1}{2\pi}\int_\mu^{\Lambda_{\text{WKB}}}\delta_l(\omega)\,d\omega-\frac{1}{2\pi}\int_0^\infty\{\chi_l(\Lambda_{\text{WKB}},r)+\kappa_l(r)[m^2(r)-\mu^2]$$

$$+\rho_l(r)[m^2(r)-\mu^2]^2\}dr .$$

$$(3.28)$$

In this equation, χ_l is the contribution from the WKB phase shift above Λ_{WKB}:

$$\chi_l(r)=\frac{m_l^0(r)^2-m_l(r)^2}{4}-\frac{1}{2}\Lambda_{\text{eff}}(r)\sqrt{\Lambda_{\text{eff}}(r)^2-m_l(r)^2}+\frac{1}{2}\Lambda_{\text{eff}}^0(r)\sqrt{\Lambda_{\text{eff}}^0(r)^2-m_l^0(r)^2}$$

$$+\frac{1}{2}m_l(r)^2\ln\frac{\Lambda_{\text{eff}}(r)+\sqrt{\Lambda_{\text{eff}}(r)^2-m_l(r)^2}}{\mu}-\frac{1}{2}m_l^0(r)^2\ln\frac{\Lambda_{\text{eff}}^0(r)+\sqrt{\Lambda_{\text{eff}}^0(r)^2-m_l^0(r)^2}}{\mu} ,$$

$$(3.29)$$

where

$$m_l^0(r)^2=\mu^2+\frac{l(l+1)}{r^2} , \quad m_l(r)^2=m^2(r)+\frac{l(l+1)}{r^2} ,$$

$$\Lambda_{\text{eff}}^0(r)=\max(\Lambda_{\text{WKB}},\Omega_0(r)) ,$$

$$\Lambda_{\text{eff}}(r)=\max(\Lambda_{\text{WKB}},\Omega(r)) ,$$

and Λ_{WKB} denotes the energy above which the phase shift is computed by the WKB method. The remaining terms in the last integral come from finite parts of the counterterm with

$$\kappa_l(r)=\int_0^\infty\frac{s^2|j_l(s)|^2-\frac{1}{2}}{\sqrt{s^2+(\mu r)^2}}ds ,$$

$$(3.30)$$

and

$$\rho_l(r)=-\frac{r^2}{4}\int_0^\infty\frac{s^2|j_l(s)|^2}{[s^2+(\mu r)^2]^{3/2}}ds .$$

$$(3.31)$$

Equation (3.28) indicates that $\Delta F_{1,\text{ren}}^l$ can now be computed by first summing over the bound state energies, then the continuum state contributions can be evaluated by explicitly computing the exact phase shift only up to Λ_{WKB}, beyond which the WKB method provides an analytical expression that accounts for contributions at higher energies. Note that while the WKB procedure entails an approximation, its accuracy can be made such that the difference between the exact and the WKB results is smaller than the desired precision. Finally, summation over l yields

$$\Delta F_{1,\text{ren}}=\sum_l(2l+1)\Delta F_{1,\text{ren}}^l .$$

$$(3.32)$$

Since ΔF_T is not divergent, it can be computed exactly using Eqs. (3.14) and (3.16), or by replacing the exact phase shift δ_l above a certain energy scale by the approximate WKB phase shift δ_l^{WKB} given by Eq. (3.25).

IV. APPLICATION AND DISCUSSION

In Ref. [7] these methods have been used to calculate the free energy of an instanton configuration which determines the decay rate in a first-order phase transition. The computation of ΔF_1 will be described first. It is found that the accuracy available on conventional computers prevents a precise determination of this quantity when it is straightforwardly evaluated as in Eq. (2.8)— that is, by doing the bare sum and subtracting the counterterm, without a decomposition into partial waves. When ΔF_1 is computed exactly, by utilizing such a decomposition, very high numerical accuracy is still required because for each l the bare sum and $F_l^{ct}(\Lambda)$ must be evaluated at a large value of the cutoff Λ. But since both quantities diverge as a function of Λ, we find that convergence with reasonable accuracy is still difficult to attain. In contrast, evaluation of $\Delta F_{1,\text{ren}}$ using the WKB improved method consisting of Eqs. (3.28) and (3.32) converges rapidly for much lower values of the cutoff Λ_{WKB} and typically only the first 50 partial waves need to be summed; the parameters required for convergence are very much dependent on the nature of $V''(\tilde\phi(r))$ and the
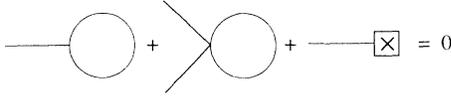
FIG. 1. Renormalization scheme for the divergent one-loop graphs with one vertex. A box with a cross denotes a counterterm insertion.

FIG. 2. Renormalization scheme for the divergent graphs at one-loop order with two vertices.

values we have quoted come from the potentials we have examined.

The exact computation of ΔF_T can be performed by evaluating Eqs. (3.14) and (3.17), but at high temperatures it is found that several hundred partial waves must be summed to attain convergence. When the exact phase shift is replaced by the approximate WKB expression at high energies, there is a reduction in the computational burden and the same number of angular momentum channels must be summed. The improvement is not marked as it was for $\Delta F_{1,\mathrm{ren}}$ in part because ΔF_T is not renormalized. The results of these computations are summarized in Tables IV and V of Ref. [7].

In summary, we have elucidated a method for the exact evaluation the effective action to one loop. The WKB extrapolation scheme was devised to reduce the computational effort. These methods enable an efficient calculation of the free energy associated with a phase transition, as detailed above. However, the applicability of this method is not limited to this example. Rather, it can be utilized in a broader variety of problems involving the nonperturbative evaluation of observables in a nonuniform background in quantum field theory [8] as well as in classical systems [9]. It can also be generalized to encompass theories with fermions [10].

ACKNOWLEDGMENTS

APPENDIX: ONE-LOOP RENORMALIZATION OF THE SCALAR FIELD THEORY

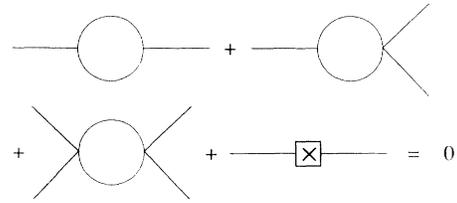This appendix discusses the one-loop renormalization of the scalar field theory described by Eq. (2.1). The clas-

sical vacuum ϕ_v satisfies

$$V'(\phi_v)=0 \quad \text{and} \quad \mu^2=V''(\phi_v)>0 \ . \tag{A1}$$

At one loop the only divergent graphs are those with one and two vertices corresponding to quadratic and logarithmic divergences, respectively.

It is convenient to adopt a renormalization scheme where the counterterms are chosen to exactly cancel the divergent graphs as shown in Figs. 1 and 2. These conditions are imposed at zero external momenta; this choice has the advantage that the one-loop contribution to the effective potential V_1 satisfies

$$V_1(\phi_v)=V_1'(\phi_v)=V_1''(\phi_v)=0 \ , \tag{A2}$$

so that Eq. (A1) is unchanged at one loop. Then the counterterm Lagrangian to be added to Eq. (2.1) is

$$\mathcal{L}_{\mathrm{ct}}=\alpha[V''(\phi)-\mu^2]+\tfrac{1}{2}\beta[V''(\phi)-\mu^2]^2 \ , \tag{A3}$$

where

$$\alpha=\frac{1}{2}\int \frac{i}{k^2-\mu^2+i\epsilon}\frac{d^4k}{(2\pi)^4} \ ,$$

and

$$\beta=\frac{1}{2}\int \frac{i}{(k^2-\mu^2+i\epsilon)^2}\frac{d^4k}{(2\pi)^4} \ .$$

The terms in Eq. (A3) involving α and β renormalize the graphs with one and two external vertices, respectively. These divergent integrals can be suitably regularized by imposing a momentum cutoff Λ.

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