Exact critical bubble free energy and the effectiveness of effective potential approximations

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To calculate the temperature at which a first-order cosmological phase transition occurs, one must calculate $F_c(T)$, the free energy of a critical bubble configuration. $F_c(T)$ is often approximated by the classical energy plus an integral over the bubble of the effective potential; one must choose a method for calculating the effective potential when $V''<0$. We test different effective potential approximations at one loop. The agreement is best if one pulls a factor μ^4/T^4 into the decay rate prefactor [where $\mu^2=V''(\phi_f)$], and takes the real part of the effective potential in the region $V''<0$. We perform a similar analysis on the one-dimensional kink.

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I. INTRODUCTION

Thermal tunneling and the critical bubble free energy

A scalar field theory whose potential V has two local minima may tunnel out of the false vacuum (ϕ_f) by the nucleation and subsequent growth of bubbles of true vacuum (ϕ_t). While we will refer to V as the "classical" potential, it may arise in part from integrating out other particles in the theory, e.g., gauge bosons [1], so V may have implicit temperature (T) dependence. The nucleation rate per unit volume in the static limit ($RT\gg 1$) is calculated in the Gaussian approximation (i.e., to one-loop order) to be [2–4]

$$\frac{\Gamma}{\mathcal{V}}=\frac{1}{\mathcal{V}}\frac{|\omega_-|}{\pi T}\frac{1}{2}T\left|\frac{\det[\partial^2+V''(\bar\phi)]}{\det[\partial^2+\mu^2]}\right|^{-1/2}e^{-E_c/T}, \quad (1.1)$$

where $\mu^2=V''(\phi_f)$. E_c is the classical energy of the critical bubble, a static and spherically symmetric field configuration $\bar\phi(r)$, of radius R, which extremizes the classical action [5] subject to period boundary conditions in Euclidean time. The determinants range over a complete basis of fluctuations about the classical solution [$\bar\phi(r)$ or ϕ_f], subject to the same boundary conditions. $\omega_-^2<0$ is the eigenvalue of the "breathing" mode about $\bar\phi(r)$. The second term on the right-hand side of Eq. (1.1) is from Affleck [3], and the $\frac{1}{2}$ is from analytically continuing the breathing mode integration [2].

With the periodic boundary conditions

$$\det[\partial^2+V''(\bar\phi)]=\exp\left\{\sum_{n=-\infty}^{\infty}\sum_j \ln[(2\pi nT)^2+\omega_j^2]\right\}, \quad (1.2)$$

where the ω_j^2 are eigenvalues of $[-\nabla^2+V''(\bar\phi)]$, and the $(\omega_j^0)^2$ are eigenvalues of $[-\nabla^2+\mu^2]$. We use the identity [6]

$$\frac{T}{2}\sum_n \ln[(2\pi nT)^2+\omega^2] =\frac{\omega}{2}+T\ln(1-e^{-\omega/T})+\mathcal{C} =T\ln\left[2\sinh\left[\frac{\omega}{2T}\right]\right]+\mathcal{C}\,. \quad (1.3)$$

The constants $\mathcal{C}$ cancel out in Eq. (1.1). The ω_- contribution is then traditionally pulled back into the prefactor. The three "translation" modes ($n=0$ and $\omega_0=0$) are not treated correctly above; they actually give $\mathcal{V}(E_c/2\pi T)^{3/2}$ in the prefactor [2], and the remaining ω_0 contribution (from $n\neq 0$ modes) gives T^3 in the prefactor. This gives

$$\frac{\Gamma}{\mathcal{V}}=\frac{T^4}{2\pi}\left[\frac{E_c}{2\pi T}\right]^{3/2}\frac{|\omega_-|/2T}{\sin(|\omega_-|/2T)}e^{-F_c^{\rm trad}/T}, \quad (1.4)$$

where the "traditional" bubble free energy

$$F_c^{\rm trad}\equiv E_c+\Delta F_{1+T}^{\rm trad}\equiv E_c+\Delta F_1^{\rm trad}+\Delta F_T^{\rm trad}, \quad (1.5)$$

$$\Delta F_1^{\rm trad}=\sum_j{}'\frac{\omega_j-\omega_j^0}{2}+F^{\rm ct}, \qquad \Delta F_T^{\rm trad}=\sum_j{}'\,T\ln\left[\frac{1-e^{-\omega_j/T}}{1-e^{-\omega_j^0/T}}\right]. \quad (1.6)$$

Primes on the sums in Eq. (1.6) indicate the omission of the translation and breathing modes ($\omega_j, j=1-4$). Counterterms $F^{\rm ct}$ are discussed below.

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We now define[1]

$$F_c^{\rm sub} \equiv E_c + \Delta F_{1+T}^{\rm sub} \equiv E_c + \Delta F_1^{\rm sub} + \Delta F_T^{\rm sub} \ , \qquad (1.7)$$

$$\Delta F_1^{\rm sub} \equiv \Delta F_1^{\rm trad} \ ,$$
$$\Delta F_T^{\rm sub} \equiv \Delta F_T^{\rm trad} - 4T \ln(T/\mu) \ . \qquad (1.8)$$

Now Eq. (1.4) becomes

$$\frac{\Gamma}{\mathcal{V}} = \frac{\mu^4}{2\pi} \left[\frac{E_c}{2\pi T} \right]^{3/2} \frac{|\omega_-|/2T}{\sin(|\omega_-|/2T)} e^{-F_c^{\rm sub}/T} \ . \qquad (1.9)$$

We will find that the effective potential approximation most closely approximates $F_c^{\rm sub}$.

The effective potential

The sums in Eq. (1.6) are often approximated by treating the fluctuations locally as plane waves to get an effective potential $V_{1+T} = V_1 + V_T$, then integrating $[V_{1+T}(\bar\phi) - V_{1+T}(\phi_f)]$ over all space. No attempt is made to remove the four translation and breathing modes. In Eq. (1.6) one substitutes

$$\sum_j \to \int d^3\mathbf{x} \int_0^\Lambda \frac{d^3\mathbf{k}}{(2\pi)^3} \ ,$$
$$\omega_j \to \sqrt{\mathbf{k}^2 + V''(\bar\phi)}, \quad \omega_j^0 \to \sqrt{\mathbf{k}^2 + \mu^2} \ , \qquad (1.10)$$

and one finds, with $m^2 \equiv V''(\phi)$,

$$V_1(\phi) = \frac{1}{64\pi^2} \left\{ m^4 \ln \left[\frac{m^2}{\mu^2} \right] - \frac{3}{2} m^4 + 2m^2\mu^2 - \frac{1}{2}\mu^4 \right\} \ , \qquad (1.11)$$

$$V_T(\phi) = \frac{T^4}{2\pi^2} I(m/T) \ ,$$
$$I(y) \equiv \int_0^\infty dx\, x^2 \ln(1 - e^{-\sqrt{x^2+y^2}}) \ . \qquad (1.12)$$

The expansion of $I(y)$ for real $y < 2\pi$ is [6,7]

$$I(y) = \frac{-\pi^4}{45} + \frac{\pi^2}{12} y^2 - \frac{\pi}{6} y^3 - \frac{y^4}{32} \left[\ln y^2 - c_3 + \sum_{k=1}^{\infty} \frac{4(2k)!\zeta(2k+1)}{k!(k+2)!} \left[\frac{-y^2}{16\pi^2} \right]^k \right] \ , \qquad (1.13)$$

where $c_3 = \frac{3}{2} + 2\ln(4\pi) - 2\gamma \approx 5.4076$. We choose a renormalization scheme in which all divergent graphs are precisely canceled by counterterms so that at zero external momenta, $V_1(\phi_f) = V_1'(\phi_f) = V_1''(\phi_f) = 0$ (and there is no wave-function renormalization) [8], specifically

$$F^{\rm ct} = \frac{-1}{64\pi^2} \int d^3\mathbf{x} \left\{ [4\Lambda^4 + \tfrac{1}{2}\mu^4] + m^2[4\Lambda^2 - 2\mu^2] + m^4 \left[2 - \ln \left[\frac{4\Lambda^2}{\mu^2} \right] \right] \right\} \Bigg|_{m^2=\mu^2}^{m^2=V''} \ . \qquad (1.14)$$

In the region $m^2 < 0$ we must modify these results to give a real answer. For V_1 we will always take the real part of Eq. (1.11). For V_T let us keep the first equation of Eq. (1.12), but replace $I(m/T)$ by $I^{(\rm neg)}(|m|/T)$ where

$$I^{(\rm neg)}(Y) \equiv \frac{-\pi^4}{45} - \frac{\pi^2}{12} Y^2 + Y^3[a + b\ln(Y^2)] - \frac{Y^4}{32}[\ln(Y^2) - c_3 + c] + \cdots \ . \qquad (1.15)$$

Methods we consider are then parametrized by $\{a,b,c\}$. The most common and obvious method (A) is to take the real part of Eq. (1.13), corresponding to $\{a=b=c=0\}$. Another method (B), proposed in Ref. [9], replaces the lower limit of integration in Eq. (1.10) by $k = {\rm Im}\{m\}$ (eliminating fluctuations with wavelengths longer than the bubble thickness), and corresponds to $\{a = \frac{4}{9} - \frac{1}{3}\ln(2), b = \frac{1}{6}, c = 0\}$.

The derivative expansion

For configurations $\bar\phi(\mathbf{x})$ which vary slowly, the effective potential approximation is the leading term in a derivative expansion of the free energy. The next term (at high T) is [10,11]

$$\Delta F_T^{\rm der} - \Delta F_T^{\rm pot} = \frac{T}{192\pi} \int d^3\mathbf{x}\, m^{-1} \nabla^2(m^2) \ , \qquad (1.16)$$

and again we take the real part (Method A) when necessary. More terms are given explicitly in Ref. [11]; they become increasingly divergent at $m^2 = 0$, where the derivation breaks down (because an integration by parts becomes invalid). Also, no attempt is made to omit the problematic modes. The usefulness of Eq. (1.16) is thus highly suspect, but we note that derivative corrections are predicted to be $O(T^1)$.

Scales, approximations, and goals

Our generic tree-level potential will be quartic in ϕ with $\phi_f = 0$, $V''(0) = \mu^2$, and $\phi_t = \sigma$. By rescaling [12] $\phi = \sigma\tilde\phi$, $x = \tilde x/\mu$, and $T = \mu\tilde T$ we can rewrite the four-action S_0 as

[1]This is somewhat like removing the lowest 4 ω_j^0's from the sums in Eq. (1.6), in addition to the lowest 4 ω_j's, since their contribution to $F_c^{\rm trad}$ is $-4[\mu/2 + T\ln(1 - e^{-\mu/T})] \approx 4T\ln(T/\mu)$.

$$S_0=\frac{E_c}{T} = \left[\frac{\sigma}{\mu}\right]^2 \frac{1}{\tilde{T}} \int d^3\tilde{\mathbf{x}} \left\{ \frac{1}{2}\left[\frac{d\tilde{\phi}}{d\tilde{r}}\right]^2 - \left[\frac{1}{2}\tilde{\phi}^2 - \frac{2\kappa+1}{3}\tilde{\phi}^3 + \frac{\kappa}{2}\tilde{\phi}^4\right]\right\} . \quad (1.17)$$

$\kappa \geq 1$ is a dimensionless parameter; $\kappa \to 1$ (degenerate minima) is the thin-wall limit, while larger κ gives thicker bubbles. With tildes indicating dimensionless results,

$$E_c=(\sigma^2/\mu)\tilde{E}_c, \quad \Delta F_{1,T}=\mu\Delta\tilde{F}_{1,T} . \quad (1.18)$$

The loop expansion [13] is an expansion in $(\mu/\sigma)^2$ and $\tilde{T}$. It is sometimes claimed that higher loops should eliminate the complex terms in $F_c^{\rm pot}$, but this cannot be generally true since the higher-loop contributions are suppressed by these arbitrary parameters. Henceforth we will drop the tildes and work in the rescaled theory (i.e., set $\mu=\sigma=1$).

We always use the static approximation [14] ($RT \gg 1$) and the one-loop approximation. In Sec. III we will use the thin-wall approximation, $R \gg 1$. At times we will make high-temperature expansions, requiring $T \geq 1$ (note the thin-wall and high-temperature limits together imply the static limit). We are examining the validity of the effective potential approximation.

In this paper we will study several systems: the one-dimensional kink, the thin-wall bubble, and two thick-wall bubbles. We will calculate ΔF_1 and ΔF_T for each system exactly [$F_c^{\rm sub}$ in Eq. (1.8)], in the effective potential approximation [$F_c^{\rm pot}$ from Eqs. (1.11) and (1.12), using different methods to calculate $I^{(\rm neg)}$ in Eq. (1.15)], and using the next term of the derivative expansion [$F_c^{\rm der}$ from Eq. (1.16)].

II. THE ONE-DIMENSIONAL KINK

Classical results

We warm up by calculating the free energy of a kink in one spatial dimension [15]:

$$\frac{d^2\bar{\phi}}{dx^2}=V'(\bar{\phi}) , \quad \frac{d\bar{\phi}}{dx}=-\sqrt{2V(\bar{\phi})} , \quad V(\phi)=\tfrac{1}{2}\phi^2(1-\phi)^2 . \quad (2.1)$$

The potential is that of Eq. (1.17) with $\kappa=1$. The kink solution is (up to an arbitrary shift in coordinate)

$$\bar{\phi}(x)=\tfrac{1}{2}[1-\tanh(\tfrac{1}{2}x)] , \quad V''[\bar{\phi}(x)]=1-\tfrac{3}{2}\mathrm{sech}^2(\tfrac{1}{2}x) . \quad (2.2)$$

Equation (2.1) allows us to convert integrals over x into integrals over ϕ:

$$\int_{-\infty}^{\infty} dx \to \int_0^1 \frac{d\phi}{\phi(1-\phi)} . \quad (2.3)$$

For example, the classical energy is

$$E_c^{\rm 1D}=\int_0^1 \frac{d\phi}{\phi(1-\phi)}\phi^2(1-\phi)^2=\tfrac{1}{6} . \quad (2.4)$$

Note that in one dimension (1D) [compare with Eq. (1.18)] $E_c=\mu\sigma^2\tilde{E}_c$ and $\Delta F_{1,T}=\mu\Delta\tilde{F}_{1,T}$, so with scales restored $E_c^{\rm 1D}=\mu\sigma^2/6$.

Exact results from the eigenvalue sum

The solutions to the eigenvalue equations (setting $\mu=1$) are known [15,16]:

$$\omega_s^0=\sqrt{(k_s^0)^2+1}, \quad \omega_1=0, \quad \omega_2=\sqrt{3}/2 , \quad \omega_{s>2}=\sqrt{(k_s)^2+1} , \quad k_s^0=\frac{\pi s}{L}, \quad k_s=\frac{\pi s-\delta(k_s)}{L} , \quad \delta(k)=2\pi-2\arctan(k)-2\arctan(2k) , \quad (2.5)$$

where we have imposed vanishing boundary conditions on a box of length L, so s is a positive integer. We drop the translation mode eigenvalue ω_1; there is no negative eigenvalue in 1D. In the continuum limit

$$\Delta F_1^{\rm trad}=\frac{\sqrt{3}}{4}+\int_0^\Lambda \frac{dk}{\pi}\frac{d\delta}{dk}\frac{\sqrt{k^2+1}}{2}-\frac{3}{2\pi}+F^{\rm ct} , \quad \Delta F_T^{\rm trad}=T\ln(1-e^{-\sqrt{3}/2T}) + \int_0^\infty \frac{dk}{\pi}\frac{d\delta}{dk}T\ln(1-e^{-\sqrt{k^2+1}/T}) . \quad (2.6)$$

In our renormalization scheme the 1D counterterms analogous to Eq. (1.14) are

$$F^{\rm ct}=\frac{-1}{16\pi}\int dx\{[4\Lambda^2+1]+m^2[2+2\ln(4\Lambda^2)] - m^4\}\Big|_{m^2=1}^{m^2=V''} = \frac{1}{8\pi}[3+6\ln(4\Lambda^2)] . \quad (2.7)$$

(This differs from Ref. [15] by $3/8\pi$ due to different renormalization schemes; also note their $m^2 \equiv \mu^2/2$.) We define $\Delta F_1^{\rm sub}\equiv\Delta F_1^{\rm trad}$ and $\Delta F_T^{\rm sub}\equiv\Delta F_T^{\rm trad}-T\ln(T/\mu)$, and find

$$\Delta F_1^{\rm sub}=\frac{1}{4\sqrt{3}}-\frac{9}{8\pi}=-0.2138 , \quad (2.8)$$

$$\Delta F_{1+T}^{\rm sub}=-(\ln\sqrt{12})T+\frac{3}{2\pi}\ln(T)+\frac{6c_1-3}{8\pi} + \frac{3\zeta(3)}{32\pi^3}T^{-2}+\cdots , \quad (2.9)$$

where $c_1=1+2\ln(4\pi)-2\gamma \approx 4.9076$, and $\zeta(3)\approx 1.2021$. These results are in the row marked "sub" of Table I.

TABLE I. Kink free energy in low- and high-T regimes.

Method	ΔF_1	ΔF_{1+T} $T\ln(T)$	T	$\ln(T)$	1	T^{-1}	T^{-2}
sub	−0.2138	0	−1.2425	0.4775	1.0522	0	0.0036
pot(A)	−0.0916	0	−2.1145	0.4475	1.0522	0	0.0036
der(A)	−0.0916	0	−2.1730	0.4775	1.0522	0	0.0036
pot(B)	−0.0916	0.4495	−1.7222	0.4775	1.0522	0.0045	0.0036

1D effective potential and derivative expansion results

The 1D effective potential for real m is [7]

$$V_1=\frac{-m^2}{8\pi}\ln(m^2)+\frac{m^4-1}{16\pi}, \quad V_T=\frac{T^2}{\pi}\hat{I}(m/T) , \quad (2.10)$$

$$\hat{I}(y)=\frac{-\pi^2}{6}+\frac{\pi y}{2}+\frac{y^2}{8}[\ln(y^2)-c_1] - \frac{\zeta(3)y^4}{64\pi^2}+\cdots . \quad (2.11)$$

For $m^2<0$ we replace $\hat{I}(m/T)$ by $\hat{I}^{(\text{neg})}(|m|/T)$ where

$$\hat{I}^{(\text{neg})}(Y)=\frac{-\pi^2}{6}+Y[\hat{a}+\hat{b}\ln(Y^2)] - \frac{Y^2}{8}[\ln(Y^2)-c_1+\hat{c}]+\cdots . \quad (2.12)$$

Method A gives $\{\hat{a}=\hat{b}=\hat{c}=0\}$ and method B gives $\{\hat{a}=1-\ln(2), \hat{b}=-1/2, \hat{c}=0\}$.

We integrate (the real part of) V_1 from Eq. (2.10) over all space, using Eq. (2.3), to get $\Delta F_1^{\text{pot}(A)}=-0.916$, which differs significantly from $\Delta F_1^{\text{sub}}=-0.2138$ (note each result is renormalization dependent, but the difference is not). This difference, which was calculated in Ref. [15], dominates the low-T regime.

A similar integral for the high-T expansion gives

$$\Delta F_{1+T}^{\text{pot}(A)}=\ln[2(\sqrt{3}-\sqrt{2})^{\sqrt{6}}]T+\frac{3}{2\pi}\ln(T) + \frac{6c_1-3}{8\pi}+\frac{3\zeta(3)}{32\pi^3}T^{-2}+\cdots , \quad (2.13)$$

as shown in the line marked "pot(A)" of Table I. Note that the difference between the true result and the effective potential approximation no longer lies in the constant term, but only (as far as we have taken the expansion) in the T term. It is

$$\Delta F_{1+T}^{\text{sub}}-\Delta F_{1+T}^{\text{pot}(A)}=-\ln[4\sqrt{3}(\sqrt{3}-\sqrt{2})^{\sqrt{6}}]T =0.8720\,T . \quad (2.14)$$

The next term of the derivative expansion [analogous to Eq. (1.16)] is

$$\Delta F_T^{\text{der}}-\Delta F_T^{\text{pot}}=\frac{T}{96}\int dx\, m^{-3}\nabla^2(m^2) =\frac{\sqrt{6}}{48}\ln(\sqrt{3}-\sqrt{2})T =-0.0585\,T \quad (2.15)$$

as incorporated in the third line of Table I. It is a very poor approximation to Eq. (2.14)!

Results from method B are given in the fourth line of Table I; these are also unsatisfactory. In fact, the choice $\{\hat{a}=1.940, \hat{b}=\hat{c}=0\}$ in Eq. (2.12) would give the correct ("sub") results, but it is not clear if there is any physics in this choice.

III. THE THIN-WALL CRITICAL BUBBLE

Classical results

For κ close to (but larger than) unity in Eq. (1.17), the solution to

$$\nabla^2\bar{\phi}=V'(\bar{\phi}) \quad (3.1)$$

is a thin-wall bubble, given approximately by the kink solution in the radial coordinate, Eq. (2.2) with $x=r-R$ and $R\gg1$ [2]. The tree-level critical bubble energy has volume and surface terms:

$$E_c=4\pi\int r^2dr\left[\frac{1}{2}\left[\frac{d\bar{\phi}}{dr}\right]^2+V(\bar{\phi}(r))\right] \approx -\frac{4}{3}\pi R^3|V(1)|+4\pi R^2E_c^{1\text{D}} , \quad (3.2)$$

where $E_c^{1\text{D}}=\frac{1}{6}$ was given in Eq. (2.4), and $|V(1)|=(\kappa-1)/6$. We extremize to find the bubble radius R and energy E_c:

$$R=\frac{2}{\kappa-1}, \quad E_c=\frac{8\pi}{9(\kappa-1)^2}=\frac{2\pi R^2}{9} . \quad (3.3)$$

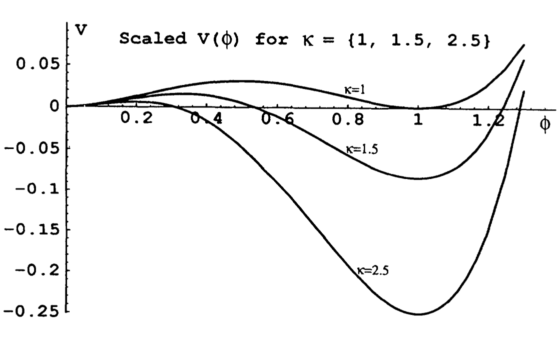

FIG. 1. The potential $V(\phi)$ for several κ's.

The wall thickness is of order 1 (i.e., μ^{-1}). It can also be shown [2] that $\omega_-^2 \approx -2/R^2$, so the static and thin-wall limits imply that the third factor of Eq. (1.4) is near unity.

Exact results for a domain wall

In the thin-wall limit the surface free-energy density $f_{1,T}=\Delta F_{1,T}/(4\pi R^2)$ of the bubble wall equals that of a planar domain wall [17]. We can thus solve the eigenvalue equation in Cartesian coordinates, using Eq. (2.5) for the radial wave number k_r, and plane waves for the tangential k_t, to get

$$
\begin{aligned}
f_1^{\rm sub} &= \int_0^{\Lambda} \frac{k_t dk_t}{2\pi}\left\{\frac{k_t}{2}+\frac{\sqrt{k_t^2+3/4}}{2}-\frac{\sqrt{\Lambda^2+1}}{2\pi}\delta(\sqrt{\Lambda^2-k_t^2})\right.\\
&\qquad \left. +\int_0^{\sqrt{\Lambda^2-k_t^2}}\frac{dk_r}{\pi}\left[\frac{-2}{k_r^2+1}+\frac{-4}{4k_r^2+1}\right]\frac{\sqrt{k_t^2+k_r^2+1}}{2}\right\}+\frac{3\Lambda^2}{8\pi^2}-\frac{3}{32\pi^2}\ln(4\Lambda^2)\\
&= \frac{-1}{32\pi^2}\left[\frac{\pi}{\sqrt{3}}+6\right]=-0.024\,74\ ,\\
f_T^{\rm sub} &= T\int_0^{\infty}\frac{k_t dk_t}{2\pi}\left\{\ln[1-e^{-k_t/T}]+\ln[1-e^{-\sqrt{k_t^2+3/4}/T}]\right.\\
&\qquad \left. +\int_0^{\infty}\frac{dk_r}{\pi}\left[\frac{-2}{k_r^2+1}+\frac{-4}{4k_r^2+1}\right]\ln[1-e^{-\sqrt{k_t^2+k_r^2+1}/T}]\right\}\ .
\end{aligned}
\tag{3.4}
$$

We have performed the f_T integration numerically, and fit to an expansion in T^{-1}; the results are shown in Table II in the row marked "sub."[2]

Effective potential and derivative expansion results

Results from integrating the effective potential and the next term of the derivative expansion, over the bubble [again using Eq. (2.3)] are shown in the rest of Table II. Using the general $I^{(\rm neg)}$ of Eq. (1.15) gives

$$
\begin{aligned}
f_{1+T}^{\rm pot} &= -\frac{1}{4}T^2-(0.0518\,b)T\ln(T)\\
&\quad +(0.1545+0.0259\,a-0.0242\,b)T\\
&\quad -(0.0190)\ln(T)+(-0.056\,12-0.000\,514\,c)\ .
\end{aligned}
\tag{3.5}
$$

Matching this to the true $f_{1+T}^{\rm sub}$ gives the coefficients $\{a,b,c\}$ shown in the first line ($\kappa=1$) of Table III.[3]

TABLE II. Thin-wall bubble free energy density for low and high T.

Method	f_1	f_{1+T}: T^2	$T\ln(T)$	T	$\ln(T)$	1	T^{-1}	T^{-2}
sub	−0.024 74	−1/4	0	0.152 15	−0.019 00	−0.03712	0	−0.000 12
pot(A)	−0.006 61	−1/4	0	0.154 52	−0.019 00	−0.056 12	0	−0.000 12
der(A)	−0.006 61	−1/4	0	0.151 87	−0.019 00	−0.056 12	0	−0.000 12
pot(B)	−0.006 61	−1/4	0.008 64	0.164 09	−0.019 00	−0.056 12	0.000 06	−0.000 12

[2]These results are also useful for the study of second-order phase transitions, in which the domain-wall free energy density is set to zero [17]. Restoring units,

$$
f[\bar\phi^{\rm wall}]=\mu\left[\frac{\sigma^2}{6}-\frac{T_c^2}{4}+0.152\,15\,\mu T_c-0.019\,00\,\mu^2\ln(T_c/\mu)-\cdots\right]=0\ ,
$$

giving, for $\mu \ll \sigma$, $T_c=\sqrt{2/3}\sigma+0.3\mu+\cdots$. That is, the critical temperature is a bit higher than the leading result which is in the literature.

[3]First subtracting the derivative correction of Eq. (1.16) from $\Delta F_{1+T}^{\rm sub}$ would give a values of 0.0109, 0.3877, and 0.5128, respectively. For the kink it gives $\hat a=2.070$. These results are no more enlightening.

TABLE III. $I^{(\text{neg})}$ parameters that make $\Delta F^{\text{pot}}_{1+T}=\Delta F^{\text{sub}}_{1+T}$.

κ	a	b	c
1	−0.0913	0	−36.974
1.5	0.2834	0	−1.424
2.5	0.4188	0	−0.180

We see "derivative corrections" are $O(T)$. The derivative expansion prediction, f^{der}_{1+T} from Eq. (1.16), is a reasonable approximation to them in this case.

IV. THICK-WALL CRITICAL BUBBLES

Classical results

From Eq. (1.17), the (scaled) potential (Fig. 1) is

$$V=\frac{1}{2}\phi^2-\frac{2\kappa+1}{3}\phi^3+\frac{\kappa}{2}\phi^4 . \tag{4.1}$$

Larger $\kappa>1$ gives thicker bubbles. The minima are at $\phi=0$ and $\phi=1$, with $V''(0)=1$ and $V''(1)=2\kappa-1$. The bubble profile is the solution to

$$\bar{\phi}''+2\bar{\phi}'/r=\bar{\phi}(1-\bar{\phi})(1-2\kappa\bar{\phi}) . \tag{4.2}$$

Figure 2 plots $\bar{\phi}(r)$ and $V''(r)$ for $\kappa=1.5$ and $\kappa=2.5$. From Ref. [12], the classical energy is, approximately,

$$E_c\approx\frac{4.85\alpha}{\kappa}\left[1+\frac{\alpha}{4}\left[1+\frac{2.4}{1-\alpha}+\frac{0.26}{(1-\alpha)^2}\right]\right] ,$$
$$\alpha\equiv\frac{9\kappa}{(1+2\kappa)^2} . \tag{4.3}$$

Exact, effective potential, and derivative expansion results

Our method of calculating the exact free energy F_c^{sub}, formally given by Eq. (1.7), is described in Ref. [8]. The results for $\kappa=1.5$ are in Table IV, and for $\kappa=2.5$ in Table V,[4] along with effective potential and derivative expansion approximations. Thin-wall predictions are also shown for two values of R: one chosen to give the correct T^2 coefficient ("Thin-1"), and one given by Eq. (3.3) ("Thin-2"). Finally, the parameters in $I^{(\text{neg})}$ needed to match the effective potential approximation to the exact result are given in Table III.

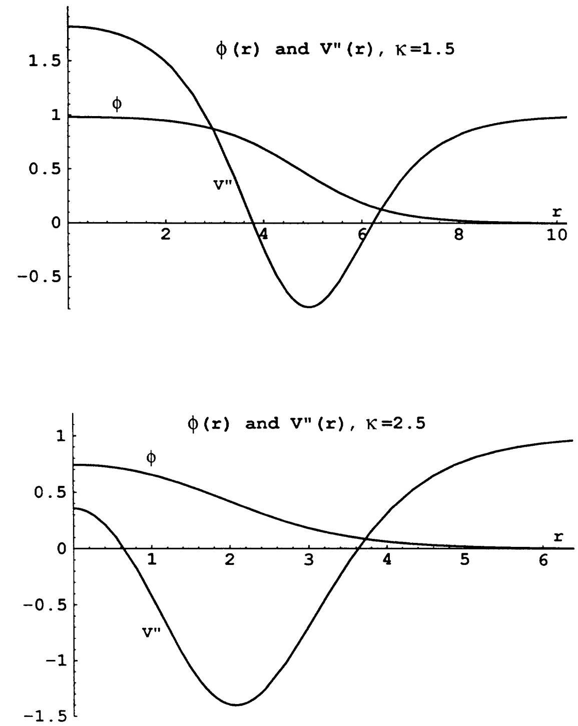

FIG. 2. Thick-wall bubble profiles $\phi(r)$ and $V''(r)$.

V. CONCLUSIONS: A NEW PREFACTOR AND DERIVATIVE CORRECTIONS

We have tested the effective potential approximation to the critical bubble free energy. The agreement is best if one pulls a factor of μ^4/T^4 into the decay rate prefactor, Eq. (1.9), and takes the real part of the effective potential in the region $V''<0$ (method A). That is, $F_c^{\text{pot}(A)}$ closely approximates $F_c^{\text{sub}}\equiv F_c^{\text{trad}}-4T\ln(T/\mu)$. Table III shows that no single set of $I^{(\text{neg})}$ parameters $\{a,b,c\}$ does con-

TABLE IV. Thick-wall bubble free energy for $\kappa=1.5$.

Method	ΔF_1	ΔF_{1+T} T^2	$T\ln(T)$	T	$\ln(T)$	1
sub	−2.13	−78.61	0	49.52	−5.193	−15.64
pot(A)	−2.65	−78.61	0	45.47	−5.193	−16.12
der(A)	−2.65	−78.61	0	43.98	−5.193	−16.12
pot(B)	−2.65	−78.61	4.76	49.73	−5.193	−16.12
Thin-1	−1.81	−78.61	0	47.84	−5.974	−17.65
Thin-2	−4.97	−50.27	0	30.59	−3.820	−7.46

[4]In our fit to the data we allowed a T^{-2} term, not shown, and constrained the T^2, $T\ln(T)$, and $\ln(T)$ terms.

TABLE V. Thick-wall bubble free energy for $\kappa=2.5$.

Method	ΔF_1	ΔF_{1+T} T^2	$T\ln(T)$	T	$\ln(T)$	1
sub	-1.34	-24.90	0	17.17	-1.408	-4.60
pot(A)	-1.009	-24.90	0	14.05	-1.408	-4.64
der(A)	-1.009	-24.90	0	13.35	-1.408	-4.64
pot(B)	-1.009	-24.90	2.48	15.60	-1.408	-4.64
Thin-1	-0.572	-24.90	0	15.15	-1.892	-5.59
Thin-2	-0.553	-5.59	0	3.40	-0.424	-0.83

sistently better than method A. With scales restored, $E_c=O(\sigma^2/\mu)$, $\Delta F^{\text{sub}}_{1+T}=O(T^2/\mu)$, and "derivative corrections" are

$$\Delta F^{\text{sub}}_{1+T}-\Delta F^{\text{pot}(A)}_{1+T}=O(T)\ . \tag{5.1}$$

This difference is numerically fairly small, and very poorly predicted by the derivative expansion [Eq. (1.16)]. In summary,

$$\frac{\Gamma}{\mathcal{V}}=X\frac{\mu^4}{2\pi}\left[\frac{E_c}{2\pi T}\right]^{3/2}\frac{|\omega_-|/2T}{\sin(|\omega_-|/2T)}e^{-F_c^{\text{pot}(A)}/T}\ , \tag{5.2}$$

where $X=e^{[F_c^{\text{pot}(A)}-F_c^{\text{sub}}]/T}$ is a dimensionless number representing derivative corrections. For the thick wall bubbles we examined, X was between 10^{-1} and 10^{-2} (but in the thin-wall limit it appears that $X>1$).

In 1D, where $\Delta F^{\text{sub}}_{1+T}$ is only $O(T)$, derivative corrections [still $O(T)$, and numerically larger] are much more significant than in 3D.

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